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Int. J. of Applied Mechanics and Engineering, 2018, vol.23, No.2, pp.385-400
DOI: 10.2478/ijame-2018-0022

NUMERICAL INVESTIGATION OF THE FREE VIBRATION OF PARTIALLY CLAMPED RECTANGULAR PLATES

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This paper studies the free vibration characteristics of rectangular plates with partially clamped edges around the corners using the finite element method. ANSYS Parametric Design Language (APDL) was utilized to produce the finite element (FE) models and to run the analysis. The FE models were used to obtain the plate first natural frequency and mode shape. A comprehensive investigation of the effect of the plate geometric parameters and different boundary condition properties on the natural frequency and mode shapes is presented. The results showed that the vibration characteristics of the structure are greatly dependent on the plate size and the constraint properties.

Key words: elastic plates, plates with partially clamped edges, natural frequencies and mode shapes, finite element analysis.

1. Introduction

Rectangular elastic plates [1, 2] are widely used in engineering applications, e.g., printed circuit boards and solar collecting panels. The exact solutions of free vibration of elastic plates are possible only for few cases, i.e., for plate structures with well-defined boundary conditions. However, real-life vibration problems may pose some difficulties in satisfying discontinuous boundary conditions, e.g. plate with partially clamped edges. In these cases, only approximate solutions, e.g. Rayleigh methods, are available [3]. Gorman *et al*. [4-10] used his superposition method (or Gorman's method) and experimental techniques to discuss the free vibration of several geometric and materials configurations of rectangular plates with discontinuous boundary conditions, i.e. plates with point supports and with partially constrained edges. Gajendar [11] studied the bending of squared plates with two opposite simply-supported edges and two partially fixed edges using the superposition method proposed by Kurata and Hatano [12]. Abrahams *et al*. [13, 14] discussed the static and dynamic deflection of a thin elastic strip (or infinite) plates with two partially clamped edges using the matrix Wiener-Hopf equation. Norita [15] employed a series-type solution to investigate the free vibration of rectangular plates with various shapes and boundary conditions, such as plates with partially-clamped edges. Wei *et al*. [16] explored the natural frequencies of partially-supported

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plates using the discrete singular convolution method. Zhou *et al*. [17, 18] analyzed the natural frequency of moderately thick rectangular plates using the Chebyshev polynomial as the admissible function in the Ritz method. Recently, Gharaibeh *et al*. [19, 20] used a combination between series solutions and the Ritz method to solve for the first natural frequency and mode shape of a squared elastic plate.

In the open literature, there is a lack of quantitative analysis concerning the effect of plate sizes and boundary condition properties on the plate with partially clamped edges natural frequencies as well as mode shapes. In addition, there is barely any information on the bending behavior of the vibrating plate. Therefore, the current work aims to thoroughly investigate the free vibration characteristics, i.e., natural frequencies and mode shapes of a thin elastic rectangular isotropic plate with partially clamped edges around the corners using the finite element method. In this paper, we first start by introducing the partially fixed plate problem and the resultant boundary conditions properties. Subsequently, the finite element modeling details are presented. Finally, a comprehensive discussion about the effect of plate geometric parameters and the constraint properties on the plate free vibration, i.e., first natural frequency and mode shape, is presented in detail. The terms a "partially fixed" and a "partially clamped" plate will be used interchangeably throughout the text.

2. Description of the problem

The problem of a rectangular thin elastic plate with partially clamped edges around the corners is presented in Fig.1. The elastic plate width is a, length is b and the thickness is h. The plate material system is isotropic with an elastic modulus E, mass density ρ and Poisson's ratio ν.

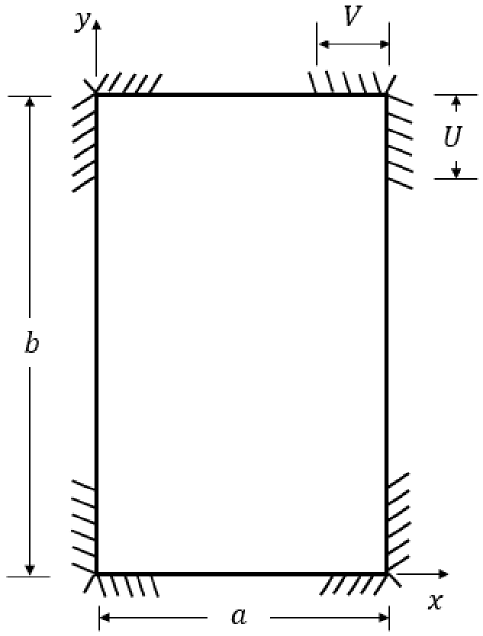

Fig.1. Two rigid support schemes (A) Case 1: Four supports symmetrically distributed along the plate diagonals, (B) Case2: Four supports symmetrically distributed about the plate central axis.

The elastic plate under consideration is partially fixed around its four corners. As seen in Fig.1, U is the ratio between the edge fixed length in the y-direction to the plate length (b). In addition, V is the normalized fixed edge length in the x-direction to the plate width (a). Both fixed lengths prevent the plate translational and rotational motions and the remaining portions of the plate edges are completely free. It is very important to emphasize that U and V are dimensionless quantities.

In the present paper, an extensive three-dimensional finite element analysis is performed to examine the first natural frequency of a partially clamped thin elastic isotropic rectangular plate at different geometric parameters and boundary condition properties (U and V). The first natural frequency (ω_1) of the elastic plate under consideration, in *rad/sec*, can be generally expressed as [21]

$$\omega_1 = \frac{\lambda_1^2}{a^2}\sqrt{\frac{D}{\rho h}} \tag{2.1}$$

where λ_1^2 is the non-dimensional free-vibration eigenvalue of the plate and $D = \dfrac{Eh^3}{12\left(1-\nu^2\right)}$ is the plate flexural rigidity. A point to remember is that the first natural frequency (f_1) in Hz is $f_1 = \omega_1 / 2\pi$.

Therefore, the plate non-dimensional free-vibration eigenvalue can be written as

$$\lambda_1^2 = \omega_1 a^2 \sqrt{\frac{\rho h}{D}}\ . \tag{2.2}$$

To keep the present analysis general, the effect of the plate aspect ratio b/a and the normalized fixed length U as well as V, for both support cases, on the plate non-dimensional free-vibration eigenvalue λ_1^2 is investigated. In this way, the effect of the plate isotropic material properties is eliminated. Hence, it can be generalized for any other isotropic plate material system.

3. Finite element modeling and verification

3.1. Finite element model

ANSYS release 17.0 was used to build the FE model of the present plate problem. ANSYS Parametric Design Language (APDL) was implemented to write the FE code. This well-written code was used to define the plate geometric details, material system and boundary condition properties as well as to run the analysis type required in the present paper. Only three-dimensional hexahedron elements, identified as SOLID185 in ANSYS, were selected to generate the FE mesh through the engineered choice of the isoparametric mapping concept. This FE model is shown in Fig.2.

