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Research Article

Numerical Investigation of the Time-Fractional Whitham–Broer–Kaup Equation Involving without Singular Kernel Operators

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This paper aims to implement an analytical method, known as the Laplace homotopy perturbation transform technique, for the result of fractional-order Whitham–Broer–Kaup equations. The technique is a mixture of the Laplace transformation and homotopy perturbation technique. Fractional derivatives with Mittag-Leffler and exponential laws in sense of Caputo are considered. Moreover, this paper aims to show the Whitham–Broer–Kaup equations with both derivatives to see their difference in a real-world problem. The efficiency of both operators is confirmed by the outcome of the actual results of the Whitham–Broer–Kaup equations. Some problems have been presented to compare the solutions achieved with both fractional-order derivatives.

1. Introduction

In engineering and applied sciences and technology, fractional partial differential equations (FPDEs) containing nonlinearities define many phenomena, ranging from gravitation to dynamics. The nonlinear FPDEs are significant tools analyzed to model nonlinear dynamical behaviour in many areas such as plasma physics, mathematical biology, fluid dynamics, and solid-state physics. The widely held dynamical schemes can be denoted by an appropriate set of FPDEs. Moreover, it is well identified that FPDEs solve mathematical models, such as Poincare conjecture and Calabi conjecture models [1–12].

It has been determined that the nonlinear development of shallow-water waves in fluid mechanics is described by a coupled system of Whitham–Broer–Kaup (WBK) equations [13]. Whitham [14], Broer [15], and Kaup [16] proposed the coupled scheme of the aforementioned equations. The aforementioned equations define shallow-water wave propagation with various spreading relations, as shown in [17]. The governing equations for the respective phenomena in classic order are provided by

$$D_{\mathfrak{J}}^{\beta}\mu(\varphi,\mathfrak{J})+\mu(\varphi,\mathfrak{J})\frac{\partial\mu(\varphi,\mathfrak{J})}{\partial\varphi}+\frac{\partial\mu(\varphi,\mathfrak{J})}{\partial\varphi}+g\frac{\partial\nu(\varphi,\mathfrak{J})}{\partial\varphi}=0,$$
$$D_{\mathfrak{J}}^{\beta}\nu(\varphi,\mathfrak{J})+\mu(\varphi,\mathfrak{J})\frac{\partial\nu(\varphi,\mathfrak{J})}{\partial\varphi}+\nu(\varphi,\mathfrak{J})\frac{\partial\mu(\varphi,\mathfrak{J})}{\partial\varphi}+p\frac{\partial^3\mu(\varphi,\mathfrak{J})}{\partial\varphi^3}-g\frac{\partial^2\nu(\varphi,\mathfrak{J})}{\partial\varphi^2}=0,\quad 0<\beta\le 1,\tag{1}$$

where $\mu(\varphi,\mathfrak{J})$ and $\nu(\varphi,\mathfrak{J})$ show the height and horizontal velocity that diverges from the liquid's equilibrium position, respectively, and pg are constant, signified in different diffusion powers. Over the last few decades, there has been a lot of research into solutions to such nonlinear PDEs. So, many researchers have created a variety of mathematical methods to investigate the analytical results of nonlinear PDEs. The HPM was used by Biazar and Khah [18] to solve the coupled schemes of the Burger and Brusselator problems. Amjad et al. [19] applied the solution of standard order coupled with fractional-order Whitham–Broer–Kaup equation by Laplace decomposition technique. Noor et al. [20] used the homotopy perturbation technique to investigate the results of much classical order of PDEs. Whitham–Broer–Kaup equations are used by other scholars who implemented several numerical techniques, such as residual power series technique [4], reduced differential transformation technique [21], Adomian decomposition technique [7], homotopy perturbation technique [22, 23], Lie symmetry analysis [24, 25], exp-function technique [26], G'/G-expansion technique [27], and homotopy analysis technique [28].

Fractional calculus (FC) is a new mathematical approach for describing models of nonlocal behaviour. Fractional derivatives have mathematically described many other physical problems in recent years; these representations have yielded excellent outcomes in the simulation of real-world issues. Some basic definitions of fractional operators were given by Riesz, Coimbra, Hadamard, Riemann–Liouville, Grunwald–Letnikov, Weyl, Caputo, Fabrizio, and Atangana Baleanu, among others [29–31]. To investigate the solutions of nonlinear FPDEs, some well-known techniques for finding actual results have been develop, for example, the homotopy perturbation transform technique [32, 33], the invariant subspace technique [34], the Hermite colocation technique [35], the q-homotopy analysis transform method [36], the optimal homotopy asymptotic technique [37], the homotopy analysis Sumudu transform technique [38], the Adomian decomposition technique [39], the Pade approximation and homotopy-Pade method [39], and the Sumudu transform series expansion technique [40]. The Laplace homotopy perturbation transform technique is a mixture of the homotopy perturbation technique introduced by Liao [41] and of the Laplace transformation [42].

The rest of the paper is arranged as follows. Section 2 discusses the basic definitions from fractional calculus. Section 3 is introduced to the fundamental methodology of HPTM. Sections 4 and 5 are the implementations of the techniques for the CF and AN operators. The conclusion of the work is written in Section 6.

2. Fractional Calculus

This section provides some fundamental concepts of fractional calculus.

Definition 1 (see [42]). The Liouville–Caputo operator is given by

$$D_{\mathfrak{J}}^{\beta}u(\varphi,\mathfrak{J})=\frac{1}{\Gamma(n-\beta)}\int_0^{\mathfrak{J}}(\mathfrak{J}-\theta)^{n-1}u^{(n)}(\varphi,\theta)\mathrm{d}\theta,\quad n-1<\beta<n,\tag{2}$$

where $u^{(n)}(\varphi,\theta)$ is the derivative of integer nth order of $u(\varphi,\theta)$, $n\in\mathbb{N}$ and $n-1<\beta\le n$. If $0<\beta\le 1$, then we defined the Laplace transformation for the Caputo fractional derivative as follows:

$$\mathscr{L}\left[D_{\mathfrak{J}}^{\beta}u(\varphi,\mathfrak{J})\right](s)=s^{\beta}\mathscr{L}[u(\varphi,\mathfrak{J})](s)-s^{\beta-1}[u(\varphi,0)].\tag{3}$$

Definition 2 (see [42]). The Caputo–Fabrizio operator (CF) is defined by

$$D_{\mathfrak{J}}^{\beta}u(\varphi,\mathfrak{J})=\frac{(2-\beta)M(\beta)}{2(n-\beta)}\int_0^{\mathfrak{J}}\exp\left(-\beta\frac{(\mathfrak{J}-\theta)}{n-\beta}\right)u^{(n)}(\varphi,\theta)\mathrm{d}\theta,\tag{4}$$

where $M(\beta)$ is a normalization form and $M(0)=M(1)=1$. The exponential law is used as the nonsingular kernel in this fractional operator.

If $0<\beta\le 1$, then we define the Caputo–Fabrizio of Laplace transformation for the fractional derivative is given as

$$\mathscr{L}\left[D_{\mathfrak{J}}^{\beta}u(\varphi,\mathfrak{J})\right](s)=\left(\frac{s\mathscr{L}[u(\varphi,\mathfrak{J})](s)-u(\varphi,0)}{s+\beta(1-s)}\right).\tag{5}$$

Definition 3 (see [42]). The fractional generalized Mittag-Leffler law with the sense of Atangana–Baleanu operator is defined as follows:

$$D_{\mathfrak{J}}^{\beta}u(\varphi,\mathfrak{J})=\frac{B(\beta)}{1-\beta}\int_0^{\mathfrak{J}}E_{\beta}\left[-\beta\frac{(\mathfrak{J}-\theta)^{\beta}}{n-\beta}\right]u^{(n)}(\varphi,\theta)\mathrm{d}\theta,\tag{6}$$

where $B(\beta)$ is a normalization function with $B(0)=B(1)=1$. The Mittag-Leffler law is used as a nonsingular and nonlocal kernel in this fractional operator.

If $0 < \beta \leq 1$, then we express the Laplace transformation for the Atangana–Baleanu operator fractional derivative as

$$\mathscr{L}\left[D_{\mathfrak{F}}^{\beta} u(\varphi, \mathfrak{F})\right](s) = \left(\frac{s\mathscr{L}[u(\varphi, \mathfrak{F})](s) - u(\varphi, 0)}{s^{\beta}(1-\beta) + \beta}\right). \quad (7)$$

3. Implementation of the LHPTM for the Solution of Fractional Partial Differential Equation

The LHPTM is general methodology and can be written as follows.

The main procedure of this technique is defined as follows:

Step 1: let us consider the following equation:

$$\frac{\partial^{\beta}}{\partial \mathfrak{F}^{\beta}} F(\varphi, \mathfrak{F}) = \Lambda(F(\varphi, \mathfrak{F})) + \Xi(F(\varphi, \mathfrak{F})) + u(\varphi, \mathfrak{F}), \quad \beta = 1, 2, 3, \ldots, \quad (8)$$

under the initial condition

$$\frac{\partial^{i}}{\partial \mathfrak{F}^{i}} F(\varphi, 0) = z_i(\varphi), \quad \frac{\partial^{\beta-1}}{\partial \mathfrak{F}^{\beta-1}} F(\varphi, 0) = 0, \quad i = 0, 1, 2, 3, \ldots, \beta - 2, \quad (9)$$

where $u(\varphi, \mathfrak{F})$ is a known function, β is the order of the derivative, Λ represents a linear differential operator, and Ξ is the general nonlinear differential operator.

Step 2: using both sides Laplace transformation operator of (8), we obtain

$$\mathscr{L}[f(\varphi, \mathfrak{F})] = s^{\beta}\mathscr{L}[\Lambda(F(\varphi, \mathfrak{F}))] + s^{\beta}\mathscr{L}[\Xi(F(\varphi, \mathfrak{F}))] + s^{\beta}\mathscr{L}[u(\varphi, \mathfrak{F})], \quad (10)$$

Laplace transformation is applied to Caputo–Fabrizio (5) and Atangana–Baleanu (7) operators.

Step 3: on both sides, using the inverse Laplace transformation of equation (10), we obtain

$$F(\varphi, \mathfrak{F}) = H(\varphi, \mathfrak{F}) + \mathscr{L}^{-1}\left[s^{\beta}\mathscr{L}[\Lambda(F(\varphi, \mathfrak{F}))] + s^{\beta}\mathscr{L}[\Xi(F(\varphi, \mathfrak{F}))]\right]. \quad (11)$$

Step 4: applying the homotopy producer, the result of the above equations in a series form is defined by

$$F(\varphi, \mathfrak{F}, k) = \sum_{n=0}^{\infty} k^n F_n(\varphi, \mathfrak{F}), \quad F(\varphi, \mathfrak{F}) = \lim_{k \longrightarrow 1} F(\varphi, \mathfrak{F}, k), \quad (12)$$

and the nonlinear terms can be expressed as

$$\Xi F(\varphi, \mathfrak{F}) = \sum_{n=1}^{\infty} k^n h_n(f(\varphi, \mathfrak{F})), \quad (13)$$

where $k \in (0, 1]$ is an embedding parameter and $h_n(F(x, \mathfrak{F}))$ are He's polynomials that can be provided by

$$h_n(F_0, \ldots, F_n) = \frac{1}{n!}\frac{\partial^n}{\partial k^k}\left[\Xi\left(\sum_{j=0}^{n} k^j F_j(\varphi, \mathfrak{F})\right)\right], \quad n \in \mathbb{N}. \quad (14)$$

Finally, the LHPTM is achieved by coupling the decomposition technique which is defined by

$$\sum_{j=0}^{\infty} k^n F_n(\varphi, \mathfrak{F}) = T(\varphi, \mathfrak{F}) + k\frac{1}{(m-1)!}\int_0^{\mathfrak{F}} (\mathfrak{F} - \tau)^{m-1} \cdot \left[u(\varphi, \mathfrak{F}) + \Lambda\left(\sum_{n=0}^{\infty} k^n F_n(\varphi, \mathfrak{F})\right) + \sum_{n=0}^{\infty} k^n h_n(F(\varphi, \mathfrak{F}))\right] d\tau, \quad (15)$$

where $T(\varphi, \mathfrak{F}) = H(\varphi, \mathfrak{F})$.

