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Numerical investigation of fractional-order Kersten–Krasil'shchik coupled KdV–mKdV system with Atangana–Baleanu derivative

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Abstract

In this article, we present a fractional Kersten–Krasil'shchik coupled KdV-mKdV nonlinear model associated with newly introduced Atangana–Baleanu derivative of fractional order which uses Mittag-Leffler function as a nonsingular and nonlocal kernel. We investigate the nonlinear behavior of multi-component plasma. For this effective approach, named homotopy perturbation, transformation approach is suggested. This scheme of nonlinear model generally occurs as a characterization of waves in traffic flow, multi-component plasmas, electrodynamics, electromagnetism, shallow water waves, elastic media, etc. The main objective of this study is to provide a new class of methods, which requires not using small variables for finding estimated solution of fractional coupled frameworks and unrealistic factors and eliminate linearization. Analytical simulation represents that the suggested method is effective, accurate, and straightforward to use to a wide range of physical frameworks. This analysis indicates that analytical simulation obtained by the homotopy perturbation transform method is very efficient and precise for evaluation of the nonlinear behavior of the scheme. This result also suggests that the homotopy perturbation transform method is much simpler and easier, more convenient and effective than other available mathematical techniques.

Keywords: Laplace transform; Homotopy perturbation method; Korteweg–de Vries nonlinear system; Atangana–Baleanu operator

1 Introduction

Many researchers have been working on various aspects of fractional derivatives in recent years. Caputo and Fabrizio modified the existing Caputo derivative to develop the Caputo–Fabrizio fractional derivative [1–5] based on a nonsingular kernel. Because of its advantages, numerous researchers utilized this operator to investigate various types of fractional-order partial differential equations [6–9]. To address this issue, Atangana and Baleanu proposed a new fractional operator called Atangana–Baleanu derivative, which combines Caputo and Riemann–Liouville derivatives. Because of the existence of the Mittag-Leffler kernel, which is a generalization of the exponential kernel, this new Atangana–Baleanu derivative has a long memory. Moreover, the Atangana–Baleanu operator outperforms other operators, and different scientific models have been successfully

solved. Many advances have been made in fractional calculus over the last few years by borrowing ideas from classical calculus, but it does not remain easy. Scholars have the main concern to obtain a numerical solution; for this, numerous efficient methodologies have been constructed for fractional differential equations, such as the Adomian decomposition transform method [10], variational iteration transform method [11, 12], optimal homotopy asymptotic method [13], the homotopy perturbation method [14, 15], etc.

Fractional coupled systems are widely applied to study complex behavior of plasma contains multi components such as ions, free electrons, atoms, etc. Many researchers made efforts to study this behavior numerically. In this direction, recently Paul Kersten and Joseph Krasil'shchik studied KdV equation and modified KdV equation and proposed absolute complexity between coupled KdV–mKdV nonlinear systems for studying the behavior of nonlinear systems [16–19]. Numerous variations of this Kersten–Krasil'shchik coupled KdV–mKdV nonlinear system have been introduced by many researchers [20–27]. Among these variations, the mathematical model for describing the behavior of multi-component plasma for waves propagating in positive χ axis, known as nonlinear fractional Kersten–Krasil'shchik coupled KdV–mKdV system, is given by

$$
\begin{aligned}
&{}^{\text{ABC}}D_\eta^\sigma \psi + \psi_{3\chi} - 6\psi\psi_\chi + 3\phi\phi_{3\chi} + 3\phi_\chi\phi_{2\chi} - 3\psi_\chi\phi^2 \\
&\quad + 6\psi\phi\phi_\chi = 0, \quad \eta > 0, \chi \in R, 0 < \sigma \leq 1, \\
&{}^{\text{ABC}}D_\eta^\sigma \phi + \phi_{3\chi} - 3\phi^2\phi_\chi - 3\psi\phi_\chi + 3\psi_\chi\phi = 0, \quad \eta > 0, \chi \in R, 0 < \sigma \leq 1,
\end{aligned}
\tag{1}
$$

where η is temporal coordinate and χ is spatial coordinate. The factor σ represents order of the fractional operator. This operator is studied in the Caputo form. When $\sigma = 1$, a fractional coupled system converts to the classical system as follows:

$$
\begin{aligned}
&\psi_\eta + \psi_{3\chi} - 6\psi\psi_\chi + 3\phi\phi_{3\chi} + 3\phi_\chi\phi_{2\chi} - 3\psi_\chi\phi^2 + 6\psi\phi\phi_\chi = 0, \quad \eta > 0, \chi \in R, \\
&\phi_\eta + \phi_{3\chi} - 3\phi^2\phi_\chi - 3\psi\phi_\chi + 3\psi_\chi\phi = 0, \quad \eta > 0, \chi \in R.
\end{aligned}
\tag{2}
$$

If we put $\phi = 0$, then the Kersten–Krasil'shchik coupled KdV–mKdV system converts into the well-known KdV system

$$
\psi_\eta + \psi_{3\chi} - 6\psi\psi_\chi = 0, \quad \eta > 0, \chi \in R.
\tag{3}
$$

If we put $\psi = 0$, then the Kersten–Krasil'shchik coupled KdV–mKdV system converts into the well-known modified KdV system

$$
\phi_\eta + \phi_{3\chi} - 3\phi^2\phi_\chi = 0, \quad \eta > 0, \chi \in R.
\tag{4}
$$

In view of that, the Kersten–Krasil'shchik coupled KdV–mKdV system can be assumed to be a combination of a KdV system and a mKdV system represented by (2) to (4).

In this study, we also consider a fractional nonlinear two component homogeneous time fractional coupled third-order KdV system as follows:

$$
\begin{aligned}
&D_\eta^\sigma \psi - \psi_{3\chi} - \psi\psi_\chi - \phi\phi_\chi = 0, \quad \eta > 0, \chi \in R, 0 < \sigma \leq 1, \\
&D_\eta^\sigma \phi + 2\phi_{3\chi} - \psi\phi_\chi = 0, \quad \eta > 0, \chi \in R, 0 < \sigma \leq 1,
\end{aligned}
\tag{5}
$$

where η is temporal coordinate and χ is spatial coordinate, σ is a factor representing the order of the fractional operator. This operator is studied in the Caputo form. When $\sigma = 1$, a fractional coupled system converts to the classical system

$$
\begin{aligned}
&\psi_\eta - \psi_{3\chi} - \psi\psi_\chi - \phi\phi_\chi = 0, \quad \eta > 0, \chi \in R, \\
&\phi_\eta + 2\phi_{3\chi} - \psi\phi_\chi = 0, \quad \eta > 0, \chi \in R.
\end{aligned}
\tag{6}
$$

He [28–30] proposed the homotopy perturbation method for solving numerous linear and nonlinear initial and boundary value problems. Several researchers have looked into using the homotopy perturbation method to solve nonlinear equations in engineering and science [31–33]. Because of the difficulties caused by nonlinear terms, the Laplace transform is completely incapable of handling nonlinear equations. Recently, various methods for dealing with such nonlinearities, such as the Laplace decomposition algorithm [34, 35] and the homotopy perturbation transform method (HPTM) [36, 37], have been proposed to produce highly effective techniques for solving many nonlinear problems. The primary goal of this paper is to use an effective homotopy perturbation method modification to overcome the deficiency. For solving the system of KdV equations, we use the homotopy perturbation transform method. All conditions can be met using this method. One or two iteration steps also yield very accurate results over a wide range. The proposed homotopy perturbation transform method provides the solution in a rapid convergent series, which may lead to a closed solution [38–41].