As mentioned earlier, the plate material system is considered to be isotropic. In the present analysis, the material properties of aluminum T6061 alloy of $E = 68.9\,GPa$ and $\nu = 0.33$ were plugged into the FEA program. In addition, and to simulate the boundary condition results from the partial clamping of the edges, the FE model was constrained at the required clamping locations in all directions, i.e., all degrees of freedom, translational and rotational, are set to zero.

The FE modal analysis, available in ANSYS, was thoroughly used to calculate the structure first natural frequency (f_1) and mode shape for different geometric and constraint parameters. Then Eq.(2.2) was employed to obtain the plate non-dimensional free-vibration eigenvalue for each configuration.

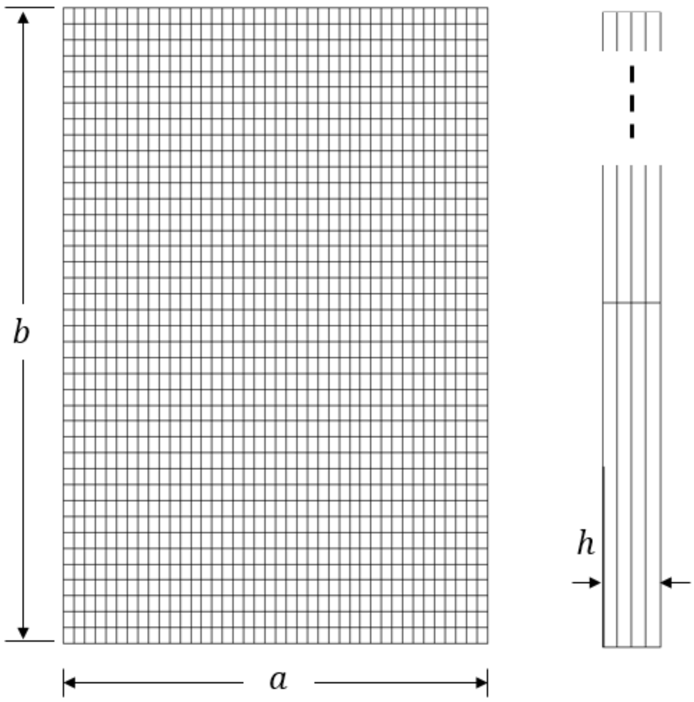

Fig.2. The FE model mesh showing plate in-plane dimensions (left) and a zoomed view of the plate thickness (right).

3.2. Mesh density study

A mesh density study was conducted to ensure best numerical solution accuracy results with a minimum solution time. For this study, the plate configuration of $b/a=3$, $U=0.2$ and $V=0.2$ along with the Aluminum T6061 alloy material system ($a=0.01m, h=0.002m$) was considered. Seven mesh levels with different mesh sizes were tested as listed in Tab.1. For each mesh level, the first natural frequency was computed from FEA and compared with the next finer mesh level value. The results of this study are depicted in Fig.3.

Table 1. The mesh size specifications of the seven FE models of the mesh density study.

Model #	**Number of elements**	**Number of nodes**	**Width-to-element size ratio (a/e)**	**First Natural Frequency [Hz]**
1	*108*	*196*	*6*	*432.92*
2	*432*	*676*	*12*	*265.14*
3	*972*	*1444*	*18*	*253.08*
4	*1728*	*2500*	*24*	*250.82*
5	***2700***	***3844***	***30***	***250.13***
6	*3888*	*5476*	*36*	*249.97*
7	*5292*	*7396*	*42*	*249.91*

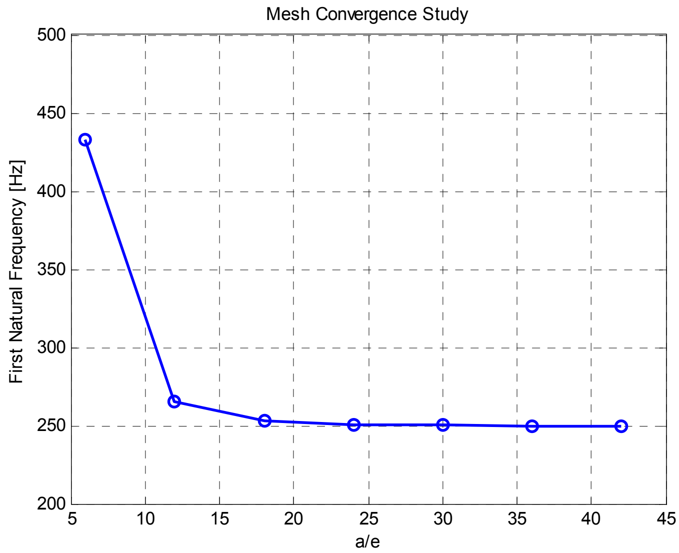

Fig.3. Mesh density study results.

Apparently, at mesh level 5 with the plate width-to-element size ratio ($a/e = 30$) the first natural frequency of the simply supported plate under consideration reaches a converged value with a relative approximate error to the next finer mesh level of less than *1*%. Consequently, this mesh level 5 was adopted throughout the present analysis.

3.3. FE model verification analysis

The verification analysis of the present FE model was performed by producing the FE model by means of the previously described modeling procedure for specific cases available in the literature and by reproducing their results. Here, the FE model non-dimensional eigenvalue $\left(\lambda_1^2\right)$ results of a fully clamped plate $\left(U = V = 0.5\right)$ were compared to those reported in Leissa [1] for different plate aspect ratio values, as listed in Tab.2. A shown in this table, the FE results of $\left(\lambda_1^2\right)$ at different plate sizes (b/a values) are in an excellent agreement with the values of reference [1] with a relative error less than *1*%. Therefore, this FE model was adopted throughout the current study with confidence.

Table 2. Comparison between present FE results Ref. [22] non-dimensional eigenvalue data for several plate aspect ratios.

$\boldsymbol{b/a}$	λ_1^2 **– Ref. [22]**	λ_1^2 **– Present FEA**	$\%\text{Error} = \left\lvert\frac{\text{FEA-Ref. [22]}}{\text{Ref. [22]}}\right\rvert \times 100\%$
1.0	*35.99*	*35.93*	*0.17*
1.5	*27.00*	*26.99*	*0.04*
2.0	*24.56*	*24.67*	*0.05*
2.5	*23.76*	*23.64*	*0.51*
3.0	*23.19*	*23.20*	*0.04*

4. Results and discussions

The present FE model was used to generate the first natural frequency and first mode shape data of the current problem for several plate aspect ratios and boundary condition properties.

Figure 4 shows the non-dimensional eigenvalue $\left(\lambda_1^2\right)$ as a function of the normalized constrained (or clamped) length in the y-direction (U) at different normalized fixed lengths in the x-direction (V). Each subfigure is plotted at certain plate aspect ratio (b/a). From this figure, it appears that λ_1^2 (therefore the first natural frequency) exponentially increases as U increases, generally.