The terms, comparing with the same powers of k, produce results of many orders. The initial estimated of the approximation is $T(\varphi, \mathfrak{F})$, which is actually the Taylor series for the exact result of order β.

Using the aforementioned technique, we solved the fractional-order Whitham–Broer–Kaup equations in the Atangana–Baleanu and Caputo–Fabrizio senses using the LHPM.

4. Implementation of Caputo–Fabrizio Operator

Example 1. Let us consider the coupled system of fractional-order WBKEs in the CFC sense:

$$
\begin{aligned}
&{}^{CF}D^{\beta}_{\mathfrak{F}}\mu(\varphi,\mathfrak{F})+\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}=0,\\
&{}^{CF}D^{\beta}_{\mathfrak{F}}\nu(\varphi,\mathfrak{F})+\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}+\nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+3\frac{\partial^3\mu(\varphi,\mathfrak{F})}{\partial\varphi^3}-\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2}=0,\quad 0<\beta\le 1,-1<\mathfrak{F}\le 1,-10\le\varphi\le 10,
\end{aligned}
\tag{16}
$$

under the initial conditions,

$$
\begin{aligned}
\mu(\varphi,0)&=\frac{1}{2}-8\tanh(-2\varphi),\\
\nu(\varphi,0)&=16-16\tanh^2(-2\varphi).
\end{aligned}
\tag{17}
$$

Using the Laplace transformation to equation (16), we obtain

$$
\begin{aligned}
\frac{s\mathscr{L}[\mu(\varphi,\mathfrak{F})](s)-\mu(\varphi,0)}{s+\beta(1-s)}&=-\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right],\\
\frac{s\mathscr{L}[\nu(\varphi,\mathfrak{F})](s)-\nu(\varphi,0)}{s+\beta(1-s)}&=-\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}+\nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+3\frac{\partial^3\mu(\varphi,\mathfrak{F})}{\partial\varphi^3}-\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2}\right].
\end{aligned}
\tag{18}
$$

Simplify the above equation and use the initial conditions (17), and we obtain

$$
\begin{aligned}
\mathscr{L}[\mu(\varphi,\mathfrak{F})]&=\frac{\mu(\varphi,0)}{s}-\frac{s+\beta(1-s)}{s}\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right],\\
\mathscr{L}[\nu(\varphi,\mathfrak{F})]&=\frac{\nu(\varphi,0)}{s}-\frac{s+\beta(1-s)}{s}\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}+\nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+3\frac{\partial^3\mu(\varphi,\mathfrak{F})}{\partial\varphi^3}-\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2}\right].
\end{aligned}
\tag{19}
$$

The inverse Laplace transformation is implemented to (19), and we obtain

$$
\begin{aligned}
\mu(\varphi,\mathfrak{F})&=\mu(\varphi,0)-\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right\}\right],\\
\nu(\varphi,\mathfrak{F})&=\nu(\varphi,0)-\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}+\nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+3\frac{\partial^3\mu(\varphi,\mathfrak{F})}{\partial\varphi^3}-\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2}\right\}\right].
\end{aligned}
\tag{20}
$$

The LHPTM is used in (20), and we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mu_n(\varphi,\mathfrak{F})&=\mu(\varphi,0)-p\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{\sum_{n=0}^{\infty}p^nA_n+\frac{\partial}{\partial\varphi}\sum_{n=0}^{\infty}p^n\mu_n(\varphi,\mathfrak{F})+\frac{\partial}{\partial\varphi}\sum_{n=0}^{\infty}p^n\nu_n(\varphi,\mathfrak{F})\right\}\right],\\
\sum_{n=0}^{\infty}\nu_n(\varphi,\mathfrak{F})&=\nu(\varphi,0)-p\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{\sum_{n=0}^{\infty}p^nB_n+\sum_{n=0}^{\infty}p^nC_n+3\frac{\partial^3}{\partial\varphi^3}\sum_{n=0}^{\infty}p^n\mu_n(\varphi,\mathfrak{F})-\frac{\partial^2}{\partial\varphi^2}\sum_{n=0}^{\infty}p^n\nu_n(\varphi,\mathfrak{F})\right\}\right].
\end{aligned}
\tag{21}
$$

The nonlinear can be found with the help of He's polynomial and can be defined as

$$\begin{aligned} A_0 &= \mu_0 \frac{\partial \mu_0}{\partial \varphi}, \\ A_1 &= \mu_0 \frac{\partial \mu_1}{\partial \varphi} + \mu_1 \frac{\partial \mu_0}{\partial \varphi}, \\ B_0 &= \mu_0 \frac{\partial \nu_0}{\partial \beta}, \\ B_1 &= \mu_0 \frac{\partial \nu_1}{\partial \beta} + \mu_1 \frac{\partial \nu_0}{\partial \beta}, \\ C_0 &= \nu_0 \frac{\partial \mu_0}{\partial \varphi}, \\ C_1 &= \nu_0 \frac{\partial \mu_1}{\partial \varphi} + \nu_1 \frac{\partial \mu_0}{\partial \varphi}. \end{aligned} \tag{22}$$

Comparing the coefficient of p, we have

$$\begin{aligned} p^0\colon \mu_0(\varphi, \mathfrak{F}) &= \mu(\varphi, 0) = \frac{1}{2} - 8\ \tanh(-2\varphi), \\ p^0\colon \nu_0(\varphi, \mathfrak{F}) &= \nu(\varphi, 0) = 16 - 16\tanh^2(-2\varphi), \\ p^1\colon \mu_1(\varphi, t) &= -\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{A_0 + \frac{\partial}{\partial\varphi}\mu_0(\varphi, \mathfrak{F}) + \frac{\partial}{\partial\varphi}\nu_0(\varphi, \mathfrak{F})\right\}\right] = -8\ \sec h^2(-2\varphi)[\beta\mathfrak{F} + (1-\beta)], \\ p^1\colon \nu_1(\varphi, \mathfrak{F}) &= -p\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{B_0 + C_0 + 3\frac{\partial^3}{\partial\varphi^3}\mu_0(\varphi, \mathfrak{F}) - \frac{\partial^2}{\partial\varphi^2}\nu_0(\varphi, \mathfrak{F})\right\}\right] \\ &= -32\ \sec h^2(-2\varphi)\tanh(-2\varphi)[\beta\mathfrak{F} + (1-\beta)], \\ p^2\colon \mu_2(\varphi, \mathfrak{F}) &= -\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{A_1 + \frac{\partial}{\partial\varphi}\mu_1(\varphi, \mathfrak{F}) + \frac{\partial}{\partial\varphi}\nu_1(\varphi, \mathfrak{F})\right\}\right] \\ &= -16\ \sec h^2(-2\varphi)\Big(4\ \sec h^2(-2\varphi) - 8\ \tanh^2(-2\varphi) + 3\ \tanh(-2\varphi)\Big)\left[(1-\beta)^2 + 2\beta(1-\beta)\mathfrak{F} + \frac{\beta^2\mathfrak{F}^2}{2}\right], \\ p^2\colon \nu_2(\varphi, \mathfrak{F}) &= -p\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{B_1 + C_1 + 3\frac{\partial^3}{\partial\varphi^3}\mu_1(\varphi, \mathfrak{F}) - \frac{\partial^2}{\partial\varphi^2}\nu_1(\varphi, \mathfrak{F})\right\}\right] \\ &= -32\ \sec h^2(-2\varphi)\Big\{40\ \sec h^2(-2\varphi)\tanh(-2\varphi) + 96\ \tanh(-2\varphi) - 2\ \tanh^2(-2\varphi) \\ &\quad -32\ \tanh^3(-2\varphi) - 25\ \sec h^2(-2\varphi)\Big\}\left[(1-\beta)^2 + 2\beta(1-\beta)\mathfrak{F} + \frac{\beta^2\mathfrak{F}^2}{2}\right], \\ &\vdots . \end{aligned} \tag{23}$$

We can calculate few terms of (16) which can be written as

$$\mu(\varphi,\mathfrak{F}) = \mu_0(\varphi,\mathfrak{F}) + \mu_1(\varphi,\mathfrak{F}) + \mu_2(\varphi,\mathfrak{F}) + \cdots,$$

$$\nu(\varphi,\mathfrak{F}) = \nu_0(\varphi,\mathfrak{F}) + \nu_1(\varphi,\mathfrak{F}) + \nu_2(\varphi,\mathfrak{F}) + \cdots,$$

$$\begin{aligned}\mu(\varphi,\mathfrak{F}) &= \frac{1}{2} - 8\tanh(-2\varphi) - 8\sec h^2(-2\varphi)[\beta\mathfrak{F} + (1-\beta)] - 16\sec h^2(-2\varphi)\\ &\times\left(4\sec h^2(-2\varphi) - 8\tanh^2(-2\varphi) + 3\tanh(-2\varphi)\right)\left[(1-\beta)^2 + 2\beta(1-\beta)\mathfrak{F} + \frac{\beta^2\mathfrak{F}^2}{2}\right] + \cdots,\end{aligned} \tag{24}$$

$$\begin{aligned}\nu(\varphi,\mathfrak{F}) &= 16 - 16\tanh^2(-2\varphi) - 8\sec h^2(-2\varphi)[\beta\mathfrak{F} + (1-\beta)] - 32\sec h^2(-2\varphi)\{40\sec h^2(-2\varphi)\tanh(-2\varphi)\\ &+96\tanh(-2\varphi) - 2\tanh^2(-2\varphi) - 32\tanh^3(-2\varphi) - 25\sec h^2(-2\varphi)\}\\ &\times\left[(1-\beta)^2 + 2\beta(1-\beta)\mathfrak{F} + \frac{\beta^2\mathfrak{F}^2}{2}\right] + \cdots.\end{aligned}$$

The exact solution of (16) is

$$\begin{aligned}\mu(\varphi,\mathfrak{F}) &= \frac{1}{2} - 8\tanh\left\{-2\left(\varphi - \frac{\mathfrak{F}}{2}\right)\right\},\\ \nu(\varphi,\mathfrak{F}) &= 16 - 16\tanh^2\left\{-2\left(\varphi - \frac{\mathfrak{F}}{2}\right)\right\}.\end{aligned} \tag{25}$$

Figure 1 shows the actual and approximate solutions of $\mu(\varphi,\mathfrak{F})$ at $\beta = 1$, and Figure 2 shows the actual and approximate solutions of $\nu(\varphi,\mathfrak{F})$ at $\beta = 1$. Figures 3 and 4 show that the first graph has a different fractional order with respect to φ and the second graph has a different fractional order with respect to $\mathfrak{F}$ of Example 1. Tables 1 and 2 show the different fractional-order β of $\mu(\varphi,\mathfrak{F})$ and $\nu(\varphi,\mathfrak{F})$. Tables 3 and 4 show the comparisons with different methods.