2 Basic definitions

Definition 2.1 The fractional-order Caputo derivative is defined by

$$
{}^{C}D_{\Im}^{\sigma}\{f(\Im)\} = \frac{1}{\Gamma(n-\sigma)} \int_0^{\Im} (\Im - k)^{n-\sigma-1} f^n(k)\, dk, \quad \text{where } n-1 < \sigma \leq n, n \in N.
$$

Definition 2.2 The Laplace transformation connected with fractional Caputo derivative ${}^{\mathrm{LC}}D_{\Im}^{\sigma}\{f(\Im)\}$ is expressed by

$$
\mathbb{L}\{{}^{\mathrm{LC}}D_{\Im}^{\sigma}\{f(\Im)\}\}(s) = \frac{1}{s^{n-\sigma}}\left[s^n \mathbb{L}\{f(\Im)\}(s) - s^{n-1}f(x,0) - \cdots - f^{n-1}(x,0)\right].
$$

Definition 2.3 In the Caputo sense the Atangana–Baleanu derivative is defined as

$$
{}^{\mathrm{ABC}}D_{\Im}^{\sigma}\{f(\Im)\} = \frac{A(\sigma)}{1-\sigma} \int_a^{\Im} f'(k) E_\sigma\left[-\frac{\sigma}{1-\sigma}(1-k)^\sigma\right] dk,
$$

where $A(\sigma)$ is a normalization function such that $A(0) = A(1) = 1, f \in H^1(a,b), b > a, \sigma \in [0,1]$ and E_σ represents the Mittag-Leffler function.

Definition 2.4 The Atangana–Baleanu derivative in the Riemann–Liouville sense is defined as

$$
{}^{\mathrm{ABC}}D_{\Im}^{\sigma}\{f(\Im)\} = \frac{A(\sigma)}{1-\sigma} \frac{d}{d\Im} \int_a^{\Im} f(k) E_\sigma\left[-\frac{\sigma}{1-\sigma}(1-k)^\sigma\right] dk.
$$

Definition 2.5 The Laplace transform connected with the Atangana–Baleanu operator is defined as

$$^{AB}D_\Im^\sigma\{f(\Im)\}(s) = \frac{A(\sigma)s^\sigma\mathbb{L}\{f(\Im)\}(s) - s^{\sigma-1}f(0)}{(1-\sigma)(s^\sigma + \frac{\sigma}{1-\sigma})}.$$

Definition 2.6 Consider $0 < \sigma < 1$, and f is a function of σ, then the fractional-order integral operator of σ is given as

$$^{ABC}I_\Im^\sigma\{f(\Im)\} = \frac{1-\sigma}{A(\sigma)}f(\Im) + \frac{\sigma}{A(\sigma)\Gamma(\sigma)}\int_a^\Im f(k)(\Im - k)^{\sigma-1}\, dk.$$

3 The general methodology of HPTM
In this section, the HPTM for the general form of FPDEs

$$D_\eta^\sigma\psi(\chi,\eta) + M\psi(\chi,\eta) + N\psi(\chi,\eta) = h(\chi,\eta), \quad \eta > 0, 0 < \sigma \leq 1, \tag{7}$$

with the initial condition

$$\psi(\chi,0) = g(\chi), \tag{8}$$

where is $D_\eta^\sigma\psi(\chi,\eta) = \frac{\partial^\sigma}{\partial\eta^\sigma}$ the Caputo fractional derivative of order σ, M, and N, are linear and nonlinear functions, respectively, and h is a source operator. Using the Laplace transform of Eq. (7), we have

$$\mathbb{L}\big[D_\eta^\sigma\psi(\chi,\eta) + M\psi(\chi,\eta) + N\psi(\chi,\eta)\big] = \mathbb{L}\big[h(\chi,\eta)\big], \quad \eta > 0, 0 < \sigma \leq 1,$$

$$\mathbb{L}\big[\psi(\chi,\eta)\big] = \frac{1}{s}g(\chi) + \frac{(s^\sigma(1-\sigma) + \sigma)}{s^\sigma}\mathbb{L}\big[h(\chi,\eta)\big] \tag{9}$$

$$- \frac{(s^\sigma(1-\sigma) + \sigma)}{s^\sigma}\mathbb{L}\big[M\psi(\chi,\eta) + N\psi(\chi,\eta)\big].$$

Now, by taking the inverse Laplace transform, we get

$$\psi(\chi,\eta) = \mathbb{L}^{-1}\left[\frac{1}{s}g(\chi) + \frac{(s^\sigma(1-\sigma) + \sigma)}{s^\sigma}\mathbb{L}\big[h(\chi,\eta)\big]\right]$$

$$- \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma) + \sigma)}{s^\sigma}\mathbb{L}\big\{M\psi(\chi,\eta) + N\psi(\chi,\eta)\big\}\right], \tag{10}$$

where

$$\psi(\chi,\eta) = g(\chi) + \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma) + \sigma)}{s^\sigma}\mathbb{L}\big[h(\chi,\eta)\big]\right]$$

$$- \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma) + \sigma)}{s^\sigma}\mathbb{L}\big\{M\psi(\chi,\eta) + N\psi(\chi,\eta)\big\}\right]. \tag{11}$$

Now, the perturbation method parameter p is defined as

$$\psi(\chi,\eta) = \sum_{i=0}^\infty p^i\psi_i(\chi,\eta), \tag{12}$$

where perturbation term $p \in [0,1]$.

The nonlinear functions can be defined as

$$N\psi(\chi,\eta) = \sum_{i=0}^{\infty} p^i H_i(\psi_i), \tag{13}$$

where H_i are He's polynomials of $\psi_0, \psi_1, \psi_2, \ldots, \psi_i$ and can be determined as

$$H_i(\psi_0, \psi_1, \ldots, \psi_i) = \frac{1}{i!}\frac{\partial^i}{\partial p^i}\left[N\left(\sum_{i=0}^{\infty} p^i \psi_i\right)\right]_{p=0}, \quad i = 0, 1, 2, \ldots. \tag{14}$$

Putting equations (13) and (14) in equation (11), we have

$$\sum_{i=0}^{\infty} p^i \psi_i(\chi,\eta)$$
$$= g(\chi) + \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}[h(\chi,\eta)]\right] \tag{15}$$
$$- \left[\mathbb{L}^{-1}\left\{\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\left\{M\sum_{i=0}^{\infty} p^i \psi_i(\chi,\eta) + \sum_{i=0}^{\infty} p^i H_i(\psi_i)\right\}\right\}\right].$$

Both sides comparison coefficient of p, we have

$$p^0 : \psi_0(\chi,\eta) = g(\chi) + \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}[h(\chi,\eta)]\right],$$

$$p^1 : \psi_1(\chi,\eta) = \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}(M\psi_0(\chi,\eta) + H_0(\psi))\right], \tag{16}$$

$$p^2 : \psi_2(\chi,\eta) = \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}(M\psi_1(\chi,\eta) + H_1(\psi))\right],$$

$$\vdots \tag{17}$$

$$p^i : \psi_i(\chi,\eta) = \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}(M\psi_{i-1}(\chi,\eta) + H_{i-1}(\psi))\right], \quad i > 0, i \in N.$$

4 Numerical experiments

Example 4.1 Assume that the time fractional Kersten–Krasil'shchik coupled KdV–mKdV nonlinear system is as follows:

$$D_\eta^\sigma \phi + \phi_{3\chi} - 6\phi\phi_\chi + 3\psi\psi_{3\chi} + 3\psi_\chi\psi_{2\chi} - 3\phi_\chi\psi^2$$
$$+ 6\phi\psi\psi_\chi = 0, \quad \eta > 0, \chi \in R, 0 < \sigma \leq 1, \tag{18}$$
$$D_\eta^\sigma \psi + \psi_{3\chi} - 3\psi^2\psi_\chi - 3\phi\psi_\chi + 3\phi_\chi\psi = 0,$$

with the initial conditions

$$\phi(\chi,0) = c - 2c\operatorname{sech}^2(\sqrt{c}\chi), \quad c > 0,$$
$$\psi(\chi,0) = 2\sqrt{c}\operatorname{sech}(\sqrt{c}\chi). \tag{19}$$

Using the Laplace transform on Eq. (18) by the application of the initial condition given by Eq. (19), we get

$$
\begin{aligned}
\mathbb{L}\big[\phi(\chi,\eta)\big] &= \frac{1}{s}\big\{c - 2c\,\mathrm{sech}^2(\sqrt{c}\chi)\big\} - \frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma} \\
&\quad \times \mathbb{L}\big[\phi_{3\chi} - 6\phi\phi_\chi + 3\psi\psi_{3\chi} + 3\psi_\chi\psi_{2\chi} - 3\phi_\chi\psi^2 + 6\phi\psi\psi_\chi\big], \\
\mathbb{L}\big[\psi(\chi,\eta)\big] &= \frac{1}{s}\big\{2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\chi)\big\} \\
&\quad - \frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[\psi_{3\chi} - 3\psi^2\psi_\chi - 3\phi\psi_\chi + 3\phi_\chi\psi\big].
\end{aligned}
\tag{20}
$$