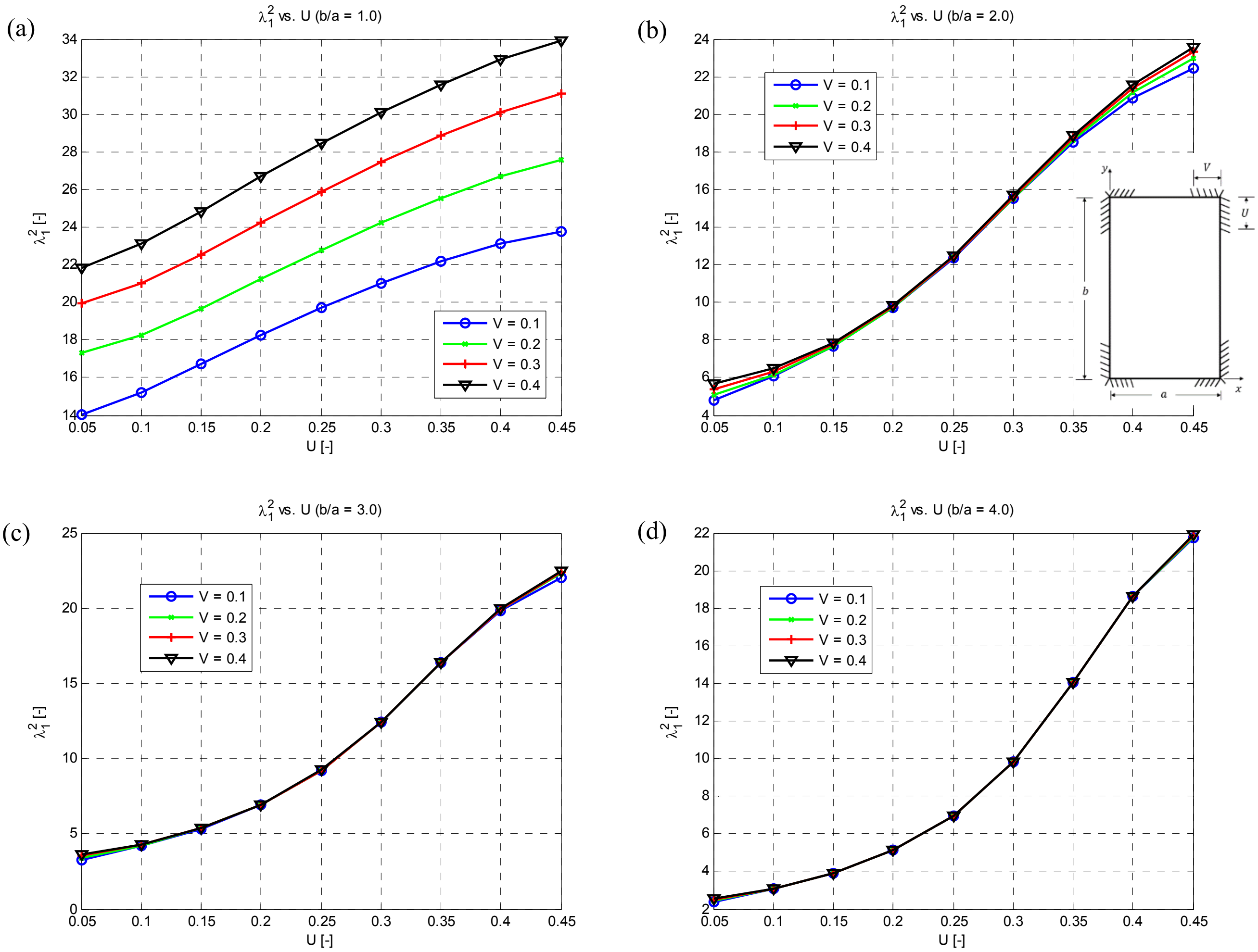

Fig.4. Effect of non-dimensionalized fixed length (U) on the plate eigenvalue $\left(\lambda_1^2\right)$ at different V values for (a) $b/a=1$ (b) $b/a=2$ (c) $b/a=3$ (d) $b/a=4$.

This could be easily explained as U becomes larger, the plate is more constrained, as shown in Fig.5, which results in higher overall stiffness of the structure and therefore, higher natural frequency is expected. In addition, this figure, shows that V effect on the eigenvalue $\left(\lambda_1^2\right)$ diminished as the plate turns into rectangular shape ($b/a>1$). This effect will be discussed later in detail.

