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Numerical Investigation on the Structure of the Zeros of the Twisted Tangent Polynomials

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Abstract

In [3], we introduced the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$. In this paper, we observe the distribution of complex roots of the twisted tangent polynomials $T_{n,w}(x)$, using numerical investigation. Finally, we give a table for the solutions of the twisted tangent polynomials $T_{n,w}(x)$.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: tangent numbers and polynomials, twisted tangent numbers and polynomials, complex roots

1 Introduction

In [3], we constructed the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$. By using these numbers and polynomials, we obtained some interesting properties. In order to study the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$, we must understand the structure of the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$. Therefore, using computer, a realistic study for the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$ is very interesting.

It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the twisted tangent polynomials $T_{n,w}(x)$ in complex plane.

The outline of this paper is as follows. In Section 2, introduce the twisted tangent polynomials $T_{n,w}(x)$. In Section 3, we describe the beautiful zeros of the twisted tangent polynomials $T_{n,w}(x)$ using a numerical investigation. Finally, we investigate the roots of the twisted tangent polynomials $T_{n,w}(x)$. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the twisted tangent polynomials $T_{n,w}(x)$.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

In [4], we introduce the tangent numbers T_n and polynomials $T_n(x)$. The tangent numbers T_n are defined by the generating function:

$$\frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}. \tag{1.1}$$

We introduce the tangent polynomials $T_n(x)$ as follows:

$$\left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,w}(x) \frac{t^n}{n!}. \tag{1.2}$$

In [4], by using p-adic integral on $\mathbb{Z}_p$, we introduced the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$.

2 twisted tangent numbers and polynomials

In this section, we introduce the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$ and investigate their properties. Let w be the p^N-th root of unity. By the meaning of (1.1) and (1.2), let us define the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$ as follows:

$$F_w(t) = \frac{2}{we^{2t}+1} = \sum_{n=0}^{\infty} T_{n,w} \frac{t^n}{n!}, \tag{2.1}$$

$$F_w(x,t) = \left(\frac{2}{we^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,w}(x) \frac{t^n}{n!}. \tag{2.2}$$

Observe that if $w = 1$, then $T_{n,w}(x) = T_n(x)$ and $T_{n,w} = T_n$(see [3-4]).

By using computer, the twisted tangent numbers $T_{n,w}$ can be determined explicitly. A few of them are

$$\begin{aligned} T_{0,w} &= \frac{2}{1+w}, \quad T_{1,w} = -\frac{4q}{(1+w)^2}, \\ T_{2,w} &= -\frac{8w}{(1+w)^3} + \frac{8w^2}{(1+w)^3}, \\ T_{3,w} &= -\frac{16w}{(1+w)^4} + \frac{64w^2}{(1+w)^4} - \frac{16w^3}{(1+w)^4}. \end{aligned}$$

The following elementary properties of tangent polynomials $T_{n,w}(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [3]-[4].

Theorem 2.1 *For any positive integer n, we have*

$$wT_{n,w}(x) = (-1)^n T_{n,w^{-1}}(2-x).$$

Theorem 2.2 *For any positive integer m(=odd), we have*

$$T_{n,w}(x) = m^n \sum_{i=0}^{m-1} (-1)^i w^i T_{n,w^m}\left(\frac{2i+x}{m}\right), \quad n \in \mathbb{Z}_+.$$

Theorem 2.3 *For $n \in \mathbb{Z}_+$, we have*

$$T_{n,w}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,w} x^{n-l}.$$

By Theorem 2.3, after some elementary calculations, we have

$$\begin{aligned} \int_a^b T_{n,w}(x)dx &= \sum_{l=0}^{n} \binom{n}{l} T_{l,w} \int_a^b x^{n-l} dx \\ &= \sum_{l=0}^{n} \binom{n}{l} T_{l,w} \left. \frac{x^{n-l+1}}{n-l+1} \right|_a^b \\ &= \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} T_{l,w} \left. x^{n-l+1} \right|_a^b. \end{aligned}$$

By Theorem 2.3, we get

$$\int_a^b T_{n,w}(x)dx = \frac{T_{n+1,w}(b) - T_{n+1,w}(a)}{n+1}. \tag{2.3}$$

Since $T_{n,w}(0) = T_{n,w}$, by (2.3), we have the following theorem.

Theorem 2.4 *For $n \in \mathbb{N}$, we have*

$$T_{n,w}(x) = T_{n,w} + n\int_0^x T_{n-1,w}(t)dt.$$

Then, it is easy to deduce that $T_{n,w}(x)$ are polynomials of degree n. Here is the list of the first twisted tangent's polynomials.

$$\begin{aligned}
T_{0,w}(x) &= \frac{2}{1+w}, \\
T_{1,w}(x) &= \frac{-4w+2x+2xw}{(1+w)^2}, \\
T_{2,w}(x) &= \frac{-8w+8w^2-8xw-8w^2x+2x^2+4wx^2+2w^2x^2}{(1+w)^3}, \\
T_{3,w}(x) &= \frac{-16w+64w^2-16w^3-24wx+24w^3x-12wx^2-24w^2x^2-12w^3x^2}{(1+w)^4} \\
&\quad + \frac{2x^3+6wx^3+6w^2x^3+2w^3x^3}{(1+w)^4}.
\end{aligned}$$