Applying the inverse Laplace transform, we get

$$
\begin{aligned}
\phi(\chi,\eta) &= c - 2c\,\mathrm{sech}^2(\sqrt{c}\chi) - \mathbb{L}^{-1}\Bigg[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma} \\
&\quad \times \mathbb{L}\big[\phi_{3\chi} - 6\phi\phi_\chi + 3\psi\psi_{3\chi} + 3\psi_\chi\psi_{2\chi} - 3\phi_\chi\psi^2 + 6\phi\psi\psi_\chi\big]\Bigg], \\
\psi(\chi,\eta) &= 2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\chi) \\
&\quad - \mathbb{L}^{-1}\Bigg[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[\psi_{3\chi} - 3\psi^2\psi_\chi - 3\phi\psi_\chi + 3\phi_\chi\psi\big]\Bigg].
\end{aligned}
\tag{21}
$$

Using HPM on Eq. (21), we get

$$
\begin{aligned}
\sum_{i=0}^{\infty}& p^i\phi_i(\chi,\eta) \\
&= c - 2c\,\mathrm{sech}^2(\sqrt{c}\chi) \\
&\quad - \Bigg[\mathbb{L}^{-1}\Bigg\{\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\Bigg(\Bigg(\sum_{n-0}^{\infty}p^i\phi_i(\chi,\eta)\Bigg)_{3\chi} + \Bigg(\sum_{i=0}^{\infty}p^iH_i(\phi)\Bigg)\Bigg)\Bigg\}\Bigg], \\
\sum_{i=0}^{\infty}& p^i\psi_i(\chi,\eta) \\
&= 2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\chi) \\
&\quad - \Bigg[\mathbb{L}^{-1}\Bigg\{\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\Bigg(\Bigg(\sum_{i=0}^{\infty}p^i\psi_i(\chi,\eta)\Bigg)_{3\chi} + \Bigg(\sum_{i=0}^{\infty}p^iH_i(\psi)\Bigg)\Bigg)\Bigg\}\Bigg].
\end{aligned}
\tag{22}
$$

Nonlinear steps given by He's polynomials $H_i(\phi)$ and $H_i(\psi)$ are given as

$$
\begin{aligned}
\sum_{i=0}^{\infty} p^iH_i(\phi) &= -6\phi\phi_\chi + 3\psi\psi_{3\chi} + 3\psi_\chi\psi_{2\chi} - 3\phi_\chi\psi^2 + 6\phi\psi\psi_\chi, \\
\sum_{i=0}^{\infty} p^iH_i(\psi) &= -3\psi^2\psi_\chi - 3\phi\psi_\chi + 3\phi_\chi\psi.
\end{aligned}
\tag{23}
$$

Values of components of He's polynomials are given by

$$H_0(\phi) = -6\phi_0(\phi_0)_\chi + 3\psi_0(\psi_0)_{3\chi} + 3(\psi_0)_\chi(\psi_0)_{2\chi} - 3(\phi_0)_\chi(\psi_0)^2 + 6(\phi_0)\psi_0(\psi_0)_\chi,$$

$$\begin{aligned}
H_1(\phi) &= -6\phi_1(\phi_0)_\chi - 6\phi_0(\phi_1)_\chi + 3\psi_1(\psi_0)_{3\chi} + 3\psi_0(\psi_1)_{3\chi} + 3(\psi_0)_\chi(\psi_1)_{2\chi} \\
&\quad + 3(\psi_0)_{2\chi}(\psi_1)_\chi - 3(\phi_1)_\chi(\psi_0)^2 + 6(\phi_0)_\chi\psi_0\psi_1 + 6(\phi_0)\psi_1(\psi_0)_\chi \\
&\quad + 6(\phi_0)\psi_0(\psi_1)_\chi \\
&\quad \times 6(\phi_1)\psi_0(\psi_0)_\chi,
\end{aligned}$$

$$\begin{aligned}
H_2(\phi) &= -6\phi_2(\phi_0)_{xi} - 6\phi_1(\phi_1)_\chi - 6\phi_0(\phi_2)_\chi + 3\psi_2(\psi_0)_{3\chi} \\
&\quad + 3\psi_1(\psi_1)_{3\chi} + 3\psi_0(\psi_2)_{3\chi} \\
&\quad + 3(\psi_0)_\chi(\psi_2)_{2\chi} + 3(\psi_1)_\chi(\psi_1)_{2\chi} + 3(\psi_2)_\chi(\psi_0)_{2\chi} \\
&\quad - 3(\phi_2)_\chi(\psi_0)^2 - 6(\phi_1)_\chi\psi_0\psi_1 \\
&\quad - 6(\phi_1)_\chi\psi_0\psi_2 + 6(\phi_2)\psi_0(\psi_0)_\chi + 6(\phi_1)\psi(\psi_0)_\chi \\
&\quad + 6(\phi_2)\psi_0(\psi_0)_\chi + 6(\phi_0)\psi_1(\psi_1)_\chi \\
&\quad + 6(\phi_1)\psi_0(\psi_1)_\chi + 6(\phi_0)\psi_0(\psi_2)_\chi,
\end{aligned}$$

$$\begin{aligned}
H_3(\phi) &= -6\phi_3(\phi_0)_\chi - 6\phi_2(\phi_1)_\chi - 6\phi_1(\phi_2)_\chi - 6\phi_0(\phi_3)_\chi \\
&\quad + 3\psi_3(\psi_0)_{3\chi} + 3\psi_2(\psi_1)_{3\chi} \\
&\quad + 3\psi_1(\psi_2)_{3\chi} + 3\psi_0(\psi_3)_{3\chi} + 3(\psi_0)_\chi(\psi_3)_{2\chi} \\
&\quad + 3(\psi_1)_\chi(\psi_2)_{2\chi} + 3(\psi_2)_\chi(\psi_1)_{2\chi} \\
&\quad \times 3(\psi_3)_\chi(\psi_0)_{2\chi} - 3(\phi_3)_\chi\psi_0^2 - 6(\phi_2)\psi_0\psi_1 \\
&\quad - 6(\phi_1)\psi_0\psi_2 - 3(\phi_1)_\chi\psi_1^2 \\
&\quad + \phi_0\psi_3(\psi_0)_\chi + 6(\phi_1)\psi_2(\psi_0)_\chi + 6(\phi_1)\psi_2(\psi_0)_\chi \\
&\quad + 6(\phi_3)\psi_0(\psi_0)_\chi + 6(\phi_0)\psi_2(\psi_1)_\chi \\
&\quad + 6(\phi_1)\psi_1(\psi_1)_\chi + 6(\phi_2)\psi_0(\psi_1)_\chi + 6(\phi_0)\psi_1(\psi_2)_\chi \\
&\quad + 6(\phi_1)\psi_0(\psi_2)_\chi + 6(\phi_0)\psi_0(\psi_3)_\chi
\end{aligned}$$

$$\vdots$$

$$(24)$$

and

$$H_0(\psi) = -3(\psi_0)^2(\psi_0)_\chi - 3\phi_0(\psi_0)_\chi + 3(\phi_0)_\chi\psi_0,$$

$$\begin{aligned}
H_1(\psi) &= -3(\psi_0)^2(\psi_1)_\chi - 6\psi_0\psi_1(\psi_0)_\chi - 3\phi_1(\psi_0)_\chi \\
&\quad - 3\phi_0(\psi_1)_\chi + 3(\phi_1)_\chi\psi_0 + 3(\phi_0)_\chi\psi_1,
\end{aligned}$$

$$\begin{aligned}
H_2(\psi) &= -3(\psi_0)^2(\psi_2)_\chi - 6\psi_0\psi_1(\psi_1)_\chi - 6\psi_0\psi_2(\psi_1)_\chi \\
&\quad - 3(\psi_2)^2(\psi_0)_\chi - 3\phi_2(\psi_0)_\chi \\
&\quad - 3\phi_1(\psi_1)_\chi - 3\phi_0(\psi_2)_\chi + 3(\phi_0)_\chi\psi_2 - 3(\phi_1)_\chi\psi_1 - 3(\phi_2)_\chi\psi_0
\end{aligned}$$