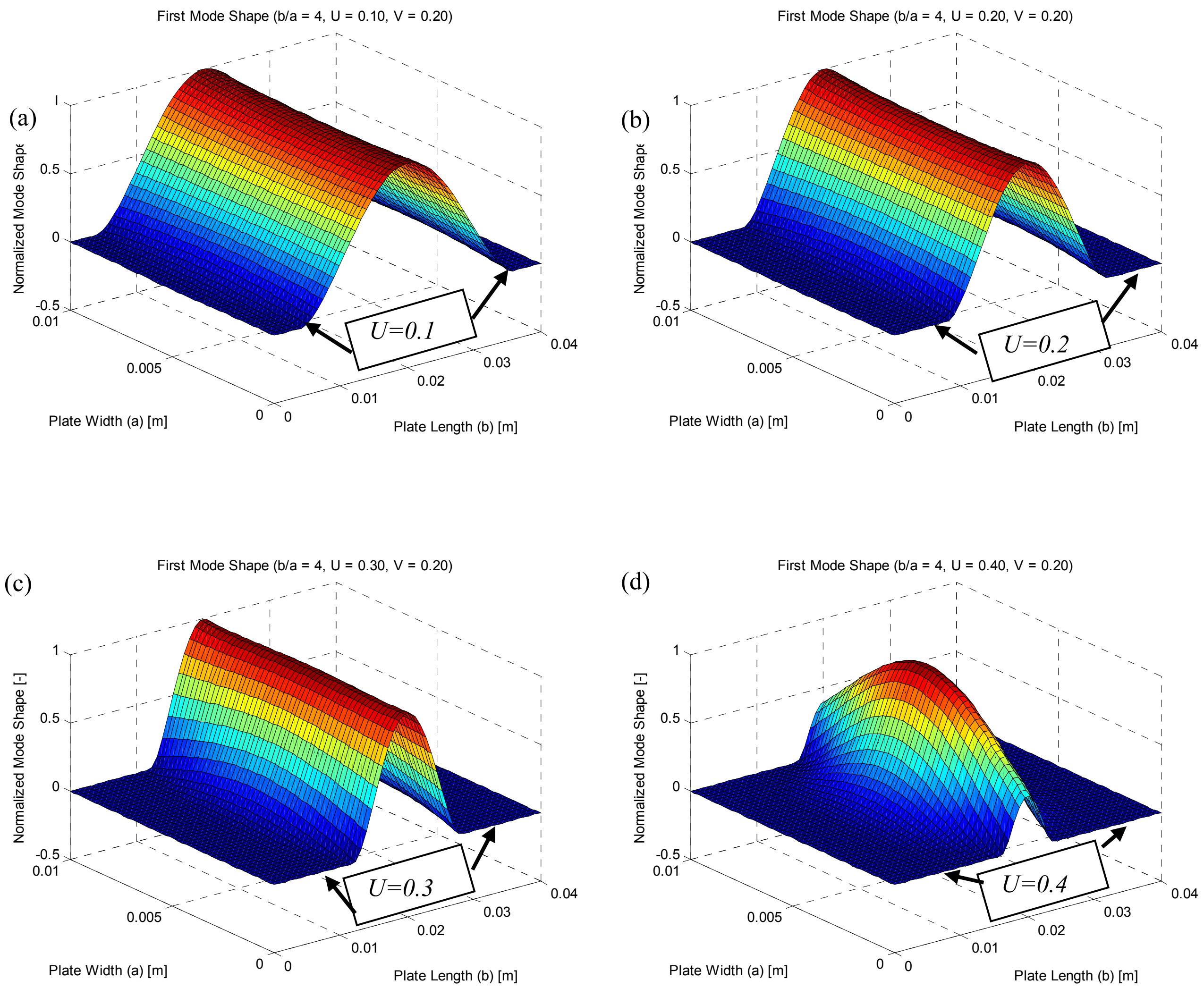

Fig.5. Plate first mode shape showing (*U*) effect on the edge bending behavior $b/a=4$, $V=0.2$ (a) $U=0.1$ (b) $U=0.2$ (c) $U=0.3$ (d) $U=0.4$.

Figure 6 depicts the effect of normalized length (U) on the plate eigenvalue $\left(\lambda_1^2\right)$ at several b/a ratios and each subplot is at certain V value. As explained, $\left(\lambda_1^2\right)$ increases as U increases exponentially for $b/a>1$ (rectangular plate) and linearly for $b/a=1$ (squared plate). Additionally, the figure proves that the plate size (or aspect ratio) has an inversely proportional effect on the structure first eigenvalue, as expected.

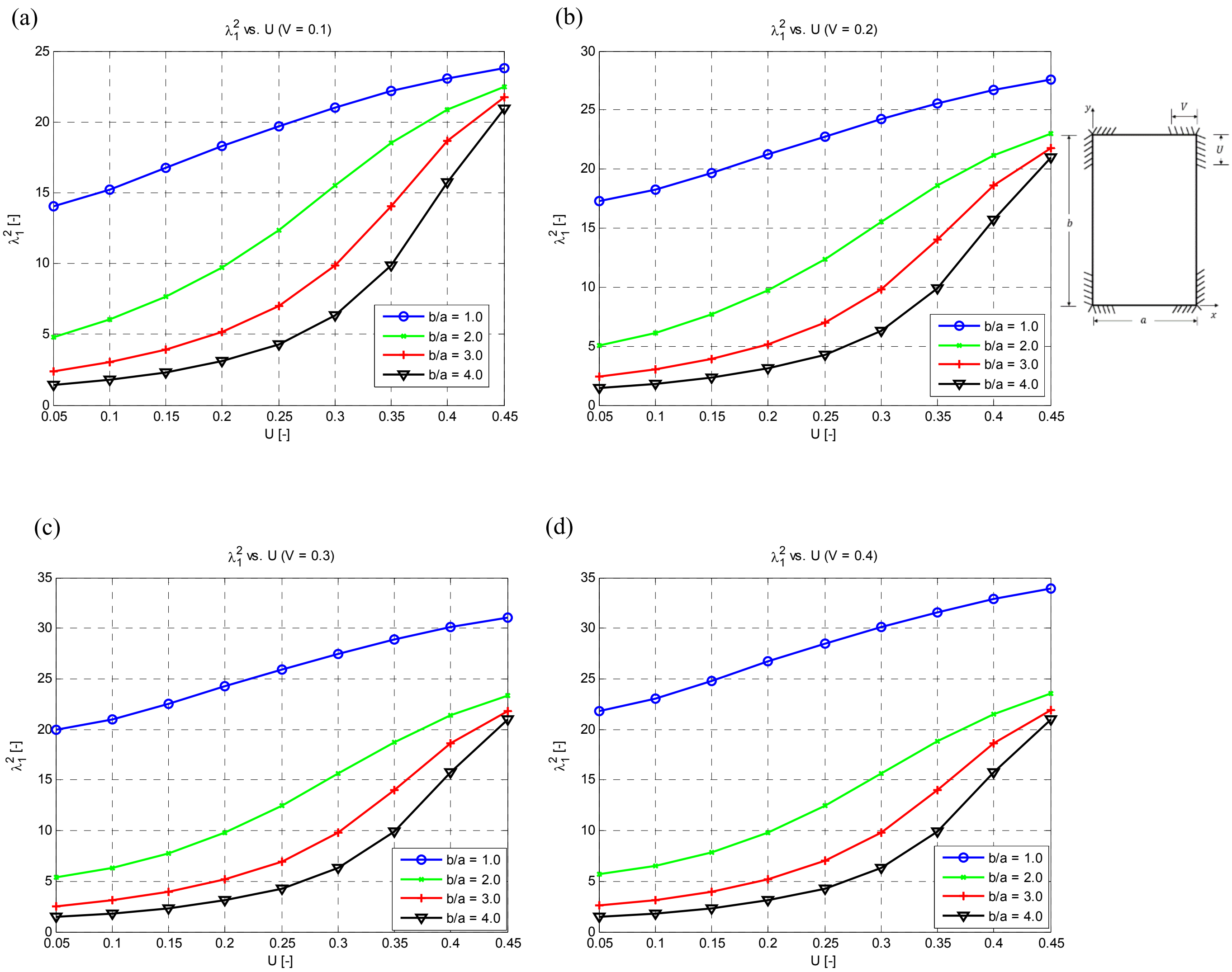

Fig.6. Effect of non-dimensionalized fixed length (U) on the plate eigenvalue $\left(\lambda_1^2\right)$ at different $\frac{b}{a}$ values for (a) $V=0.1$ (b) $V=0.2$ (c) $V=0.3$ (d) $V=0.4$.

The effect of the edge-clamping in the x-direction (V) on the plate natural frequency is presented in Fig.7 and Fig.8. Figure 7 demonstrates the effect of V on eigenvalue $\left(\lambda_1^2\right)$ at several U values while Fig.8 shows the same effect at several plate aspect ratios (b/a). For a squared plate ($b/a=1$), as shown in Fig.7a and Figs 8 a-d it appears that V has a considerable effect on the plate eigen frequency. Interestingly, this effect starts to diminish as the plate becomes rectangular ($b/a>1$), as depicted in Figs 7b-d and Figs 8a-d.