Example 2. Let us consider the coupled system of fractional-order WBKEs in the CFC sense:

$$\begin{aligned}{}^{CF}D^{\beta}_{\mathfrak{F}}\mu(\varphi,\mathfrak{F}) + \mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{1}{2}\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi} &= 0,\\ {}^{CF}D^{\beta}_{\mathfrak{F}}\nu(\varphi,\mathfrak{F}) + \mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi} + \nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} - \frac{1}{2}\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2} &= 0, \quad 0<\beta\le 1, 0<\mathfrak{F}\le 1, -100\le\varphi\le 100,\end{aligned} \tag{26}$$

under the initial conditions,

$$\begin{aligned}\mu(\varphi,0) &= \xi - \kappa\coth[\kappa(\varphi+\theta)],\\ \nu(\varphi,0) &= -\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)].\end{aligned} \tag{27}$$

Using the Laplace transformation to equation (26), we obtain

$$\begin{aligned}\frac{s\mathscr{L}[\mu(\varphi,\mathfrak{F})](s) - \mu(\varphi,0)}{s+\beta(1-s)} &= -\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{1}{2}\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right],\\ \frac{s\mathscr{L}[\nu(\varphi,\mathfrak{F})](s) - \nu(\varphi,0)}{s+\beta(1-s)} &= -\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi} + \nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} - \frac{1}{2}\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2}\right].\end{aligned} \tag{28}$$

Simplify the above equation and use the initial conditions (27), and we obtain

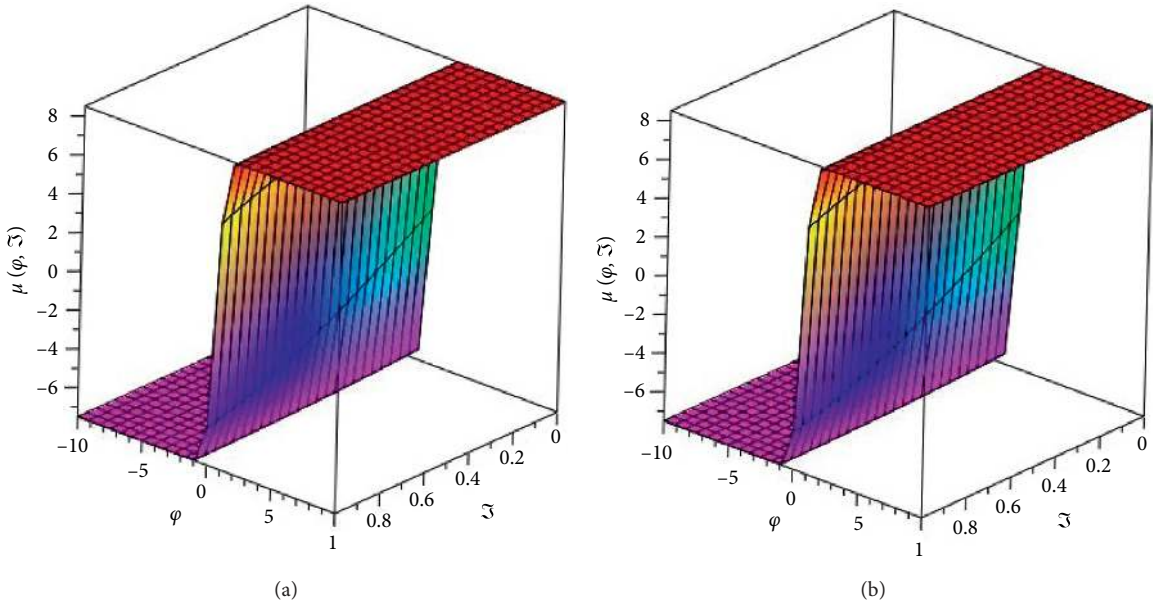

Figure 1: The exact and analytical solution with respect to the Caputo–Fabrizio operator of $\mu(\varphi, \Im)$ at $\beta = 1$ of Example 1.

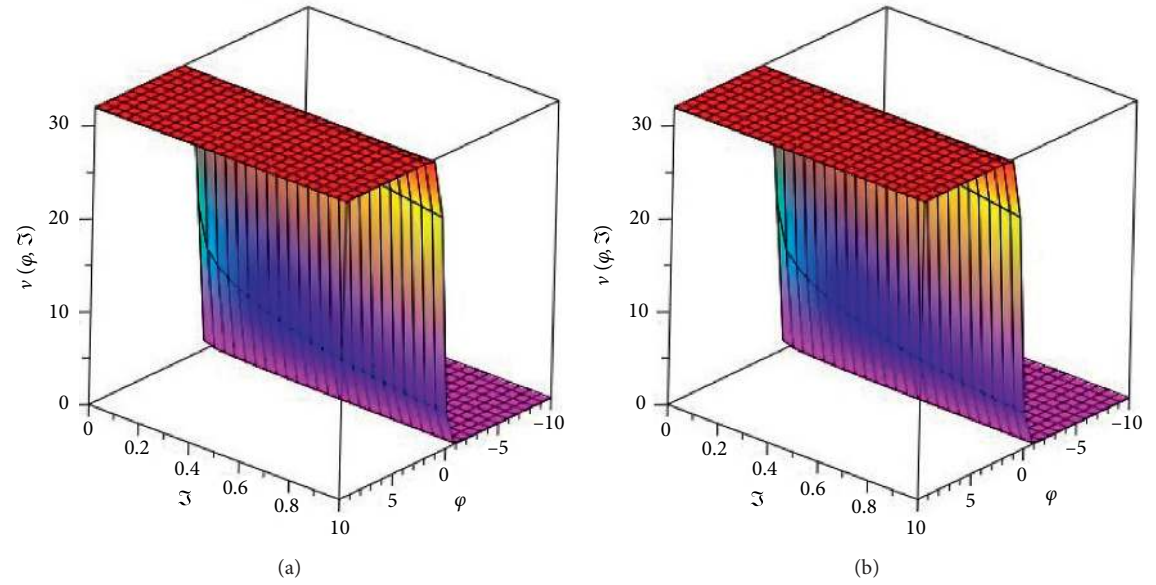

Figure 2: The exact and analytical solution with respect to the Caputo–Fabrizio operator of $\nu(\varphi, \Im)$ at $\beta = 1$ of Example 1.

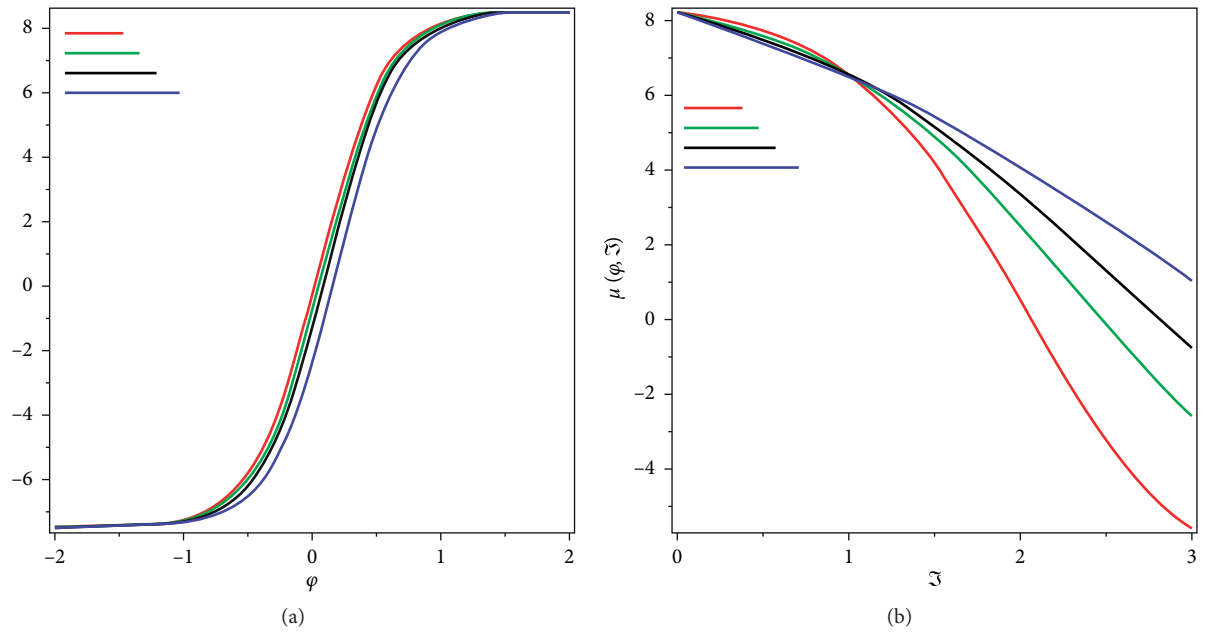

Figure 3: The different fractional-order solution with respect to the Caputo–Fabrizio operator of $\mu(\varphi, \mathfrak{F})$ at β of Example 1.

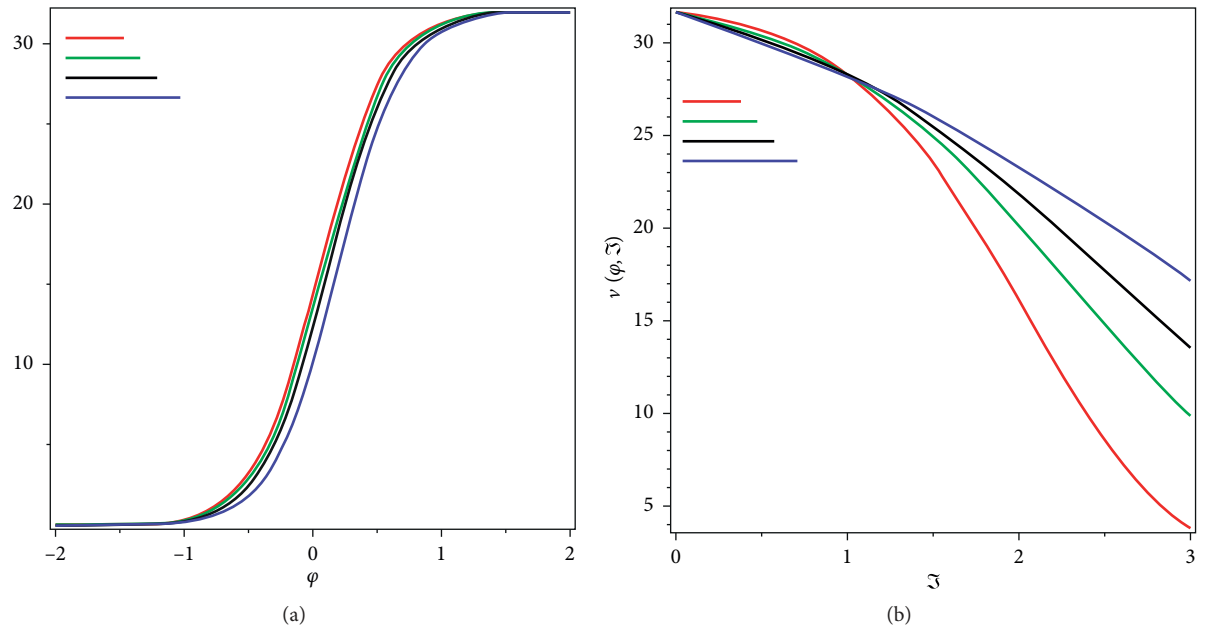

Figure 4: The different fractional-order solution with respect to the Caputo–Fabrizio operator of $\nu(\varphi, \mathfrak{F})$ at β of Example 1.