$$(25)$$

$$H_3(\psi) = -3(\psi_0)^2(\psi_3)_\chi - 6\psi_0\psi_1(\psi_2)_\chi - 6\psi_0\psi_2(\psi_1)_\chi$$

$$- 6\psi_0\psi_3(\psi_0)_\chi - 6\psi_1\psi_2(\psi_0)_\chi$$

$$- 3(\psi_3)^2(\psi_3)_\chi - 3\phi_3(\psi_0)_\chi - 3\phi_2(\psi_1)_\chi - 3\phi_1(\psi_2)_\chi - 3\phi_0(\psi_3)_\chi + 3(\phi_0)_\chi\psi_3$$

$$+ 3(\phi_1)_\chi\psi_2 + 3(\phi_2)_\chi\psi_1 + 3(\phi_3)_\chi\psi_0$$

$$\vdots$$

Comparing the coefficients of the same powers of p, we have

$$p^0 : \phi_0(\chi,\eta) = c - 2c\,\mathrm{sech}^2(\sqrt{c}\chi),$$

$$p^1 : \phi_1(\chi,\eta) = -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[(\phi_0)_{3\chi} + H_0(\phi)\big]\right] \tag{26}$$

$$= 8c^{\frac{5}{2}}\sinh(\sqrt{c}\chi)\,\mathrm{sech}^3(\sqrt{c}\chi)\left((1-\sigma) + \frac{\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right),$$

$$p^2 : \phi_2(\chi,\eta) = -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[(\phi_1)_{3\chi} + H_1(\phi)\big]\right]$$

$$= -16c^4\big[2\cosh^2(\sqrt{c}\chi) - 3\big]$$

$$\times \mathrm{sech}^4(\sqrt{c}\chi)\left((1-\sigma)^2 + \frac{\sigma^2\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{2(1-\sigma)\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right),$$

$$p^3 : \phi_3(\chi,\eta) = -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[(\phi_2)_{3\chi} + H_2(\phi)\big]\right] \tag{27}$$

$$= 128c^{11/2}\big[\cosh^2(\sqrt{c}\chi) - 3\big]\sinh(\sqrt{c}\chi)$$

$$\times \mathrm{sech}^5(\sqrt{c}\chi)\left\{(1-\sigma)^3 + \sigma(1-\sigma)(1+\sigma+2\sigma^2)\frac{\eta^\sigma}{\Gamma(\sigma+1)}\right.$$

$$\left. + \frac{3\sigma^2(1-\sigma)\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{\sigma^3\Gamma(2\sigma+1)\eta^{3\sigma}}{\Gamma(3\sigma+1)}\right\},$$

$$\vdots$$

and

$$p^0 : \psi_0(\chi,\eta) = 2\sqrt{c}\,\mathrm{sech}(\sqrt{c}\chi),$$

$$p^1 : \psi_1(\chi,\eta) = -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[(\psi_0)_{3\chi} + H_0(\psi)\big]\right],$$

$$= -4c^2\sinh(\sqrt{c}\chi)\,\mathrm{sech}^2(\sqrt{c}\chi)\left((1-\sigma) + \frac{\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right),$$

$$p^2 : \psi_2(\chi,\eta) = -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[(\psi_1)_{3\chi} + H_1(\psi)\big]\right]$$

$$= 8c^{\frac{7}{2}}\big[\cosh^2(\sqrt{c}\chi) - 2\big]$$

$$\times \mathrm{sech}^3(\sqrt{c}\chi)\left((1-\sigma)^2 + \frac{\sigma^2\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{2(1-\sigma)\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right), \tag{28}$$

$$p^3 : \psi_3(\chi, \eta)$$

$$= -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\left[(\psi_2)_{3\chi} + H_2(\psi)\right]\right]$$

$$= -16c^2\left[\cosh^5(\sqrt{c}\chi) - 6\right]\sinh(\sqrt{c}\chi)$$

$$\times \operatorname{sech}^4(\sqrt{c}\chi)\left\{(1-\sigma)^3 + \sigma(1-\sigma)\left(1+\sigma+2\sigma^2\right)\frac{\eta^\sigma}{\Gamma(\sigma+1)}\right.$$

$$\left. + \frac{3\sigma^2(1-\sigma)\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{\sigma^3\Gamma(2\sigma+1)\eta^{3\sigma}}{\Gamma(3\sigma+1)}\right\},$$

$$\vdots$$

Hence a series solution is given by

$$\phi(\chi, \eta) = \sum_{i=0}^{\infty}\phi_i(\chi, \eta)$$

$$= c - 2c\operatorname{sech}^2(\sqrt{c}\chi) + 8c^{\frac{5}{2}}\sinh(\sqrt{c}\chi)\operatorname{sech}^3(\sqrt{c}\chi)\left((1-\sigma) + \frac{\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right)$$

$$- 16c^4\left[2\cosh^2(\sqrt{c}\chi) - 3\right]$$

$$\times \operatorname{sech}^4(\sqrt{c}\chi)\left((1-\sigma)^2 + \frac{\sigma^2\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{2(1-\sigma)\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right)$$

$$+ 128c^{11/2}\left[\cosh^2(\sqrt{c}\chi) - 3\right]$$

$$\times \sinh(\sqrt{c}\chi)\operatorname{sech}^5(\sqrt{c}\chi)\left\{(1-\sigma)^3 + \sigma(1-\sigma)\left(1+\sigma+2\sigma^2\right)\frac{\eta^\sigma}{\Gamma(\sigma+1)}\right.$$

$$\left. + \frac{3\sigma^2(1-\sigma)\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{\sigma^3\Gamma(2\sigma+1)\eta^{3\sigma}}{\Gamma(3\sigma+1)}\right\} + \cdots, \tag{29}$$

and

$$\psi(\chi, \eta) = \sum_{i=0}^{\infty}\psi_i(\chi, \eta)$$

$$= 2\sqrt{c}\operatorname{sech}(\sqrt{c}\chi) - 4c^2\sinh(\sqrt{c}\chi)$$

$$\times \operatorname{sech}^2(\sqrt{c}\chi)\left((1-\sigma) + \frac{\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right) + 8c^{\frac{7}{2}}\left[\cosh^2(\sqrt{c}\chi) - 2\right]$$

$$\times \operatorname{sech}^3(\sqrt{c}\chi)\left((1-\sigma)^2 + \frac{\sigma^2\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{2(1-\sigma)\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right)$$

$$- 16c^2\left[\cosh^5(\sqrt{c}\chi) - 6\right]\sinh(\sqrt{c}\chi)\operatorname{sech}^4(\sqrt{c}\chi)$$

$$\times \left\{(1-\sigma)^3 + \sigma(1-\sigma)\left(1+\sigma+2\sigma^2\right)\frac{\eta^\sigma}{\Gamma(\sigma+1)}\right.$$

$$\left. + \frac{3\sigma^2(1-\sigma)\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{\sigma^3\Gamma(2\sigma+1)\eta^{3\sigma}}{\Gamma(3\sigma+1)}\right\} + \cdots. \tag{30}$$