To explain this interesting behavior, the first mode shapes of a squared and non-squared plate having different clamping properties were plotted in Fig.9 and Fig.10, respectively. For a squared plate (Fig.9), it can be clearly seen that as V increases the bending of the plate edge associated with V becomes harder and therefore the plate gets stiffer which further results in higher natural frequency of the plate system. However, for the rectangular plate ($b/a=3$) depicted in Fig.10, the bending behavior of the plate edge associated with V does not change for different fixed lengths (V) configurations.

Also, for the same plate, the dominant bending happens to be on the plate other pair of edges (associated with U). This discussion explains the plate shortest length clamping effect diminishing for a rectangular plate structure.

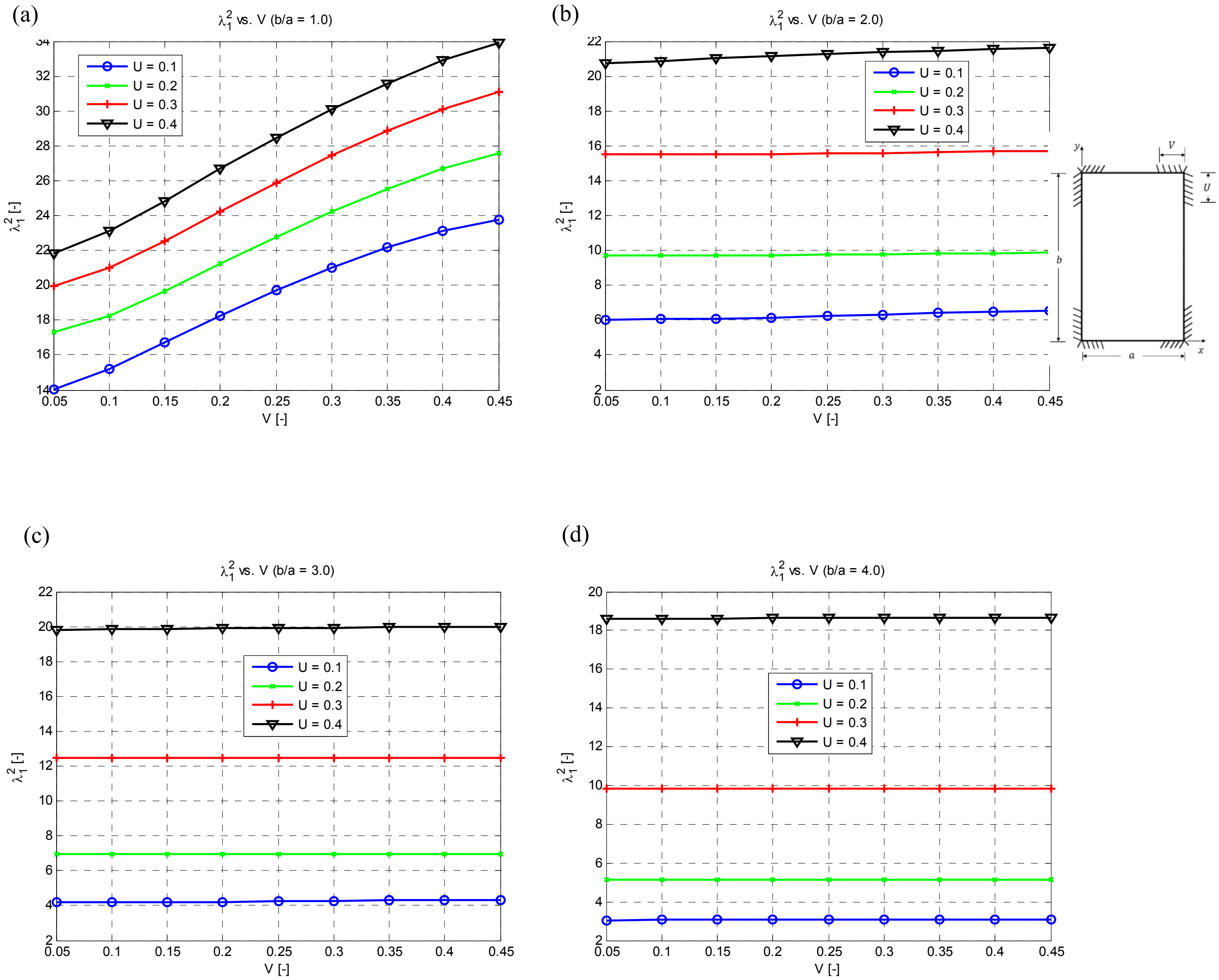

Fig.7. Effect of non-dimensionalized fixed length (V) on the plate eigenvalue $\left(\lambda_1^2\right)$ at different U values for (a) $b/a=1$ (b) $b/a=2$ (c) $b/a=3$ (d) $b/a=4$.