Table 1: HPTM for the approximate solution of Example 1.

$(\varphi, \mathfrak{F})$	$\mu(\varphi, \mathfrak{F})$ at $\beta = 0.5$	$\mu(\varphi, \mathfrak{F})$ at $\beta = 0.75$	$\mu(\varphi, \mathfrak{F})$ at $\beta = 1$	Exact solution
(0.1, 0.1)	−0.500817	−0.500795	−0.500782	−0.500782
(0.1, 0.3)	−0.500853	−0.500829	−0.50081	−0.50081
(0.1, 0.5)	−0.500878	−0.500857	−0.500837	−0.500837
(0.2, 0.1)	−0.49812	−0.498098	−0.498085	−0.498085
(0.2, 0.3)	−0.498154	−0.498131	−0.498112	−0.498112
(0.2, 0.5)	−0.498178	−0.498158	−0.498139	−0.498139
(0.3, 0.1)	−0.495491	−0.49547	−0.495458	−0.495458
(0.3, 0.3)	−0.495525	−0.495502	−0.495484	−0.495484
(0.3, 0.5)	−0.495548	−0.495529	−0.49551	−0.49551
(0.4, 0.1)	−0.49293	−0.492909	−0.492897	−0.492897
(0.4, 0.3)	−0.492963	−0.49294	−0.492922	−0.492922
(0.4, 0.5)	−0.492985	−0.492966	−0.492948	−0.492948
(0.5, 0.1)	−0.490433	−0.490413	−0.490401	−0.490401
(0.5, 0.3)	−0.490465	−0.490443	−0.490426	−0.490426
(0.5, 0.5)	−0.490487	−0.490469	−0.490451	−0.490451

Table 2: HPTM for the approximate solution of Example 1.

$(\varphi, \mathfrak{F})$	$\nu(\varphi, \mathfrak{F})$ at $\beta = 0.5$	$\nu(\varphi, \mathfrak{F})$ at $\beta = 0.75$	$\nu(\varphi, \mathfrak{F})$ at $\beta = 1$	Exact solution
(0.1, 0.1)	−0.0939215	−0.0939015	−0.09389	−0.09389
(0.1, 0.3)	−0.0939536	−0.0939319	−0.0939146	−0.0939146
(0.1, 0.5)	−0.0939757	−0.0939571	−0.0939391	−0.0939391
(0.2, 0.1)	−0.0915064	−0.091487	−0.0914759	−0.0914759
(0.2, 0.3)	−0.0915375	−0.0915165	−0.0914997	−0.0914997
(0.2, 0.5)	−0.0915589	−0.0915409	−0.0915235	−0.0915235
(0.3, 0.1)	−0.0891657	−0.0891469	−0.0891361	−0.0891361
(0.3, 0.3)	−0.0891958	−0.0891754	−0.0891592	−0.0891592
(0.3, 0.5)	−0.0892166	−0.0891992	−0.0891822	−0.0891822
(0.4, 0.1)	−0.0868965	−0.0868782	−0.0868678	−0.0868678
(0.4, 0.3)	−0.0869257	−0.0869059	−0.0868901	−0.08688901
(0.4, 0.5)	−0.0869458	−0.0869289	−0.0869125	−0.0869125
(0.5, 0.1)	−0.0846961	−0.0846784	−0.0846683	−0.0846683
(0.5, 0.3)	−0.0847244	−0.0847052	−0.0846899	−0.0846899
(0.5, 0.5)	−0.0847439	−0.0847275	−0.0847116	−0.0847116

$$\begin{aligned}
\mathscr{L}[\mu(\varphi, \mathfrak{F})] &= \frac{\mu(\varphi, 0)}{s} - \frac{s + \beta(1-s)}{s}\mathscr{L}\left[\mu(\varphi, \mathfrak{F})\frac{\partial \mu(\varphi, \mathfrak{F})}{\partial \varphi} + \frac{1}{2}\frac{\partial \mu(\varphi, \mathfrak{F})}{\partial \varphi} + \frac{\partial \nu(\varphi, \mathfrak{F})}{\partial \varphi}\right], \\
\mathscr{L}[\nu(\varphi, \mathfrak{F})] &= \frac{\nu(\varphi, 0)}{s} - \frac{s + \beta(1-s)}{s}\mathscr{L}\left[\mu(\varphi, \mathfrak{F})\frac{\partial \nu(\varphi, \mathfrak{F})}{\partial \varphi} + \nu(\varphi, \mathfrak{F})\frac{\partial \mu(\varphi, \mathfrak{F})}{\partial \varphi} - \frac{1}{2}\frac{\partial^2 \nu(\varphi, \mathfrak{F})}{\partial \varphi^2}\right].
\end{aligned} \tag{29}$$

The inverse Laplace transformation is implemented to (29), and we obtain

$$\begin{aligned}
\mu(\varphi, \mathfrak{F}) &= \mu(\varphi, 0) - \mathscr{L}^{-1}\left[\frac{s + \beta(1-s)}{s}\mathscr{L}\left\{\mu(\varphi, \mathfrak{F})\frac{\partial \mu(\varphi, \mathfrak{F})}{\partial \varphi} + \frac{1}{2}\frac{\partial \mu(\varphi, \mathfrak{F})}{\partial \varphi} + \frac{\partial \nu(\varphi, \mathfrak{F})}{\partial \varphi}\right\}\right], \\
\nu(\varphi, \mathfrak{F}) &= \nu(\varphi, 0) - \mathscr{L}^{-1}\left[\frac{s + \beta(1-s)}{s}\mathscr{L}\left\{\mu(\varphi, \mathfrak{F})\frac{\partial \nu(\varphi, \mathfrak{F})}{\partial \varphi} + \nu(\varphi, \mathfrak{F})\frac{\partial \mu(\varphi, \mathfrak{F})}{\partial \varphi} - \frac{1}{2}\frac{\partial^2 \nu(\varphi, \mathfrak{F})}{\partial \varphi^2}\right\}\right].
\end{aligned} \tag{30}$$

The LHPTM is used in (30), and we obtain

Table 3: HPTM for the approximate solution of Example 1.

$(\varphi, \mathfrak{F})$	Absolute error of ADM	Absolute error of VIM	Absolute error of OHAM	Absolute error HPTM
(0.1, 0.1)	1.04892×10^{-4}	1.23033×10^{-4}	1.07078×10^{-4}	1.67111×10^{-12}
(0.1, 0.3)	9.64474×10^{-5}	3.69597×10^{-4}	3.04565×10^{-4}	4.51196×10^{-11}
(0.1, 0.5)	8.88312×10^{-5}	6.16873×10^{-4}	4.81303×10^{-4}	2.08888×10^{-10}
(0.2, 0.1)	4.25408×10^{-4}	1.19869×10^{-4}	1.04388×10^{-4}	1.57879×10^{-12}
(0.2, 0.3)	3.91098×10^{-4}	3.60098×10^{-4}	2.97260×10^{-4}	4.26227×10^{-11}
(0.2, 0.5)	3.60161×10^{-4}	6.01006×10^{-4}	4.70138×10^{-4}	1.97328×10^{-10}
(0.3, 0.1)	9.71922×10^{-4}	1.16789×10^{-4}	1.01776×10^{-4}	1.49181×10^{-12}
(0.3, 0.3)	8.93309×10^{-4}	3.50866×10^{-4}	2.90150×10^{-4}	4.02799×10^{-11}
(0.3, 0.5)	8.22452×10^{-4}	5.85610×10^{-4}	4.59590×10^{-4}	1.86481×10^{-10}
(0.4, 0.1)	1.75596×10^{-3}	1.13829×10^{-4}	9.92418×10^{-5}	1.41043×10^{-12}
(0.4, 0.3)	1.61430×10^{-3}	3.41948×10^{-4}	2.83229×10^{-4}	3.80803×10^{-11}
(0.4, 0.5)	1.48578×10^{-3}	5.70710×10^{-4}	4.49118×10^{-4}	1.76298×10^{-10}
(0.5, 0.1)	2.79519×10^{-3}	1.10936×10^{-4}	9.67808×10^{-4}	1.33388×10^{-12}
(0.5, 0.3)	2.56714×10^{-3}	3.33274×10^{-4}	2.76492×10^{-4}	3.60145×10^{-11}
(0.5, 0.5)	2.36184×10^{-3}	5.56235×10^{-4}	4.38895×10^{-4}	1.66734×10^{-10}

Table 4: Comparison of different methods for the approximate solution of Example 1.

$(\varphi, \mathfrak{F})$	Absolute error of ADM	Absolute error of VIM	Absolute error of OHAM	Absolute error HPTM
(0.1, 0.1)	6.41419×10^{-3}	1.10430×10^{-4}	5.86860×10^{-5}	3.28081×10^{-12}
(0.1, 0.3)	5.99783×10^{-3}	3.31865×10^{-4}	3.04565×10^{-4}	8.85812×10^{-11}
(0.1, 0.5)	5.61507×10^{-3}	5.54071×10^{-4}	3.08812×10^{-4}	4.10099×10^{-10}
(0.2, 0.1)	1.33181×10^{-2}	1.07016×10^{-4}	5.56884×10^{-5}	3.07768×10^{-12}
(0.2, 0.3)	1.24441×10^{-2}	3.21601×10^{-4}	2.97260×10^{-4}	8.30963×10^{-11}
(0.2, 0.5)	1.16416×10^{-2}	5.36927×10^{-4}	2.92626×10^{-4}	3.84706×10^{-10}
(0.3, 0.1)	2.07641×10^{-2}	1.03737×10^{-4}	5.28609×10^{-5}	2.88849×10^{-12}
(0.3, 0.3)	1.93852×10^{-2}	3.11737×10^{-4}	2.90150×10^{-4}	7.79908×10^{-11}
(0.3, 0.5)	1.81209×10^{-2}	5.20447×10^{-4}	2.77382×10^{-4}	3.6107×10^{-10}
(0.4, 0.1)	2.88100×10^{-2}	1.00579×10^{-4}	5.01929×10^{-5}	2.71246×10^{-12}
(0.4, 0.3)	2.68724×10^{-2}	3.02245×10^{-4}	2.83229×10^{-4}	7.32356×10^{-11}
(0.4, 0.5)	2.50985×10^{-2}	5.04593×10^{-4}	2.63019×10^{-4}	3.39055×10^{-10}
(0.5, 0.1)	3.75193×10^{-2}	9.75385×10^{-5}	4.76741×10^{-5}	2.54828×10^{-12}
(0.5, 0.3)	3.49617×10^{-2}	2.93107×10^{-4}	2.76492×10^{-4}	6.88039×10^{-11}
(0.5, 0.5)	3.26239×10^{-2}	4.89335×10^{-4}	2.49480×10^{-4}	3.18537×10^{-10}

$$\sum_{n=0}^{\infty} \mu_n(\varphi, \mathfrak{F}) = \mu(\varphi, 0) - p\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{\sum_{n=0}^{\infty} p^n A_n + \frac{1}{2}\frac{\partial}{\partial\varphi}\sum_{n=0}^{\infty} p^n \mu_n(\varphi, \mathfrak{F}) + \frac{\partial}{\partial\varphi}\sum_{n=0}^{\infty} p\nu_n(\varphi, \mathfrak{F})\right\}\right],$$
$$\sum_{n=0}^{\infty} \nu_n(\varphi, \mathfrak{F}) = \nu(\varphi, 0) - p\mathscr{L}^{-1}\left[\frac{s+\beta(1-s)}{s}\mathscr{L}\left\{\sum_{n=0}^{\infty} p^n B_m + \sum_{n=0}^{\infty} p^n C_n - \frac{1}{2}\frac{\partial^2}{\partial\varphi^2}\sum_{n=0}^{\infty} p^n \nu_n(\varphi, \mathfrak{F})\right\}\right]. \tag{31}$$