Putting $\sigma = 1$ in (29) and (30), we get the solution of the problem

$$
\begin{aligned}
\phi(\chi,\eta) = {}& c - 2c\operatorname{sech}^2(\sqrt{c}\chi) + 8\eta c^{\frac{5}{2}}\sinh(\sqrt{c}\chi)\operatorname{sech}^3(\sqrt{c}\chi) - 8\eta^2 c^4 \\
& \times \left[2\cosh^2(\sqrt{c}\chi) - 3\right]\operatorname{sech}^4(\sqrt{c}\chi) \\
& + \frac{64}{3}\eta^3 c^{11/2}\left[\cosh^2(\sqrt{c}\chi) - 3\right]\sinh(\sqrt{c}\chi) \\
& \times \operatorname{sech}^5(\sqrt{c}\chi) - \frac{32}{3}\eta^4 c^7\left[2\cosh^4(\sqrt{c}\chi) - 15\cosh^2(\sqrt{c}\chi) + 15\right] \\
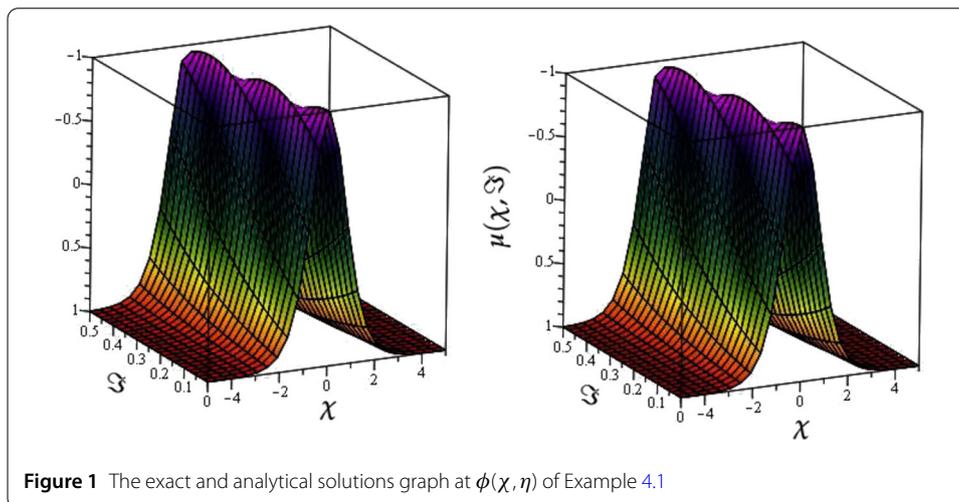& \times \operatorname{sech}^6(\sqrt{c}\chi) + \cdots ,
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
\psi(\chi,\eta) = {}& 2\sqrt{c}\operatorname{sech}(\sqrt{c}\chi) - 4\eta c^2\sinh(\sqrt{c}\chi)\operatorname{sech}^2(\sqrt{c}\chi) \\
& + 4\eta^2 c^{\frac{7}{2}}\left[\cosh^2(\sqrt{c}\chi) - 2\right] \\
& \times \operatorname{sech}^3(\sqrt{c}\chi) - \frac{8}{3}\eta^3 c^2\left[\cosh^5(\sqrt{c}\chi) - 6\right] \\
& \times \sinh(\sqrt{c}\chi)\operatorname{sech}^4(\sqrt{c}\chi) + \frac{4}{3}\eta^4 c^{13/2} \\
& \times \left[\cosh^4(\sqrt{c}\chi) - 20\cosh^2(\sqrt{c}\chi) + 24\right]\operatorname{sech}^5(\sqrt{c}\chi) - \cdots .
\end{aligned}
\tag{32}
$$

The solution represented by Eqs. (31) and (32) is similar to the exact solution in a closed form as follows:

$$
\begin{aligned}
\phi(\chi,\eta) &= c - 2c\operatorname{sech}^2\left(\sqrt{c}(\chi + 2c\eta)\right), \\
\psi(\chi,\eta) &= 2\sqrt{c}\operatorname{sech}\left(\sqrt{c}(\chi + 2c\eta)\right).
\end{aligned}
\tag{33}
$$

In Figs. 1 and 3, the actual and HPTM solutions of $\phi(\chi,\eta)$ and $\psi(\chi,\eta)$ are calculated at $\sigma = 1$. In Figs. 2 and 4, the 3D graphs for $\phi(\chi,\eta)$ and $\psi(\chi,\eta)$ for different fractional order show that the HPTM solutions derived are in a strong agreement with the actual
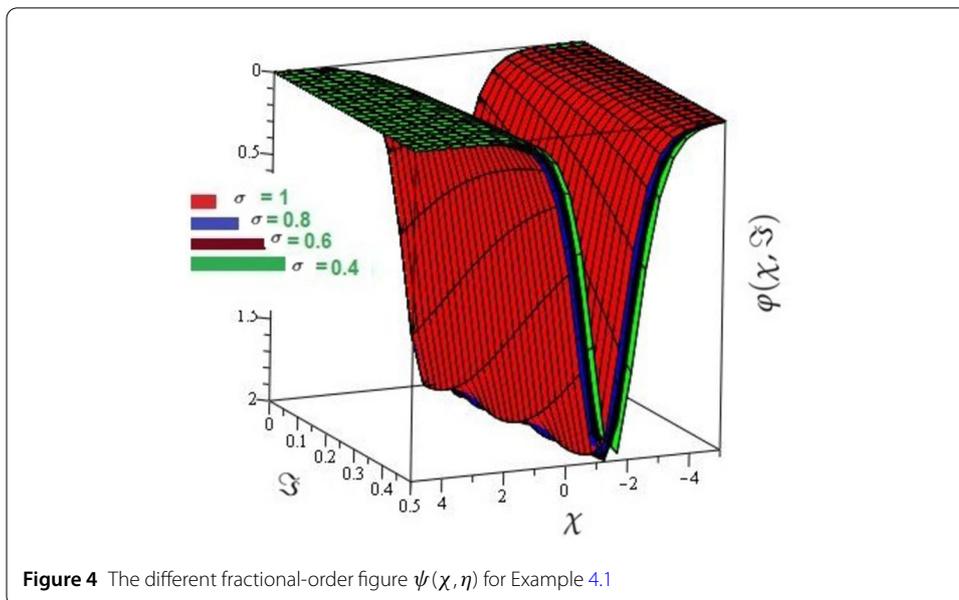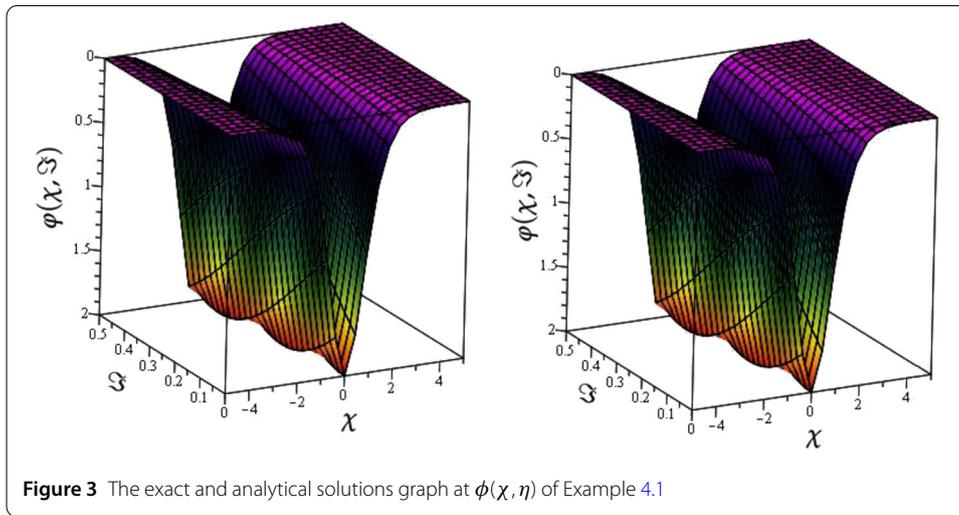

Figure 1 The exact and analytical solutions graph at $\phi(\chi,\eta)$ of Example 4.1

Figure 2 The different fractional-order figure of $\phi(\chi, \eta)$ of Example 4.1

Figure 3 The exact and analytical solutions graph at $\phi(\chi, \eta)$ of Example 4.1

Figure 4 The different fractional-order figure $\psi(\chi, \eta)$ for Example 4.1
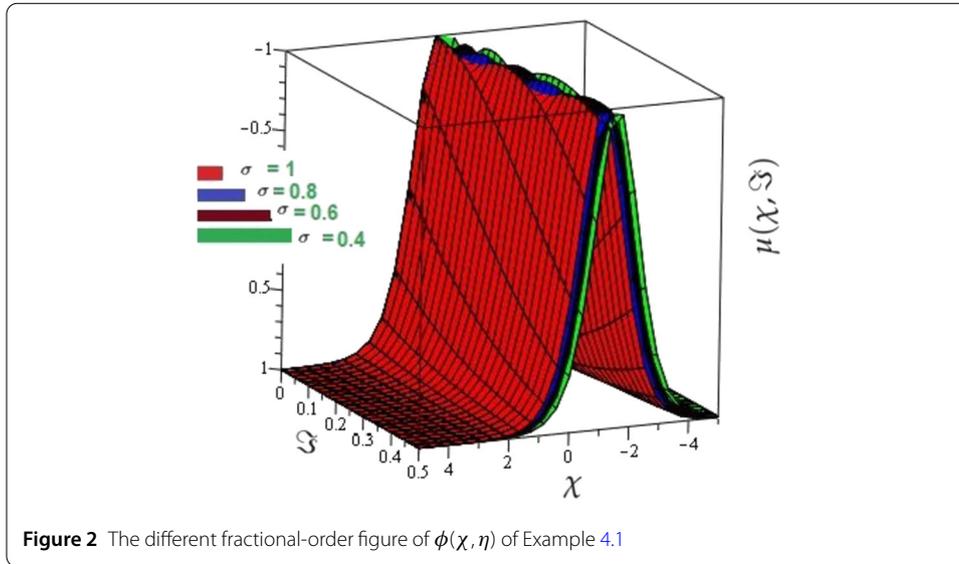
Table 1 Variation of the actual solution with HPTM solution of $\phi(\chi, \eta)$ at $\eta = 0.01$