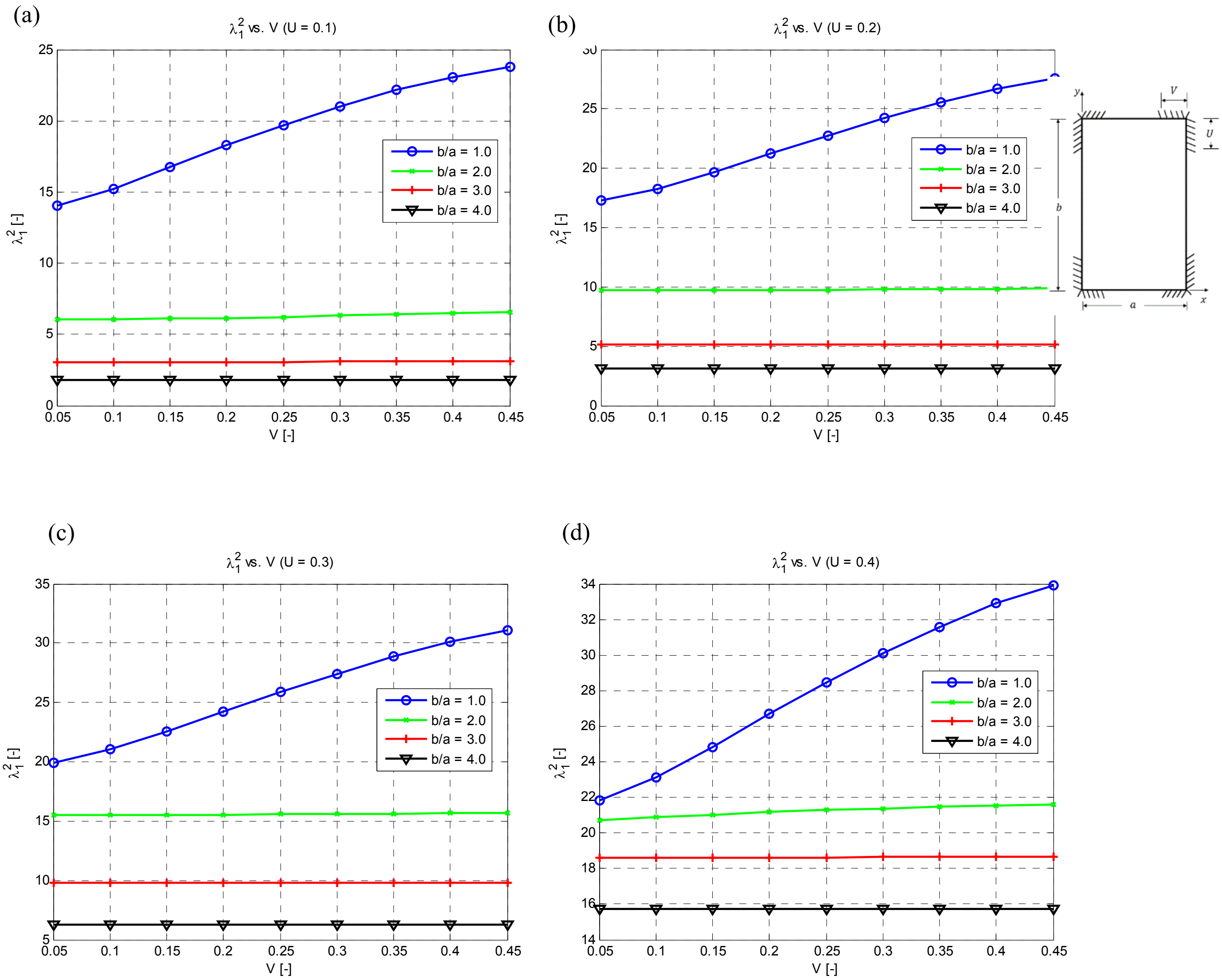

Fig.8. Effect of non-dimensionalized fixed length (V) on the plate eigenvalue $\left(\lambda_1^2\right)$ at different $\frac{b}{a}$ values for (a) $U = 0.1$ (b) $U = 0.2$ (c) $U = 0.3$ (d) $U = 0.4$.

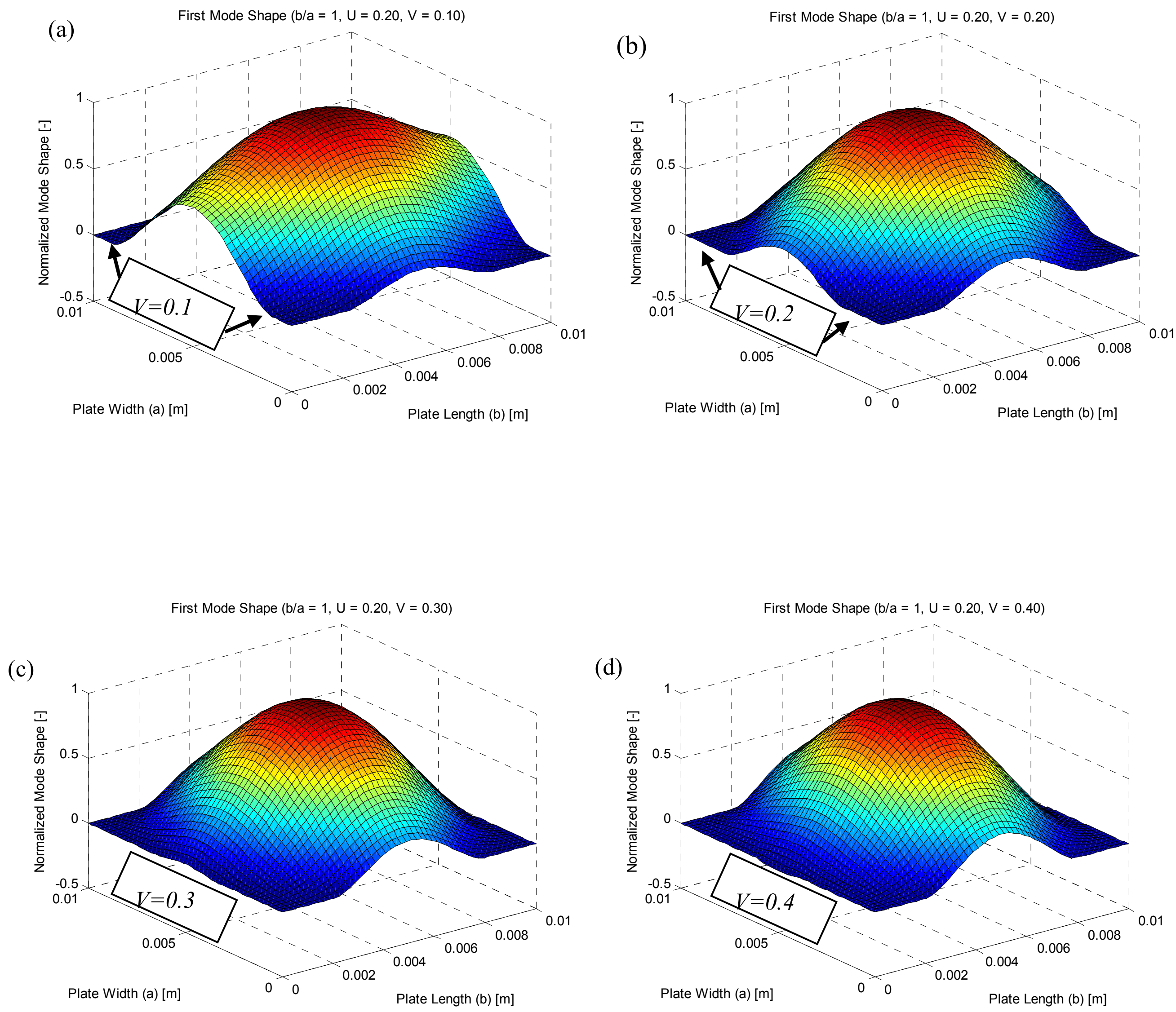

Fig.9. A squared plate first mode shape showing (V) effect on the edge bending behavior $b/a = 1$, $U = 0.2$ (a) $V = 0.1$ (b) $V = 0.2$ (c) $V = 0.3$ (d) $V = 0.4$.