The nonlinear can be found with the help of He's polynomial and can be defined as

Comparing the coefficient of p, we have

$$\begin{aligned}
A_0 &= \mu_0 \frac{\partial \mu_0}{\partial \varphi}, \\
A_1 &= \mu_0 \frac{\partial \mu_1}{\partial \varphi} + \mu_1 \frac{\partial \mu_0}{\partial \varphi}, \\
B_0 &= \mu_0 \frac{\partial \nu_0}{\partial \beta}, \\
B_1 &= \mu_0 \frac{\partial \nu_1}{\partial \beta} + \mu_1 \frac{\partial \nu_0}{\partial \beta}, \\
C_0 &= \nu_0 \frac{\partial \mu_0}{\partial \varphi}, \\
C_1 &= \nu_0 \frac{\partial \mu_1}{\partial \varphi} + \nu_1 \frac{\partial \mu_0}{\partial \varphi}.
\end{aligned} \tag{32}$$

$$\begin{aligned}
p^0\colon \mu_0(\varphi, \mathfrak{F}) &= \mu(\varphi, 0) = \xi - \kappa \coth[\kappa(\varphi + \theta)], \\
p^0\colon \nu_0(\varphi, \mathfrak{F}) &= \nu(\varphi, 0) = -\kappa^2 \operatorname{cosech}^2[\kappa(\varphi + \theta)], \\
p^1\colon \mu_1(\varphi, \mathfrak{F}) &= -\mathscr{L}^{-1}\left[\frac{s + \beta(1-s)}{s}\mathscr{L}\left\{A_0 + \frac{1}{2}\frac{\partial}{\partial \varphi}\mu_0(\varphi, \mathfrak{F}) + \frac{\partial}{\partial \varphi}\nu_0(\varphi, \mathfrak{F})\right\}\right] \\
&= -\xi\kappa^2 \operatorname{cosech}^2[\kappa(\varphi + \theta)][\beta\mathfrak{F} + (1-\beta)], \\
p^1\colon \nu_1(\varphi, \mathfrak{F}) &= -p\mathscr{L}^{-1}\left[\frac{s + \beta(1-s)}{s}\mathscr{L}\left\{B_0 + C_0 - \frac{1}{2}\frac{\partial^2}{\partial \varphi^2}\nu_0(\varphi, \mathfrak{F})\right\}\right] \\
&= -\xi\kappa^2 \operatorname{cosech}^2[\kappa(\varphi + \theta)]\coth[\kappa(\varphi + \theta)][\beta\mathfrak{F} + (1-\beta)], \\
p^2\colon \mu_2(\varphi, \mathfrak{F}) &= -\mathscr{L}^{-1}\left[\frac{s + \beta(1-s)}{s}\mathscr{L}\left\{A_1 + \frac{1}{2}\frac{\partial}{\partial \varphi}\mu_1(\varphi, \mathfrak{F}) + \frac{\partial}{\partial \varphi}\nu_1(\varphi, \mathfrak{F})\right\}\right] \\
&= -\xi\kappa^4 \operatorname{cosech}^2[\kappa(\varphi + \theta)]\left(3\coth^2([\kappa(\varphi + \theta)] - 1)\right)\left[(1-\beta)^2 + 2\beta(1-\beta)\mathfrak{F} + \frac{\beta^2\mathfrak{F}^2}{2}\right] \\
&\quad + 2\xi^2\kappa^5 \operatorname{cosech}^2[\kappa(\varphi + \theta)]\left(3\beta\left(-2\beta + 1 + \beta^2\right)\mathfrak{F} + \frac{\beta^3\mathfrak{F}^3}{6} - \frac{3\beta^2(\beta-1)\mathfrak{F}^2}{2} + 3\beta^2 - 3\beta + 1 - \beta^3\right), \\
p^2\colon \nu_2(\varphi, \mathfrak{F}) &= -p\mathscr{L}^{-1}\left[\frac{s + \beta(1-s)}{s}\mathscr{L}\left\{B_1 + C_1 - \frac{1}{2}\frac{\partial^2}{\partial \varphi^2}\nu_1(\varphi, \mathfrak{F})\right\}\right] \\
&= 2\xi\kappa^5 \operatorname{cosech}^2[\kappa(\varphi + \theta)]\left\{\xi\kappa \operatorname{cosech}^2\left(3\coth^2([\kappa(\varphi + \theta)] - 1)\right) + 2\xi\kappa \operatorname{cosech}^2\coth^2([\kappa(\varphi + \theta)])\right\} \\
&\quad \times\left(3\beta\left(-2\beta + 1 + \beta^2\right)\mathfrak{F} + \frac{\beta^3\mathfrak{F}^3}{6} - \frac{3\beta^2(\beta-1)\mathfrak{F}^2}{2} + 3\beta^2 - 3\beta + 1 - \beta^3\right) \\
&\quad - 2\xi\coth\left(3\operatorname{cosech}^2([\kappa(\varphi + \theta)] - 1)\right)\left[(1-\beta)^2 + 2\beta(1-\beta)\mathfrak{F} + \frac{\beta^2\mathfrak{F}^2}{2}\right], \\
&\quad \vdots
\end{aligned} \tag{33}$$

We can calculate few terms of (26) which can be written as

$$\mu(\varphi,\mathfrak{T}) = \mu_0(\varphi,\mathfrak{T}) + \mu_1(\varphi,\mathfrak{T}) + \mu_2(\varphi,\mathfrak{T}) + \cdots,$$

$$\nu(\varphi,\mathfrak{T}) = \nu_0(\varphi,\mathfrak{T}) + \nu_1(\varphi,\mathfrak{T}) + \nu_2(\varphi,\mathfrak{T}) + \cdots,$$

$$\begin{aligned}\mu(\varphi,\mathfrak{T}) &= \xi - \kappa\coth[\kappa(\varphi+\theta)] - \xi\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)][\beta\mathfrak{T}+(1-\beta)]\\ &\quad - \xi\kappa^4\operatorname{cosech}^2[\kappa(\varphi+\theta)]\left(3\coth^2([\kappa(\varphi+\theta)]-1)\right)\left[(1-\beta)^2 + 2\beta(1-\beta)\mathfrak{T} + \frac{\beta^2\mathfrak{T}^2}{2}\right]\\ &\quad + 2\xi^2\kappa^5\operatorname{cosech}^2[\kappa(\varphi+\theta)]\left(3\beta\left(-2\beta+1+\beta^2\right)\mathfrak{T} + \frac{\beta^3\mathfrak{T}^3}{6} - \frac{3\beta^2(\beta-1)\mathfrak{T}^2}{2} + 3\beta^2 - 3\beta + 1 - \beta^3\right) + \cdots,\end{aligned} \tag{34}$$

$$\begin{aligned}\nu(\varphi,\mathfrak{T}) &= -\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)] - \xi\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)]\coth[\kappa(\varphi+\theta)][\beta\mathfrak{T}+(1-\beta)]\\ &\quad + 2\xi\kappa^5\cos ech^2[\kappa(\varphi+\theta)]\left[\left\{\xi\kappa\operatorname{cosech}^2\left(3\coth^2([\kappa(\varphi+\theta)]-1)\right) + 2\xi\kappa\operatorname{cosech}^2\coth^2([\kappa(\varphi+\theta)])\right\}\right.\\ &\quad \times\left(3\beta\left(-2\beta+1+\beta^2\right)\mathfrak{T} + \frac{\beta^3\mathfrak{T}^3}{6} - \frac{3\beta^2(\beta-1)\mathfrak{T}^2}{2} + 3\beta^2 - 3\beta + 1 - \beta^3\right)\\ &\quad \left. -2\xi\coth\left(3\operatorname{cosech}^2([\kappa(\varphi+\theta)]-1)\right)\left[(1-\beta)^2 + 2\beta(1-\beta)\mathfrak{T} + \frac{\beta^2\mathfrak{T}^2}{2}\right]\right] + \cdots.\end{aligned}$$

The exact solution of (26) is

$$\begin{aligned}\mu(\varphi,\mathfrak{T}) &= \xi - \kappa\coth[\kappa(\varphi+\theta-\xi\mathfrak{T})],\\ \nu(\varphi,\mathfrak{T}) &= -\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta-\xi\mathfrak{T})].\end{aligned} \tag{35}$$

5. Implementation of Atangana–Baleanu Operator

Example 3. Let us consider the coupled system of fractional-order WBKEs in the ABC sense:

$$\begin{aligned}&{}^{AB}D_{\mathfrak{T}}^{\beta}\mu(\varphi,\mathfrak{T}) + \mu(\varphi,\mathfrak{T})\frac{\partial\mu(\varphi,\mathfrak{T})}{\partial\varphi} + \frac{\partial\mu(\varphi,\mathfrak{T})}{\partial\varphi} + \frac{\partial\nu(\varphi,\mathfrak{T})}{\partial\varphi} = 0,\\ &{}^{AB}D_{\mathfrak{T}}^{\beta}\nu(\varphi,\mathfrak{T}) + \mu(\varphi,\mathfrak{T})\frac{\partial\nu(\varphi,\mathfrak{T})}{\partial\varphi} + \nu(\varphi,\mathfrak{T})\frac{\partial\mu(\varphi,\mathfrak{T})}{\partial\varphi} + 3\frac{\partial^3\mu(\varphi,\mathfrak{T})}{\partial\varphi^3} - \frac{\partial^2\nu(\varphi,\mathfrak{T})}{\partial\varphi^2} = 0, \quad 0<\beta\le 1, -1<\mathfrak{T}\le 1, -10\le\varphi\le 10,\end{aligned} \tag{36}$$

under the initial conditions,

$$\mu(\varphi,0)=\frac{1}{2}-8\ \tanh(-2\varphi),$$
$$\nu(\varphi,0)=16-16\tanh^{2}(-2\varphi). \tag{37}$$