χ	Exact result	HPTM result	Absolute error
3.0	0.9810379547	0.9810379579	1.957810944E−09
2.5	0.9488741084	0.9488741131	4.588520200E−09
2.0	0.8640471736	0.8640471812	7.500972059E−09
1.5	0.6514628688	0.6514628649	3.864259000E−09
1.0	0.1853889664	0.1853888931	7.330837940E−08
0.5	−0.543605397	0-0.5436054423	4.527637000E−08
0.0	−0.999200213	0-0.9992000000	2.130000000E−07
−0.5	−0.601733281	0-0.6017333223	4.129203000E−08
−1.0	0.1342165142	0.1342164396	7.470826874E−08
−1.5	0.6252890662	0.6252890617	4.510941000E−09
−2.0	0.8531473885	0.8531473961	7.543603848E−09
−2.5	0.9446752749	0.9446752795	4.629439800E−09
−3.0	0.9794667886	0.9794667902	1.994434944E−09

Table 2 Variation of the exact result with HPTM solution of $\psi(\chi, \eta)$ at $\eta = 0.01$

χ	Exact result	HPTM result	Absolute error
3.0	0.1947410861	0.1947410860	1.001455080E−10
2.5	0.3197683274	0.3197683273	1.358172040E−10
2.0	0.5214457334	0.5214457333	6.868329502E−11
1.5	0.8349097330	0.8349097328	2.056274600E−10
1.0	1.2764098350	1.2764098350	3.489344755E−10
0.5	1.7570460420	1.7570460420	6.046615540E−10
0.0	1.9996000670	1.9996000670	3.333333500E−10
−0.5	1.7898230530	1.7898230520	4.983904460E−10
−1.0	1.3158901820	1.3158901820	3.908333123E−10
−1.5	0.8656915546	0.8656915546	3.031660000E−12
−2.0	0.5419457750	0.5419457750	7.044098959E−11
−2.5	0.3326401212	0.3326401213	5.249600400E−11
−3.0	0.2026485202	0.2026485201	2.472076832E−10

and the approximate solution. This comparison shows that the HPTM and the actual solutions are very close. As a result, the HPTM is a dependable new study that requires less computation of computations, is adaptable, and simple to use. In Tables 1 and 2, the exact result and HPTM solution of different fractional order of $\phi(\chi, \eta)$ and $\psi(\chi, \eta)$ at $\eta = 0.01$ are given.

Example 4.2 Assume homogeneous two-component time fractional coupled third-order KdV system as follows:

$$D_\eta^\sigma \phi - \phi_{3\chi} - \phi\phi_\chi - \psi\psi_\chi = 0, \quad \eta > 0, \chi \in R, 0 < \sigma \leq 1,$$
$$D_\eta^\sigma \psi + 2\psi_{3\chi} - \phi\psi_\chi = 0,$$

(34)

with the initial condition

$$\phi(\chi, 0) = 3 - 6\tanh^2\left(\frac{\chi}{2}\right),$$
$$\psi(\chi, 0) = -3c\sqrt{2}\tanh\left(\frac{\chi}{2}\right).$$

(35)

Using the Laplace transform on Eq. (34) by the application of initial condition given by Eq. (35), we get

$$\mathbb{L}\big[\phi(\chi,\eta)\big] = \frac{1}{s}\left\{3 - 6\tanh^2\left(\frac{\chi}{2}\right)\right\} + \frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}[\phi_{3\chi} - \phi\phi_\chi - \psi\psi_\chi],$$

$$\mathbb{L}\big[\psi(\chi,\eta)\big] = \frac{1}{s}\left\{-3c\sqrt{2}\tanh\left(\frac{\chi}{2}\right)\right\} - \frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}[2\psi_{3\chi} - \phi\psi_\chi]. \tag{36}$$

Applying the inverse Laplace transform, we get

$$\phi(\chi,\eta) = 3 - 6\tanh^2\left(\frac{\chi}{2}\right) + \mathbb{L}^{-1}[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}[\phi_{3\chi} - \phi\phi_\chi - \psi\psi_\chi],$$

$$\psi(\chi,\eta) = -3c\sqrt{2}\tanh\left(\frac{\chi}{2}\right) - \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}[2\psi_{3\chi} - \phi\psi_\chi]\right]. \tag{37}$$

Using HPM on Eq. (37), we get

$$\sum_{i=0}^{\infty}p^i\phi_i(\chi,\eta)$$

$$= 3 - 6\tanh^2\left(\frac{\chi}{2}\right) \tag{38}$$

$$+ \left[\mathbb{L}^{-1}\left\{\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\left(\left(\sum_{i=0}^{\infty}p^i\phi_i(\chi,\eta)\right)_{3\chi} + \left(\sum_{i=0}^{\infty}p^iH_i(\phi)\right)\right)\right\}\right],$$

$$\sum_{i=0}^{\infty}p^i\psi_i(\chi,\eta)$$

$$= -3c\sqrt{2}\tanh\left(\frac{\chi}{2}\right) \tag{39}$$

$$- \left[\mathbb{L}^{-1}\left\{\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\left(2\left(\sum_{i=0}^{\infty}p^i\psi_i(\chi,\eta)\right)_{3\chi} - \left(\sum_{i=0}^{\infty}p^iH_i(\psi)\right)\right)\right\}\right].$$

Nonlinear steps given by He's polynomials $H_i(\phi)$ and $H_i(\psi)$ are given as follows:

$$\sum_{i=0}^{\infty}p^iH_i(\phi) = \phi\phi_\chi + \psi\psi_\chi,$$

$$\sum_{i=0}^{\infty}p^iH_i(\psi) = -\phi\psi_\chi. \tag{40}$$

Values of factors of He's polynomials are given as follows:

$$H_0(\phi) = \phi_0(\phi_0)_\chi + \psi_0(\psi_0)_\chi,$$

$$H_1(\phi) = \phi_1(\phi_0)_\chi + \phi_0(\phi_1)_\chi + \psi_1(\psi_0)_\chi + \psi_0(\psi_1)_\chi,$$

$$H_2(\phi) = \phi_2(\phi_0)_\chi + \phi_1(\phi_1)_\chi + \phi_0(\phi_2)_\chi + \psi_2(\psi_0)_\chi + \psi_1(\psi_1)_\chi + \psi_0(\psi_2)_\chi, \tag{41}$$

$$H_3(\phi) = \phi_3(\phi_0)_\chi + \phi_2(\phi_1)_\chi + \phi_1(\phi_2)_\chi + \phi_0(\phi_3)_\chi + \psi_3(\psi_0)_\chi + \psi_2(\psi_1)_\chi$$

$$+ \psi_1(\psi_2)_\chi + \psi_0(\psi_3)_\chi,$$

$$\vdots$$

and

$$H_0(\psi) = -\phi_0(\psi_0)_\chi,$$

$$H_1(\psi) = -\phi_1(\psi_0)_\chi - \phi_0(\psi_1)_\chi,$$

$$H_2(\psi) = -\phi_2(\psi_0)_\chi - \phi_1(\psi_1)_\chi - \phi_0(\psi_2)_\chi,$$

$$H_3(\psi) = -\phi_3(\psi_0)_\chi - \phi_2(\psi_1)_\chi - \phi_1(\psi_2)_\chi - \phi_0(\psi_3)_\chi,$$

$$\vdots$$

(42)