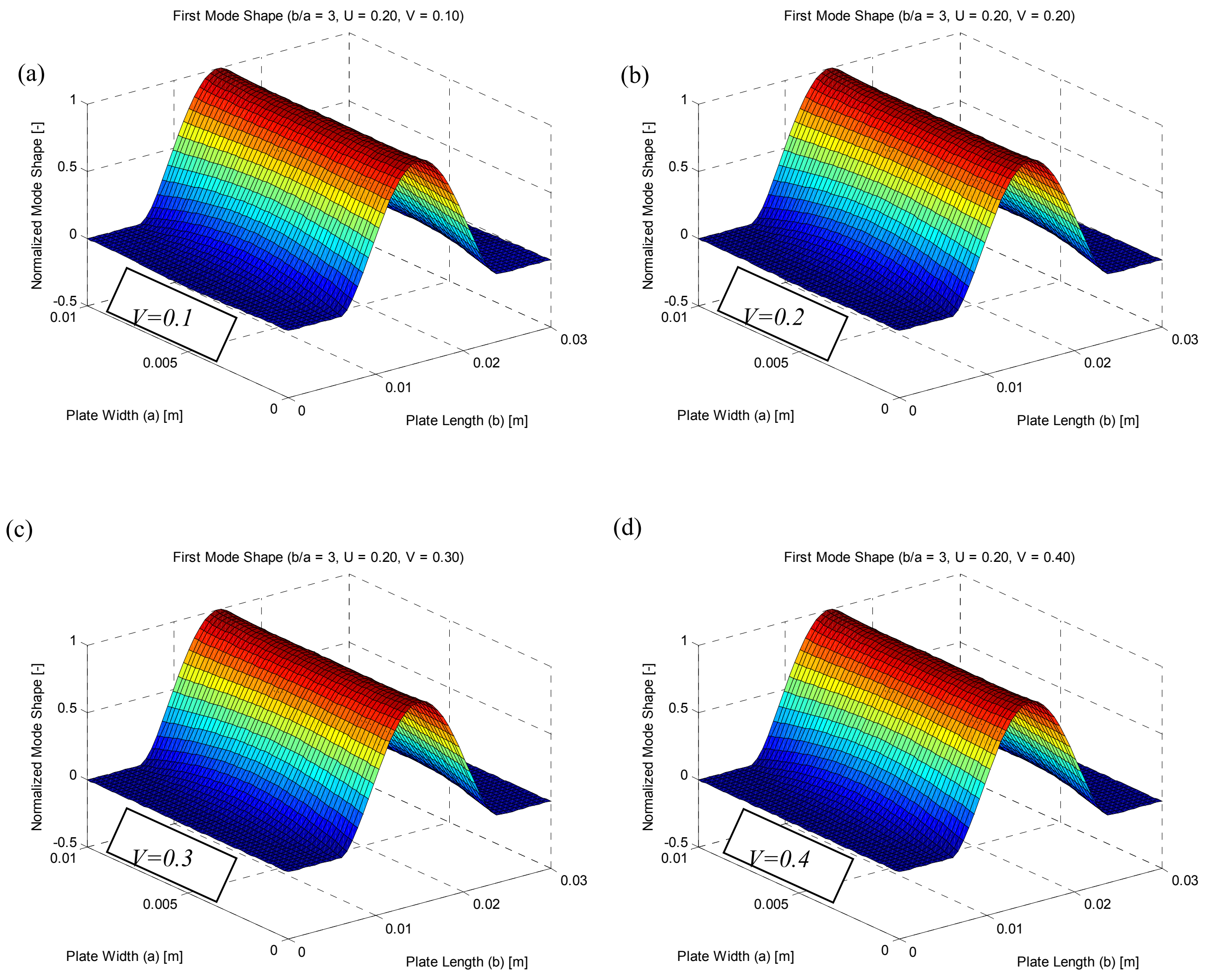

Fig.10. A non-squared plate first mode shape showing (V) effect on the edge bending behavior $b/a=3$, $U=0.2$ (a) $V=0.1$ (b) $V=0.2$ (c) $V=0.3$ (d) $V=0.4$.

For examining the effect of the plate size, i.e., the aspect ratio, on the system first non-dimensional eigenvalue, Fig.11 and Fig.12 are plotted. Figure 11 and Fig.12 illustrate the effect of b/a on eigenvalue $\left(\lambda_1^2\right)$ at different V and U values, respectively. As seen from both figures, the plate non-dimensional eigenvalue $\left(\lambda_1^2\right)$ is inversely proportional to the plate aspect ratio. This trend could be explained as follows; when the elastic plate gets larger it becomes more complaint and therefore, this would result in a lower $\left(\lambda_1^2\right)$. This behavior is well-known in the literature for the elastic rectangular plate structure free vibration problem.