Using the Laplace transformation to (36), we obtain

$$\frac{B(\beta)}{1-\beta}\frac{s\mathscr{L}[\mu(\varphi,\mathfrak{F})](s)-s^{\beta-1}\mu(\varphi,0)}{s+(\beta/(1-\beta))}=-\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right],$$
$$\frac{B(\beta)}{1-\beta}\frac{s\mathscr{L}[\nu(\varphi,\mathfrak{F})](s)-s^{\beta-1}\nu(\varphi,0)}{s+(\beta/(1-\beta))}=-\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}+\nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+3\frac{\partial^{3}\mu(\varphi,\mathfrak{F})}{\partial\varphi^{3}}-\frac{\partial^{2}\nu(\varphi,\mathfrak{F})}{\partial\varphi^{2}}\right]. \tag{38}$$

Simplify the above equation and use the initial conditions (37), and we obtain

$$\mathscr{L}[\mu(\varphi,\mathfrak{F})]=\frac{\mu(\varphi,0)}{s}-\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right],$$
$$\mathscr{L}[\nu(\varphi,\mathfrak{F})]=\frac{\nu(\varphi,0)}{s}-\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}+\nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+3\frac{\partial^{3}\mu(\varphi,\mathfrak{F})}{\partial\varphi^{3}}-\frac{\partial^{2}\nu(\varphi,\mathfrak{F})}{\partial\varphi^{2}}\right]. \tag{39}$$

The inverse Laplace transformation implement to (39), we obtain

$$\mu(\varphi,\mathfrak{F})=\mu(\varphi,0)-\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right\}\right],$$
$$\nu(\varphi,\mathfrak{F})=\nu(\varphi,0)-\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}+\nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi}+3\frac{\partial^{3}\mu(\varphi,\mathfrak{F})}{\partial\varphi^{3}}-\frac{\partial^{2}\nu(\varphi,\mathfrak{F})}{\partial\varphi^{2}}\right\}\right]. \tag{40}$$

The LHPTM is used in (40), and we obtain

$$\sum_{n=0}^{\infty}\mu_{n}(\varphi,\mathfrak{F})=\mu(\varphi,0)-p\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{\sum_{n=0}^{\infty}p^{n}A_{n}+\frac{\partial}{\partial\varphi}\sum_{n=0}^{\infty}p^{n}\mu_{n}(\varphi,\mathfrak{F})+\frac{\partial}{\partial\varphi}\sum_{n=0}^{\infty}p^{n}\nu_{n}(\varphi,\mathfrak{F})\right\}\right],$$
$$\sum_{n=0}^{\infty}\nu_{n}(\varphi,\mathfrak{F})=\nu(\varphi,0)-p\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{\sum_{n=0}^{\infty}p^{n}B_{m}+\sum_{n=0}^{\infty}p^{n}C_{n}+3\frac{\partial^{3}}{\partial\varphi^{3}}\sum_{n=0}^{\infty}p^{n}\mu_{n}(\varphi,\mathfrak{F})-\frac{\partial^{2}}{\partial\varphi^{2}}\sum_{n=0}^{\infty}p^{n}\nu_{n}(\varphi,\mathfrak{F})\right\}\right]. \tag{41}$$

The nonlinear can be found with the help of He's polynomial and can be defined as

Comparing the coefficient of p, we have

$$\begin{aligned}
A_0 &= \mu_0 \frac{\partial \mu_0}{\partial \varphi}, \\
A_1 &= \mu_0 \frac{\partial \mu_1}{\partial \varphi} + \mu_1 \frac{\partial \mu_0}{\partial \varphi}, \\
B_0 &= \mu_0 \frac{\partial \nu_0}{\partial \beta}, \\
B_1 &= \mu_0 \frac{\partial \nu_1}{\partial \beta} + \mu_1 \frac{\partial \nu_0}{\partial \beta}, \\
C_0 &= \nu_0 \frac{\partial \mu_0}{\partial \varphi}, \\
C_1 &= \nu_0 \frac{\partial \mu_1}{\partial \varphi} + \nu_1 \frac{\partial \mu_0}{\partial \varphi}.
\end{aligned} \tag{42}$$

$$\begin{aligned}
p^0\colon \mu_0(\varphi, \mathfrak{F}) &= \mu(\varphi, 0) = \frac{1}{2} - 8 \tanh(-2\varphi), \\
p^0\colon \nu_0(\varphi, \mathfrak{F}) &= \nu(\varphi, 0) = 16 - 16\tanh^2(-2\varphi), \\
p^1\colon \mu_1(\varphi, \mathfrak{F}) &= -\mathscr{L}^{-1}\left[\frac{(1-\beta)s^\beta + \beta}{B(\beta)s^\beta}\mathscr{L}\left\{A_0 + \frac{\partial}{\partial \varphi}\mu_0(\varphi, \mathfrak{F}) + \frac{\partial}{\partial \varphi}\nu_0(\varphi, \mathfrak{F})\right\}\right] \\
&= -8 \sec h^2(-2\varphi)\frac{1}{B(\beta)}\left[\frac{\beta \mathfrak{F}^\beta}{\Gamma(\beta+1)} + (1-\beta)\right], \\
p^1\colon \nu_1(\varphi, \mathfrak{F}) &= -p\mathscr{L}^{-1}\left[\frac{(1-\beta)s^\beta + \beta}{B(\beta)s^\beta}\mathscr{L}\left\{B_0 + C_0 + 3\frac{\partial^3}{\partial \varphi^3}\mu_0(\varphi, \mathfrak{F}) - \frac{\partial^2}{\partial \varphi^2}\nu_0(\varphi, \mathfrak{F})\right\}\right] \\
&= -32 \sec h^2(-2\varphi)\tanh(-2\varphi)\frac{1}{B(\beta)}\left[\frac{\beta \mathfrak{F}^\beta}{\Gamma(\beta+1)} + (1-\beta)\right], \\
p^2\colon \mu_2(\varphi, \mathfrak{F}) &= -\mathscr{L}^{-1}\left[\frac{(1-\beta)s^\beta + \beta}{B(\beta)s^\beta}\mathscr{L}\left\{A_1 + \frac{\partial}{\partial \varphi}\mu_1(\varphi, \mathfrak{F}) + \frac{\partial}{\partial \varphi}\nu_1(\varphi, \mathfrak{F})\right\}\right] \\
&= -16 \sec h^2(-2\varphi)\left(4 \sec h^2(-2\varphi) - 8 \tanh^2(-2\varphi) + 3 \tanh(-2\varphi)\right) \\
&\quad \times \frac{1}{B^2(\beta)}\left[(1-\beta)^2 + \frac{2\beta(1-\beta)\mathfrak{F}^\beta}{\Gamma(\beta+1)} + \frac{\beta^2 \mathfrak{F}^{2\beta}}{\Gamma(2\beta+1)}\right], \\
p^2\colon \nu_2(\varphi, \mathfrak{F}) &= -p\mathscr{L}^{-1}\left[\frac{(1-\beta)s^\beta + \beta}{B(\beta)s^\beta}\mathscr{L}\left\{B_1 + C_1 + 3\frac{\partial^3}{\partial \varphi^3}\mu_1(\varphi, \mathfrak{F}) - \frac{\partial^2}{\partial \varphi^2}\nu_1(\varphi, \mathfrak{F})\right\}\right] \\
&= -32 \sec h^2(-2\varphi)\{40 \sec h^2(-2\varphi)\tanh(-2\varphi) + 96 \tanh(-2\varphi) - 2 \tanh^2(-2\varphi) \\
&\quad -32 \tanh^3(-2\varphi) - 25 \sec h^2(-2\varphi)\}\left[(1-\beta)^2 + \frac{2\beta(1-\beta)\mathfrak{F}^\beta}{\Gamma(\beta+1)} + \frac{\beta^2 \mathfrak{F}^{2\beta}}{\Gamma(2\beta+1)}\right], \\
&\vdots
\end{aligned} \tag{43}$$

We can calculate few terms of (36) which can be written as

$$\begin{aligned}
\mu(\varphi,\mathfrak{F}) &= \mu_0(\varphi,\mathfrak{F}) + \mu_1(\varphi,\mathfrak{F}) + \mu_2(\varphi,\mathfrak{F}) + \cdots,\\
\nu(\varphi,\mathfrak{F}) &= \nu_0(\varphi,\mathfrak{F}) + \nu_1(\varphi,\mathfrak{F}) + \nu_2(\varphi,\mathfrak{F}) + \cdots,\\
\mu(\varphi,\mathfrak{F}) &= \frac{1}{2} - 8\,\tanh(-2\varphi) - 8\,\sec h^2(-2\varphi)\left[\frac{\beta\mathfrak{F}^\beta}{\Gamma(\beta+1)} + (1-\beta)\right] - 16\,\sec h^2(-2\varphi)\\
&\quad \left(4\,\sec h^2(-2\varphi) - 8\,\tanh^2(-2\varphi) + 3\,\tanh(-2\varphi)\right)\frac{1}{B^2(\beta)}\left[(1-\beta)^2 + \frac{2\beta(1-\beta)\mathfrak{F}^\beta}{\Gamma(\beta+1)} + \frac{\beta^2\mathfrak{F}^{2\beta}}{\Gamma(2\beta+1)}\right] + \cdots,\\
\nu(\varphi,\mathfrak{F}) &= 16 - 16\tanh^2(-2\varphi) - 8\,\sec h^2(-2\varphi)\left[\frac{\beta\mathfrak{F}^\beta}{\Gamma(\beta+1)} + (1-\beta)\right] - 32\,\sec h^2(-2\varphi)\\
&\quad \times\left\{40\,\sec h^2(-2\varphi)\tanh(-2\varphi) + 96\,\tanh(-2\varphi) - 2\,\tanh^2(-2\varphi) - 32\,\tanh^3(-2\varphi) - 25\,\sec h^2(-2\varphi)\right\}\\
&\quad \times\left[(1-\beta)^2 + \frac{2\beta(1-\beta)\mathfrak{F}^\beta}{\Gamma(\beta+1)} + \frac{\beta^2\mathfrak{F}^{2\beta}}{\Gamma(2\beta+1)}\right] + \cdots.
\end{aligned} \tag{44}$$

The exact solution of (36) is

$$\begin{aligned}
\mu(\varphi,\mathfrak{F}) &= \frac{1}{2} - 8\,\tanh\left\{-2\left(\varphi - \frac{\mathfrak{F}}{2}\right)\right\},\\
\nu(\varphi,\mathfrak{F}) &= 16 - 16\tanh^2\left\{-2\left(\varphi - \frac{\mathfrak{F}}{2}\right)\right\}.
\end{aligned} \tag{45}$$

Figure 5 shows the actual and approximate solutions of $\mu(\varphi,\mathfrak{F})$ at $\beta = 1$, and Figure 6 shows the actual and approximate solutions of $\nu(\varphi,\mathfrak{F})$ at $\beta = 1$. Figures 7 and 8 show that the first graph has a different fractional order with respect to φ and the second graph has a different fractional order with respect to $\mathfrak{F}$ of Example 3.