Comparing coefficients of the same powers of p, we have

$$p^0 : \phi_0(\chi, \eta) = 3 - 6 \tanh^2\left(\frac{\chi}{2}\right),$$

$$p^1 : \phi_1(\chi, \eta) = \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[(\phi_0)_{3\chi} + H_0(\phi)\big]\right],$$

$$= 6 \operatorname{sech}^2\left(\frac{\chi}{2}\right) \tanh\left(\frac{\chi}{2}\right)\left((1-\sigma) + \frac{\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right),$$

$$p^2 : \phi_2(\chi, \eta)$$

$$= \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[(\phi_1)_{3\chi} + H_1(\phi)\big]\right],$$

$$= 3\left[2 + 7 \operatorname{sech}^2\left(\frac{\chi}{2}\right) - 15 \operatorname{sech}^4\left(\frac{\chi}{2}\right)\right]$$

$$\times \operatorname{sech}^2\left(\frac{\chi}{2}\right)\left((1-\sigma)^2 + \frac{\sigma^2\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{2(1-\sigma)\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right),$$

$$p^3 : \phi_3(\chi, \eta) = \mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[(\phi_2)_{3\chi} + H_2(\phi)\big]\right],$$

$$\vdots$$

(43)

and

$$p^0 : \psi_0(\chi, \eta) = -3c\sqrt{2}\tanh\left(\frac{\chi}{2}\right),$$

$$p^1 : \psi_1(\chi, \eta) = -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[2(\psi_0)_{3\chi} - H_0(\psi)\big]\right],$$

$$= 3c\sqrt{2}\operatorname{sech}^2\left(\frac{\chi}{2}\right)\tanh\left(\frac{\chi}{2}\right)\left((1-\sigma) + \frac{\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right),$$

$$p^2 : \psi_2(\chi, \eta)$$

$$= -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\big[2(\psi_1)_{3\chi} - H_1(\psi)\big]\right],$$

(44)

$$= \frac{3c\sqrt{2}}{2}\left[2 + 21\operatorname{sech}^2\left(\frac{\chi}{2}\right) - 24\operatorname{sech}^4\left(\frac{\chi}{2}\right)\right]$$

$$\times \operatorname{sech}^2\left(\frac{\chi}{2}\right)\left((1-\sigma)^2 + \frac{\sigma^2\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{2(1-\sigma)\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right),$$

$$p^3 : \psi_3(\chi,\eta) = -\mathbb{L}^{-1}\left[\frac{(s^\sigma(1-\sigma)+\sigma)}{s^\sigma}\mathbb{L}\left[2(\psi_2)_{3\chi} - H_2(\psi)\right]\right],$$
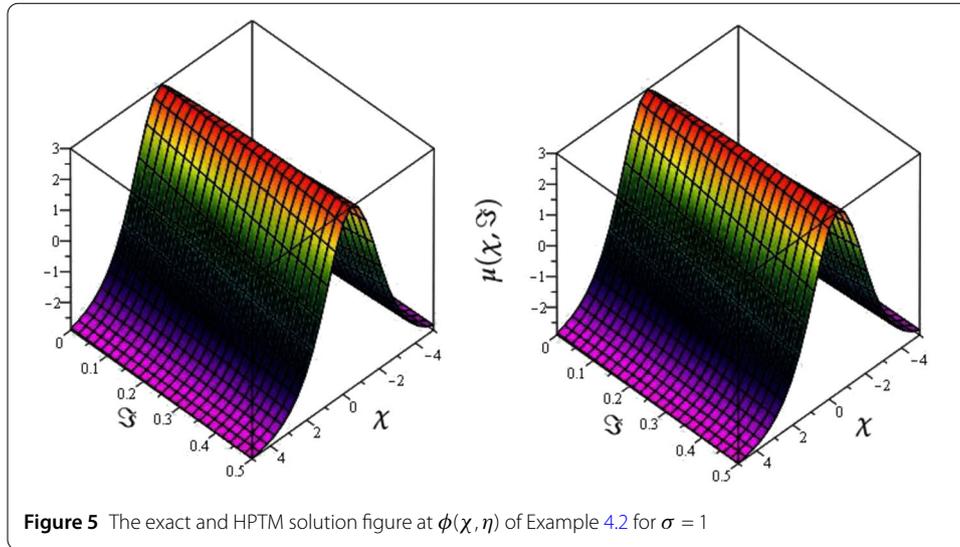
$$\vdots$$

Hence a series solution is given by

$$\phi(\chi,\eta)$$

$$= \sum_{i=0}^{\infty} \phi_i(\chi,\eta),$$

$$= 3 - 6\tanh^2\left(\frac{\chi}{2}\right) + 6\operatorname{sech}^2\left(\frac{\chi}{2}\right)\tanh\left(\frac{\chi}{2}\right)\left((1-\sigma) + \frac{\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right)$$

$$+ 3\left[2 + 7\operatorname{sech}^2\left(\frac{\chi}{2}\right) - 15\operatorname{sech}^4\left(\frac{\chi}{2}\right)\right]$$

$$\times \operatorname{sech}^2\left(\frac{\chi}{2}\right)\left((1-\sigma)^2 + \frac{\sigma^2\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{2(1-\sigma)\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right) - \cdots, \tag{45}$$

$$\psi(\chi,\eta)$$

$$= \sum_{i=0}^{\infty} \psi_i(\chi,\eta)$$

$$= -3c\sqrt{2}\tanh\left(\frac{\chi}{2}\right) + 3c\sqrt{2}\operatorname{sech}^2\left(\frac{\chi}{2}\right)\tanh\left(\frac{\chi}{2}\right)\left((1-\sigma) + \frac{\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right)$$

$$+ \frac{3c\sqrt{2}}{2}\left[2 + 21\operatorname{sech}^2\left(\frac{\chi}{2}\right) - 24\operatorname{sech}^4\left(\frac{\chi}{2}\right)\right]$$

$$\times \operatorname{sech}^2\left(\frac{\chi}{2}\right)\left((1-\sigma)^2 + \frac{\sigma^2\eta^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{2(1-\sigma)\sigma\eta^\sigma}{\Gamma(\sigma+1)}\right) + \cdots.$$

Putting $\sigma = 1$ in (45), we get the solution of the problem

$$\phi(\chi,\eta) = 3 - 6\tanh^2\left(\frac{\chi}{2}\right) + 6\eta\operatorname{sech}^2\left(\frac{\chi}{2}\right)\tanh\left(\frac{\chi}{2}\right)$$

$$+ \frac{3}{2}\eta^2\left[2 + 7\operatorname{sech}^2\left(\frac{\chi}{2}\right) - 15\operatorname{sech}^4\left(\frac{\chi}{2}\right)\right]\operatorname{sech}^2\left(\frac{\chi}{2}\right) - \cdots,$$

$$\psi(\chi,\eta) = -3c\sqrt{2}\tanh\left(\frac{\chi}{2}\right) + 3\eta c\sqrt{2}\operatorname{sech}^2\left(\frac{\chi}{2}\right)\tanh\left(\frac{\chi}{2}\right) \tag{46}$$

$$+ \frac{3c\sqrt{2}}{4}\eta^2\left[2 + 21\operatorname{sech}^2\left(\frac{\chi}{2}\right) - 24\operatorname{sech}^4\left(\frac{\chi}{2}\right)\right]\operatorname{sech}^2\left(\frac{\chi}{2}\right) + \cdots.$$

Figure 5 The exact and HPTM solution figure at $\phi(\chi, \eta)$ of Example 4.2 for $\sigma = 1$

The solution given by Eq. (46) is similar to a closed form solution

$$
\phi(\chi, \eta) = 3 - 6 \tanh^2\left(\frac{\chi + \eta}{2}\right),
$$
$$
\psi(\chi, \eta) = -3c\sqrt{2}\tanh\left(\frac{\chi + \eta}{2}\right).
$$
(47)

In Figs. 5 and 7, the actual and HPTM solutions of $\phi(\chi, \eta)$ and $\psi(\chi, \eta)$ are calculated at $\sigma = 1$. In Figs. 6 and 8, the 3D graphs for $\phi(\chi, \eta)$ and $\psi(\chi, \eta)$ for different fractional order show that the HPTM approximated solutions derived are in a strong agreement with the actual and the approximate solution. This comparison shows that the HPTM and the actual solutions are very close. As a result, the HPTM is a dependable new study that requires less computation of computations, is adaptable, and simple to use. In Tables 3 and 4, the exact result and HPTM solution of different fractional order of $\phi(\chi, \eta)$ and $\psi(\chi, \eta)$ at $\eta = 0.01$ are given.