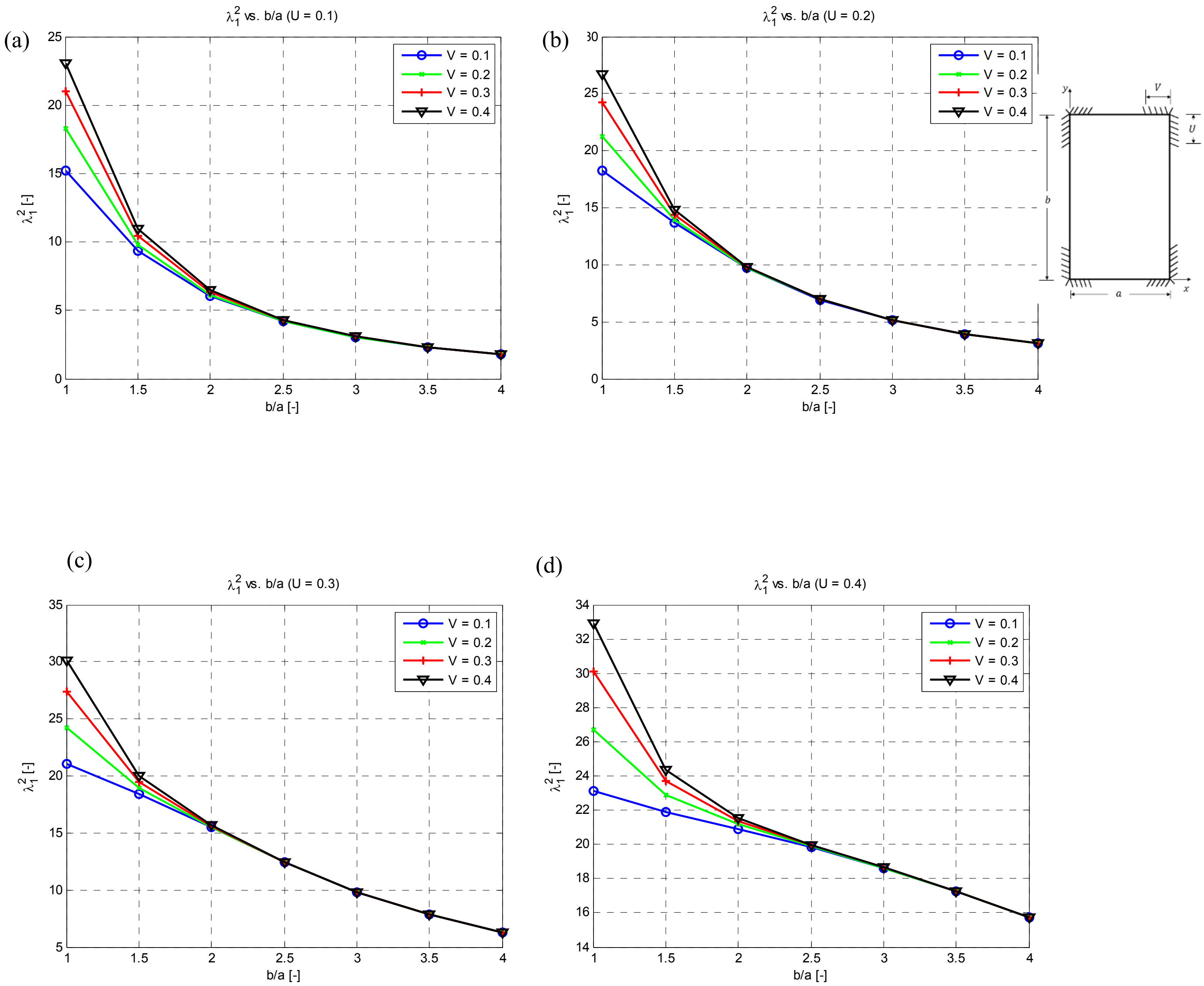

Fig.11. Effect of the plate aspect ratio (b/a) on the plate eigenvalue $\left(\lambda_1^2\right)$ at different V values for (a) $U=0.1$ (b) $U=0.2$ (c) $U=0.3$ (d) $U=0.4$.

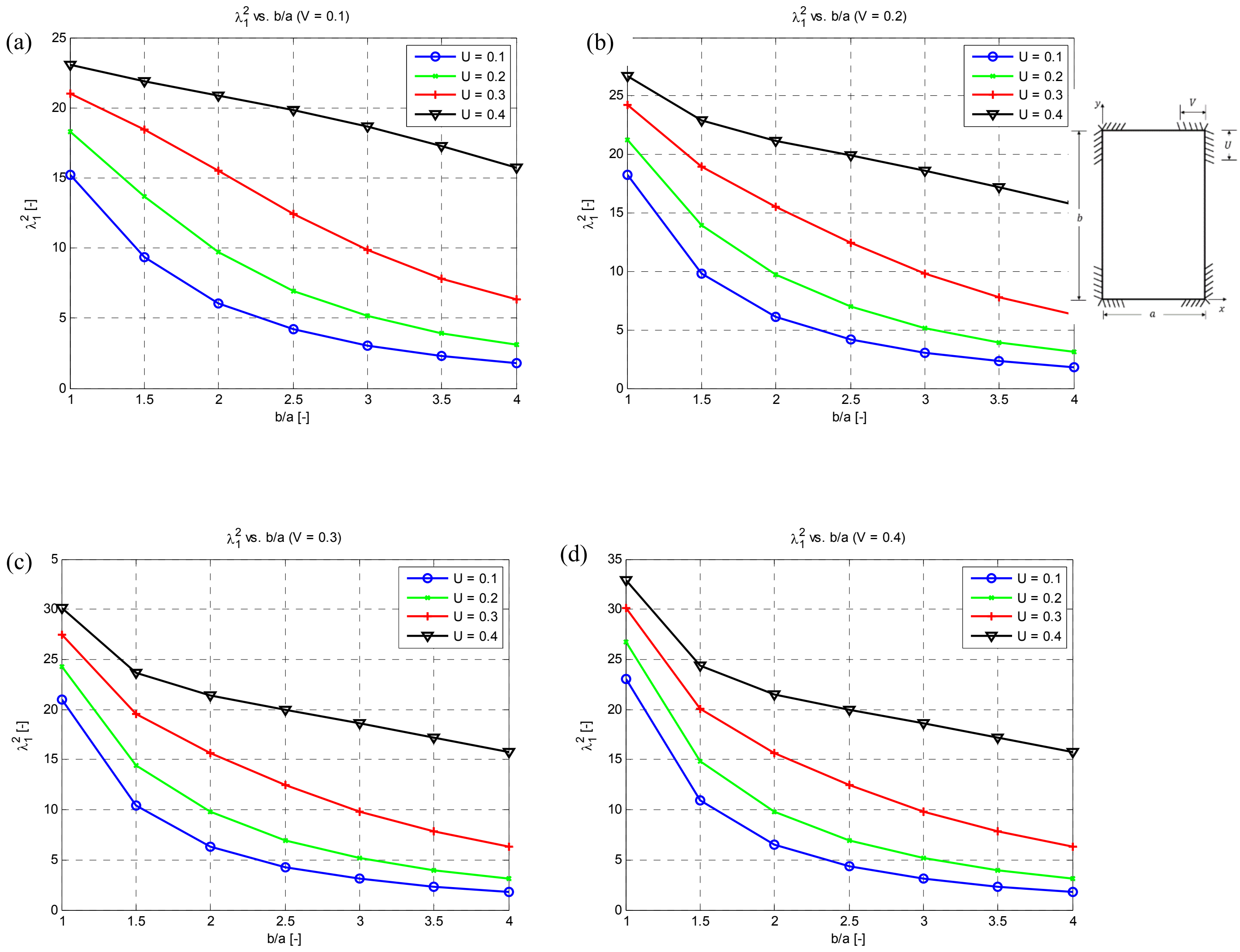

Fig.12. Effect of the plate aspect ratio (b/a) on the plate eigenvalue $\left(\lambda_1^2\right)$ at different U values for (a) $V = 0.1$ (b) $V = 0.2$ (c) $V = 0.3$ (d) $V = 0.4$.

It can be stated that the plate non-dimensional eigenvalue $\left(\lambda_1^2\right)$ generally increases as the fixed lengths increase and decrease for larger plate sizes.

Conclusions

The free vibration characteristics of an elastic rectangular plate with partially clamped edges around the corners were numerically investigated using the finite element method. ANSYS software package was employed to build the FE model of the problem and run all the required analyses for several plate geometries as well as constraint properties. The effect of the plate size, i.e., the aspect ratio, and the clamped lengths properties on the plate first non-dimensional eigenvalue, hence the first natural frequency, was discussed in detail. A thorough explanation of the effect of such parameters on the natural frequency of structure throughout the first mode shape data was presented. Finally, the results showed that the clamping properties generally affect the natural frequency of the system proportionally, while the plate aspect ratio has an inversely proportional effect on this natural frequency.

Acknowledgement

The authors wish to thank the deanship of scientific research at the Hashemite University for providing the necessary funds and tools to finish this work.

Nomenclature

a,b	– plate dimensions
b/a	– plate aspect ratio
D	– plate flexural modulus $D = \frac{Eh^3}{12\left(1-\nu^2\right)}$
E,ρ and ν	– plate materials' modulus of elasticity, density and Poisson's ratio
h	– plate thickness
U,V	– normalized clamped-edge lengths
λ_1^2	– plate free-vibration eigenvalue $\lambda_1^2 = \omega_1^2 a\sqrt{\rho h / D}$
ω_1	– plate first circular natural frequency

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Received: July 25, 2017

Revised: March 7, 2018