Example 4. Let us consider the coupled system of fractional-order WBKEs in the ABC sense:

$$\begin{aligned}
{}^{ABC}D^\beta_{\mathfrak{F}}\mu(\varphi,\mathfrak{F}) + \mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{1}{2}\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi} &= 0,\\
{}^{ABC}D^\beta_{\mathfrak{F}}\nu(\varphi,\mathfrak{F}) + \mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi} + \nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} - \frac{1}{2}\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2} &= 0, \quad 0<\beta\le 1, -1<\mathfrak{F}\le 1, -10\le\varphi\le 10,
\end{aligned} \tag{46}$$

under the initial conditions,

$$\begin{aligned}
\mu(\varphi,0) &= \xi - \kappa\,\coth[\kappa(\varphi+\theta)],\\
\nu(\varphi,0) &= -\kappa^2\mathrm{cosech}^2[\kappa(\varphi+\theta)].
\end{aligned} \tag{47}$$

Using the Laplace transformation to (46), we obtain

$$\begin{aligned}
\frac{B(\beta)}{1-\beta}\frac{s\mathscr{L}[\mu(\varphi,\mathfrak{F})](s) - s^{\beta-1}\mu(\varphi,0)}{s + (\beta/(1-\beta))} &= -\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{1}{2}\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right],\\
\frac{B(\beta)}{1-\beta}\frac{s\mathscr{L}[\nu(\varphi,\mathfrak{F})](s) - s^{\beta-1}\nu(\varphi,0)}{s + (\beta/(1-\beta))} &= -\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi} + \nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} - \frac{1}{2}\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2}\right].
\end{aligned} \tag{48}$$

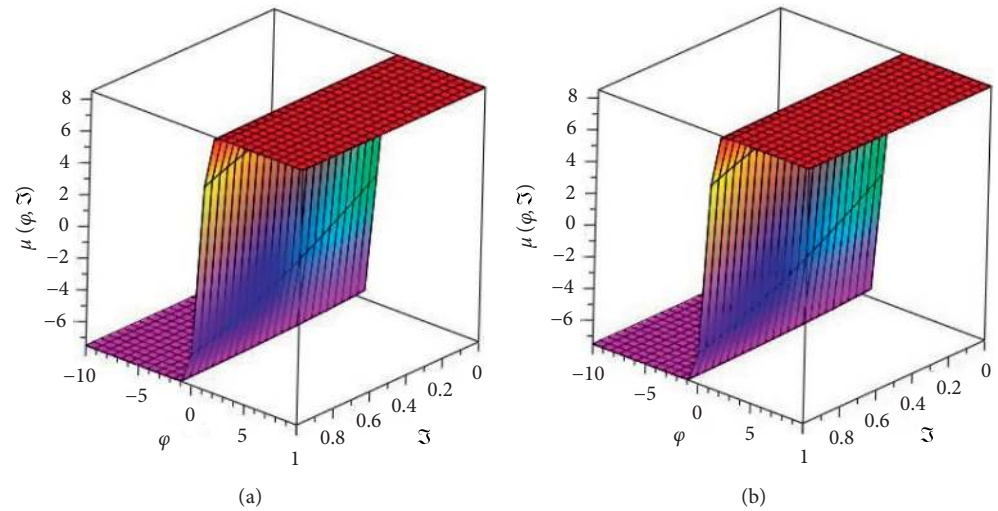

Figure 5: The exact and analytical solution with respect to the Atangana–Baleanu operator of $\mu(\varphi, \mathfrak{F})$ at $\beta = 1$ of Example 3.

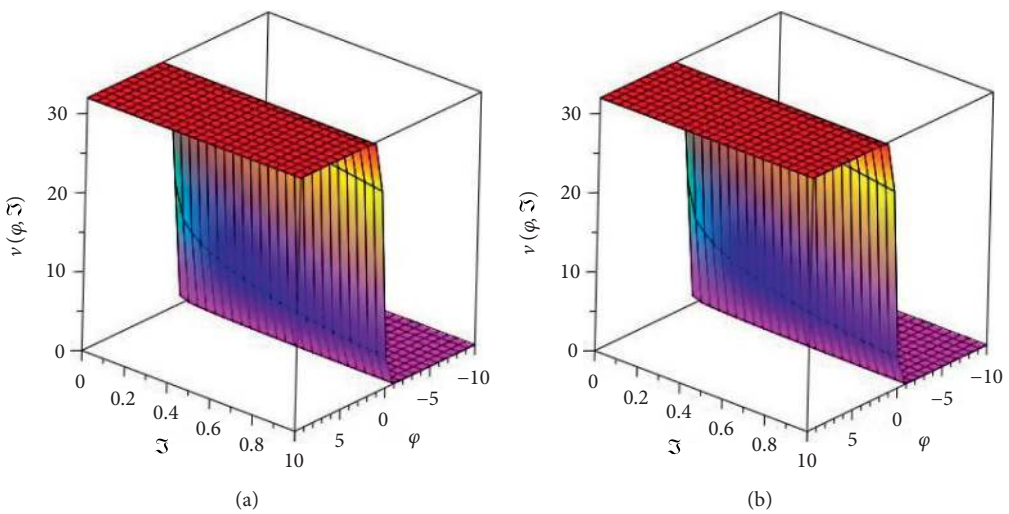

Figure 6: The exact and analytical solution with respect to the Atangana–Baleanu operator of $\nu(\varphi, \mathfrak{F})$ at $\beta = 1$ of Example 3.

Simplifying the above equation and using the initial conditions (47), we obtain

$$\begin{aligned}\mathscr{L}[\mu(\varphi,\mathfrak{F})] &= \frac{\mu(\varphi,0)}{s} - \frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{1}{2}\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right],\\ \mathscr{L}[\nu(\varphi,\mathfrak{F})] &= \frac{\nu(\varphi,0)}{s} - \frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left[\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi} + \nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} - \frac{1}{2}\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2}\right].\end{aligned} \tag{49}$$

The inverse Laplace transformation is implemented to (49), and we obtain

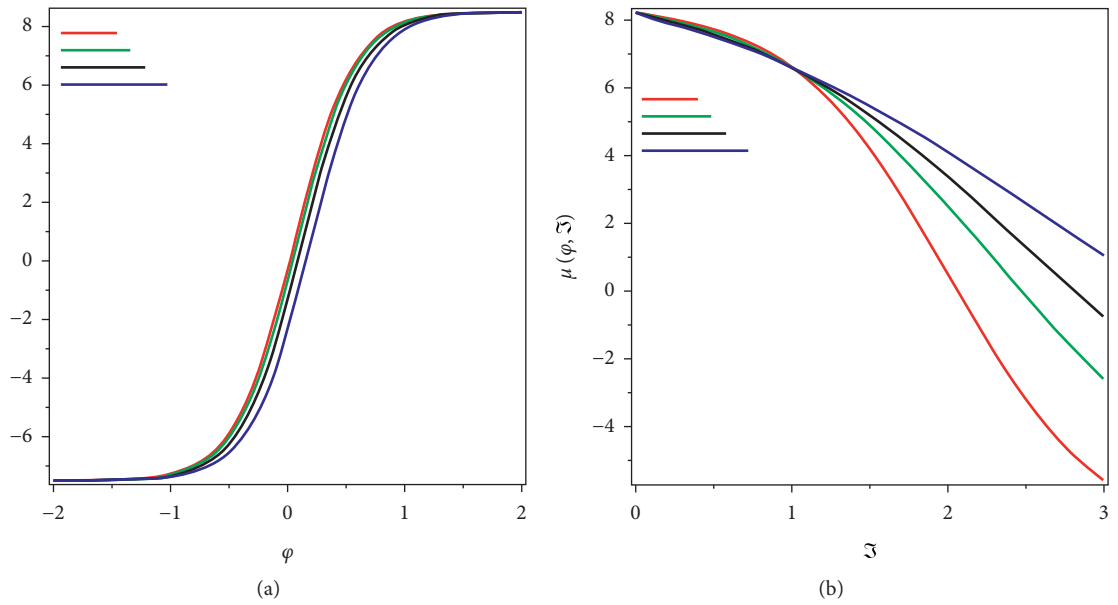

FIGURE 7: The different fractional-order solution with respect to the Atangana–Baleanu operator of $\mu(\varphi, \mathfrak{F})$ at β of Example 3.

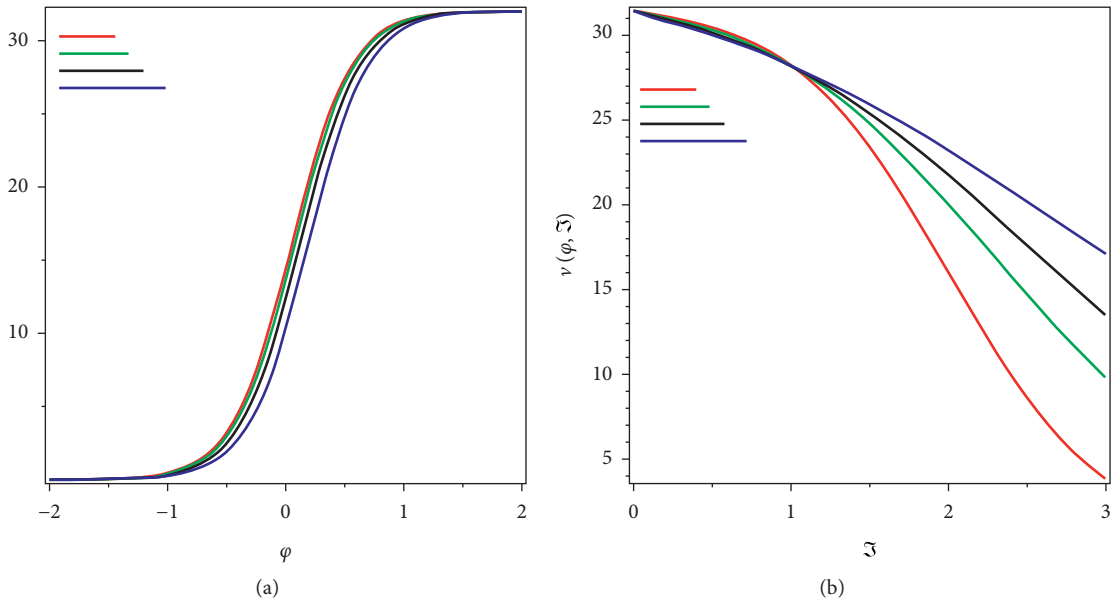

FIGURE 8: The different fractional-order solution with respect to the Atangana–Baleanu operator of $\nu(\varphi, \mathfrak{F})$ at β of Example 3.

$$\mu(\varphi,\mathfrak{F}) = \mu(\varphi,0) - \mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{\mu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{1}{2}\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} + \frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi}\right\}\right],$$
$$\nu(\varphi,\mathfrak{F}) = \nu(\varphi,0) - \mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{\mu(\varphi,\mathfrak{F})\frac{\partial\nu(\varphi,\mathfrak{F})}{\partial\varphi} + \nu(\varphi,\mathfrak{F})\frac{\partial\mu(\varphi,\mathfrak{F})}{\partial\varphi} - \frac{1}{2}\frac{\partial^2\nu(\varphi,\mathfrak{F})}{\partial\varphi^2}\right\}\right]. \tag{50}$$