5 Conclusions

In this paper, we calculated the fractional-order Kersten–Krasil'shchik coupled KdV–mKdV nonlinear system using the Laplace transform and the Atangana–Baleanu derivative. The suggested method is applied to obtain the solution of the given two problems. The HPTM solution is in close contact with the exact result of the given problems. The present scenario also calculated the results of the given problems with fractional-order derivatives. The figures of the fractional-order results achieved have shown the convergence towards the results of integer order. Furthermore, the present method is simple, straightforward and required less computational cost; the current technique can be modified to solve other fractional-order partial differential equations.
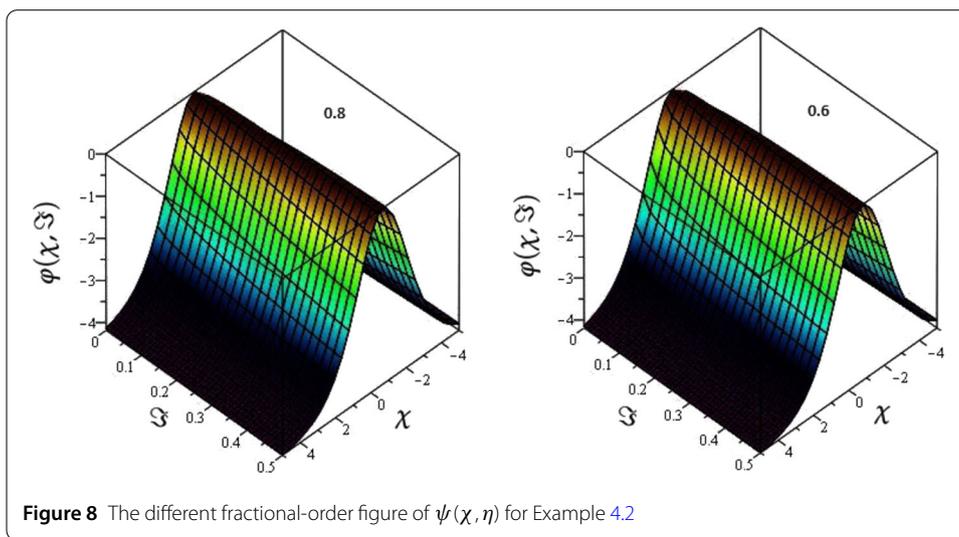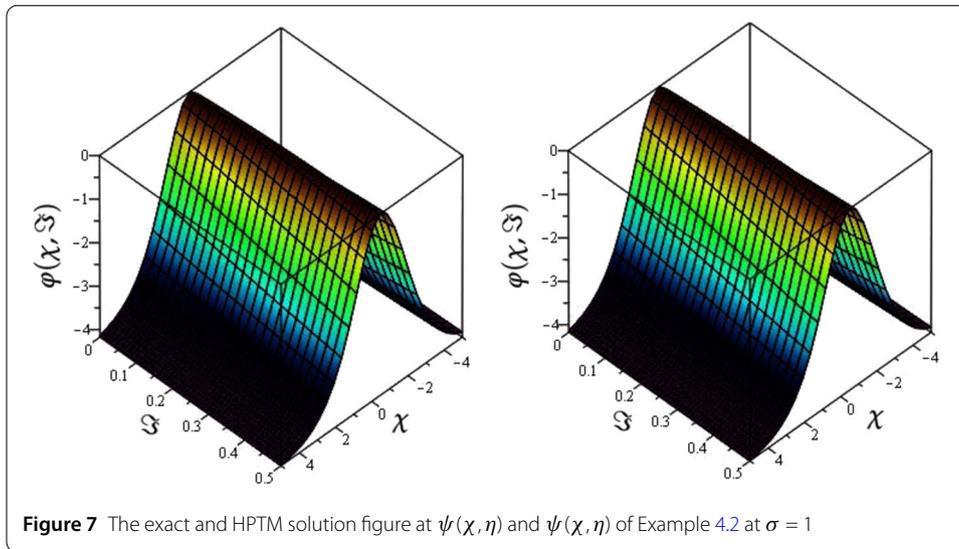
Figure 6 The different fractional-order figure of $\phi(\chi, \eta)$ for Example 4.2

Figure 7 The exact and HPTM solution figure at $\psi(\chi, \eta)$ and $\psi(\chi, \eta)$ of Example 4.2 at $\sigma = 1$

Figure 8 The different fractional-order figure of $\psi(\chi, \eta)$ for Example 4.2
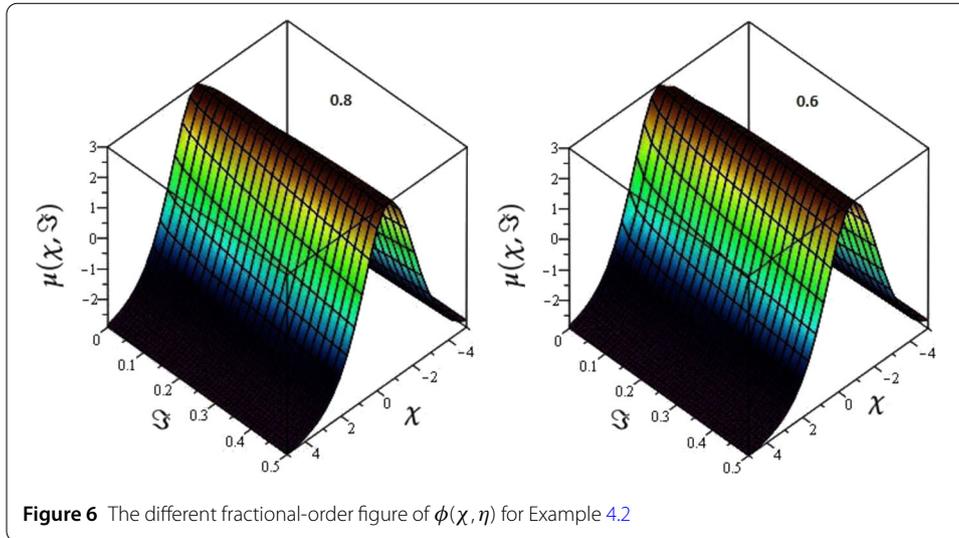
Table 3 Variation of the exact result with HPTM solution of $\phi(\chi, \eta)$ at $\eta = 0.01$

χ	Exact result	HPTM result	Absolute error
5	−2.840462381	−2.840462380	1.444186835E−09
4	−2.576135914	−2.576135916	7.505047070E−10
3	−1.915858303	−1.915858299	4.760423501E−09
2	−0.480345856	−0.480345845	1.127798505E−08
1	1.718468335	1.718468318	1.736182461E−08
0	2.999999985	2.999999910	7.500000002E−08
−1	1.718904452	1.718904435	1.715811279E−08
−2	−0.479962035	−0.479962025	1.021909119E−08
−3	−1.9156620237	−1.915662020	4.383116417E−09
−4	−2.576054184	−2.576054186	1.160759310E−10
−5	−2.840430898	−2.840430897	5.811609752E−10

Table 4 The comparison of the exact solution with HPTM solution of $\psi(\chi, \eta)$ at $\eta = 0.01$

χ	Exact result	HPTM result	Absolute error
5	−0.00003128920578	−0.00005018707030	1.889786452E−09
4	−0.00002953283553	−0.00003941505303	9.882217500E−09
3	−0.00002464651012	−0.00002354262325	1.103886870E−08
2	−0.00001563187126	−0.00001368571425	1.946157010E−08
1	−0.000009064576442	−0.000008133706541	9.3086990E−08
0	−1.060578884000000	−1.050550263000000	1.769181166E−07
−1	−0.000008032738383	−0.000008064752114	3.080452365E−08
−2	−0.00001648493229	−0.00001351283541	2.711411868E−08
−3	−0.00002345262104	−0.00002565550250	1.387225658E−08
−4	−0.00002833704423	−0.00002832282225	5.778090290E−09
−5	−0.00003218607047	−0.00003218920551	2.226059143E−09

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this article. They read and approved the final manuscript.

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