The LHPTM is used in (50), and we obtain

$$\sum_{n=0}^{\infty}\mu_n(\varphi,\mathfrak{F})=\mu(\varphi,0)-p\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{\sum_{n=0}^{\infty}p^nA_n+\frac{1}{2}\frac{\partial}{\partial\varphi}\sum_{n=0}^{\infty}p^n\mu_n(\varphi,\mathfrak{F})+\frac{\partial}{\partial\varphi}\sum_{n=0}^{\infty}p^n\nu_n(\varphi,\mathfrak{F})\right\}\right],$$
$$\sum_{n=0}^{\infty}\nu_n(\varphi,\mathfrak{F})=\nu(\varphi,0)-p\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{\sum_{n=0}^{\infty}p^nB_n+\sum_{n=0}^{\infty}p^nC_n-\frac{1}{2}\frac{\partial^2}{\partial\varphi^2}\sum_{n=0}^{\infty}p^n\nu_n(\varphi,\mathfrak{F})\right\}\right]. \tag{51}$$

The nonlinear can be found with the help of He's polynomial and can be defined as

$$\begin{aligned}
A_0&=\mu_0\frac{\partial\mu_0}{\partial\varphi},\\
A_1&=\mu_0\frac{\partial\mu_1}{\partial\varphi}+\mu_1\frac{\partial\mu_0}{\partial\varphi},\\
B_0&=\mu_0\frac{\partial\nu_0}{\partial\beta},\\
B_1&=\mu_0\frac{\partial\nu_1}{\partial\beta}+\mu_1\frac{\partial\nu_0}{\partial\beta},\\
C_0&=\nu_0\frac{\partial\mu_0}{\partial\varphi},\\
C_1&=\nu_0\frac{\partial\mu_1}{\partial\varphi}+\nu_1\frac{\partial\mu_0}{\partial\varphi}.
\end{aligned} \tag{52}$$

Comparing the coefficient of p, we have

$$p^0\colon \mu_0(\varphi,\mathfrak{F})=\mu(\varphi,0)=\xi-\kappa\coth[\kappa(\varphi+\theta)],$$
$$p^0\colon \nu_0(\varphi,\mathfrak{F})=\nu(\varphi,0)=-\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)],$$
$$p^1\colon \mu_1(\varphi,\mathfrak{F})=-\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{A_0+\frac{1}{2}\frac{\partial}{\partial\varphi}\mu_0(\varphi,\mathfrak{F})+\frac{\partial}{\partial\varphi}\nu_0(\varphi,\mathfrak{F})\right\}\right]$$
$$=-\xi\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)]\frac{1}{B(\beta)}\left[\frac{\beta\mathfrak{F}^{\beta}}{\Gamma(\beta+1)}+(1-\beta)\right],$$
$$p^1\colon \nu_1(\varphi,\mathfrak{F})=-p\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{B_0+C_0-\frac{1}{2}\frac{\partial^2}{\partial\varphi^2}\nu_0(\varphi,\mathfrak{F})\right\}\right]$$
$$=-\xi\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)]\coth[\kappa(\varphi+\theta)]\frac{1}{B(\beta)}\left[\frac{\beta\mathfrak{F}^{\beta}}{\Gamma(\beta+1)}+(1-\beta)\right],$$
$$p^2\colon \mu_2(\varphi,\mathfrak{F})=-\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{A_1+\frac{1}{2}\frac{\partial}{\partial\varphi}\mu_1(\varphi,\mathfrak{F})+\frac{\partial}{\partial\varphi}\nu_1(\varphi,\mathfrak{F})\right\}\right]$$
$$=-\xi\kappa^4\operatorname{cosech}^2[\kappa(\varphi+\theta)]\left(3\coth^2([\kappa(\varphi+\theta)]-1)\right)\frac{1}{B^2}\left[(1-\beta)^2+\frac{2\beta(1-\beta)\mathfrak{F}^{\beta}}{\Gamma(\beta+1)}+\frac{\beta^2\mathfrak{F}^{2\beta}}{\Gamma(2\beta+1)}\right]$$
$$+2\xi^2\kappa^5\operatorname{cosech}^2[\kappa(\varphi+\theta)]\frac{1}{B^3}\left[(1-\beta)^3+\frac{3\beta(1-\beta)^2\mathfrak{F}^{\beta}}{\Gamma(2\beta+1)}+\frac{3\beta^2(1-\beta)\mathfrak{F}^{2\beta}}{\Gamma(2\beta+1)}+\frac{\beta^3\mathfrak{F}^{3\beta}}{\Gamma(3\beta+1)}\right],$$

$$
\begin{aligned}
p^2\colon \nu_2(\varphi,\mathfrak{T}) &= -p\mathscr{L}^{-1}\left[\frac{(1-\beta)s^{\beta}+\beta}{B(\beta)s^{\beta}}\mathscr{L}\left\{B_1+C_1-\frac{1}{2}\frac{\partial^2}{\partial\varphi^2}\nu_1(\varphi,\mathfrak{T})\right\}\right]\\
&= 2\xi\kappa^5\operatorname{cosech}^2[\kappa(\varphi+\theta)]\left\{\xi\kappa\operatorname{cosech}^2\left(3\coth^2([\kappa(\varphi+\theta)]-1)\right)+2\xi\kappa\operatorname{cosech}^2\coth^2([\kappa(\varphi+\theta)])\right\}\\
&\quad\times\frac{1}{B^3}\left[(1-\beta)^3+\frac{3\beta(1-\beta)^2\mathfrak{T}^{\beta}}{\Gamma(2\beta+1)}+\frac{3\beta^2(1-\beta)\mathfrak{T}^{2\beta}}{\Gamma(2\beta+1)}+\frac{\beta^3\mathfrak{T}^{3\beta}}{\Gamma(3\beta+1)}\right]\\
&\quad-2\xi\coth\left(3\operatorname{cosech}^2([\kappa(\varphi+\theta)]-1)\right)\frac{1}{B^2}\left[(1-\beta)^2+\frac{2\beta(1-\beta)\mathfrak{T}^{\beta}}{\Gamma(\beta+1)}+\frac{\beta^2\mathfrak{T}^{2\beta}}{\Gamma(2\beta+1)}\right],\\
&\quad\vdots
\end{aligned}
\tag{53}
$$

We can calculate few terms of (46) which can be written as

$$
\begin{aligned}
&\mu(\varphi,\mathfrak{T})=\mu_0(\varphi,\mathfrak{T})+\mu_1(\varphi,\mathfrak{T})+\mu_2(\varphi,\mathfrak{T})+\cdots,\\
&\nu(\varphi,\mathfrak{T})=\nu_0(\varphi,\mathfrak{T})+\nu_1(\varphi,\mathfrak{T})+\nu_2(\varphi,\mathfrak{T})+\cdots,\\
&\mu(\varphi,\mathfrak{T})=\xi-\kappa\coth[\kappa(\varphi+\theta)]-\xi\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)]\frac{1}{B(\beta)}\left[\frac{\beta\mathfrak{T}^{\beta}}{\Gamma(\beta+1)}+(1-\beta)\right]\\
&\qquad-\xi\kappa^4\operatorname{cosech}^2[\kappa(\varphi+\theta)]\left(3\coth^2([\kappa(\varphi+\theta)]-1)\right)\frac{1}{B^2}\left[(1-\beta)^2+\frac{2\beta(1-\beta)\mathfrak{T}^{\beta}}{\Gamma(\beta+1)}+\frac{\beta^2\mathfrak{T}^{2\beta}}{\Gamma(2\beta+1)}\right]\\
&\qquad+2\xi^2\kappa^5\operatorname{cosech}^2[\kappa(\varphi+\theta)]\frac{1}{B^3}\left[(1-\beta)^3+\frac{3\beta(1-\beta)^2\mathfrak{T}^{\beta}}{\Gamma(2\beta+1)}+\frac{3\beta^2(1-\beta)\mathfrak{T}^{2\beta}}{\Gamma(2\beta+1)}+\frac{\beta^3\mathfrak{T}^{3\beta}}{\Gamma(3\beta+1)}\right]+\cdots,\\
&\nu(\varphi,\mathfrak{T})=-\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)]-\xi\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta)]\coth[\kappa(\varphi+\theta)]\frac{1}{B(\beta)}\left[\frac{\beta\mathfrak{T}^{\beta}}{\Gamma(\beta+1)}+(1-\beta)\right]\\
&\qquad+2\xi\kappa^5\operatorname{cosech}^2[\kappa(\varphi+\theta)]\left\{\xi\kappa\operatorname{cosech}^2\left(3\coth^2([\kappa(\varphi+\theta)]-1)\right)+2\xi\kappa\operatorname{cosech}^2\coth^2([\kappa(\varphi+\theta)])\right\}\\
&\qquad\times\frac{1}{B^3}\left[(1-\beta)^3+\frac{3\beta(1-\beta)^2\mathfrak{T}^{\beta}}{\Gamma(2\beta+1)}+\frac{3\beta^2(1-\beta)\mathfrak{T}^{2\beta}}{\Gamma(2\beta+1)}+\frac{\beta^3\mathfrak{T}^{3\beta}}{\Gamma(3\beta+1)}\right]-2\xi\coth\left(3\operatorname{cosech}^2([\kappa(\varphi+\theta)]-1)\right)\\
&\qquad\times\frac{1}{B^2}\left[(1-\beta)^2+\frac{2\beta(1-\beta)\mathfrak{T}^{\beta}}{\Gamma(\beta+1)}+\frac{\beta^2\mathfrak{T}^{2\beta}}{\Gamma(2\beta+1)}\right]+\cdots.
\end{aligned}
\tag{54}
$$

The exact solution of (46) is

$$
\begin{aligned}
\mu(\varphi,\mathfrak{T})&=\xi-\kappa\coth[\kappa(\varphi+\theta-\xi\mathfrak{T})],\\
\nu(\varphi,\mathfrak{T})&=-\kappa^2\operatorname{cosech}^2[\kappa(\varphi+\theta-\xi\mathfrak{T})].
\end{aligned}
\tag{55}
$$

6. Conclusions

In this paper, the LHPTM was considered to achieve an analytical result for the fractional-order Whitham–Broer–Kaup equations considering the Caputo–Fabrizio and Atangana–Baleanu operators. Analytical results were obtained for different fractional order β. Both suggested operators have shown to be critical mathematical tools for scientists working in numerous fields of applied sciences. The polynomial expansion well-thought-out in the LHPTM permits to achieve an infinite series result for the Whitham–Broer–Kaup equations. This technique established a general system to achieve fractional-order models' analytical results, and the results are obtained in the series form, which converges quickly; the LHPTM is used for investigating other nonlinear fractional systems of partial differential equations.

Abbrevation

HPTM: Homotopy perturbation transform method
lT: Laplace transform
FPDEs: Fractional partial differential equations
FC: Fractional calculus
WBKEs: Whitham–Broer–Kaup equations
ADM: Adomian decomposition method.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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