3 Zeros of the twisted tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the twisted tangent polynomials $T_{n,w}(x)$. We investigate the beautiful zeros of the $T_{n,w}(x)$ by using a computer. Let $w = e^{\frac{2\pi i}{N}}$ in $\mathbb{C}$. We plot the zeros of the twisted tangent polynomials $T_{n,w}(x)$ for $n = 30, N = 1, 3, 5, 7$ and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 30$ and $w = e^{\frac{2\pi i}{1}}$. In Figure 1(top-right), we choose $n = 30$ and $w = e^{\frac{2\pi i}{3}}$. In Figure 1(bottom-left), we choose $n = 30$ and $w = e^{\frac{2\pi i}{5}}$. In Figure 1(bottom-right), we choose $n = 30$ and $w = e^{\frac{2\pi i}{7}}$. Stacks of zeros of $T_{n,w}(x)$ for $1 \le n \le 30$ from a 3-D structure are presented(Figure 2). In Figure 2(left), we choose $1 \le n \le 30$ and $w = e^{\frac{2\pi i}{1}}$. In Figure 2(right), we choose $1 \le n \le 30$ and $w = e^{\frac{2\pi i}{7}}$. Our numerical results for approximate solutions of real zeros of $T_{n,w}(x)$ are displayed(Tables 1, 2).

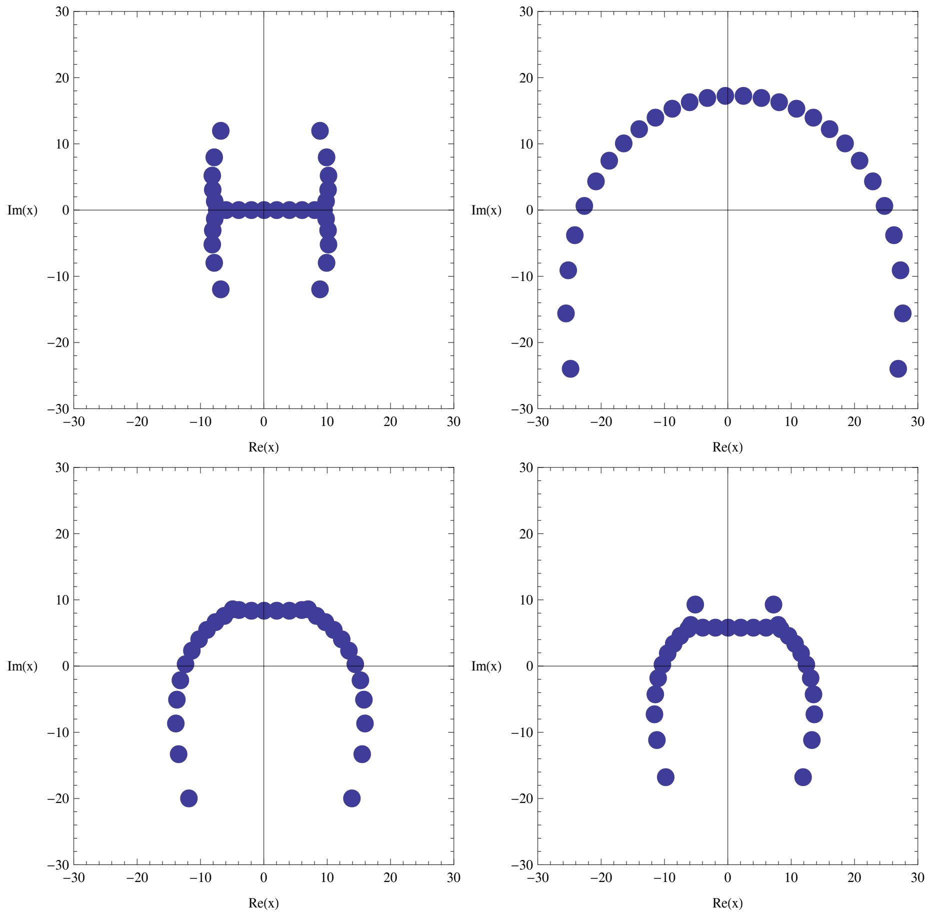

Figure 1: Zeros of $T_{n,w}(x)$

Table 1. Numbers of real and complex zeros of $T_{n,w}(x)$

degree n	$w = e^{\frac{2\pi i}{1}}$		$w = e^{\frac{2\pi i}{7}}$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	0	1
2	2	0	0	2
3	3	0	0	3
4	4	0	0	4
5	5	0	0	5
6	2	4	0	6
7	3	4	0	7
8	4	4	0	8
9	5	4	0	9
10	6	4	0	10

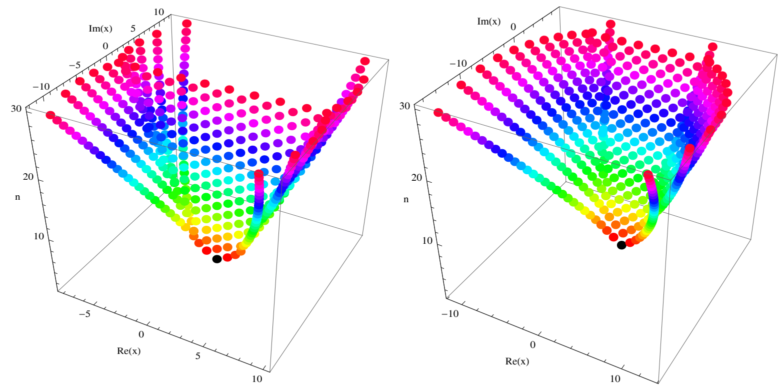

Figure 2: Stacks of zeros of $T_{n,w}(x)$ for $1 \leq n \leq 30$

Plot of real zeros of $T_{n,w}(x)$ for $1 \leq n \leq 30$ structure are presented(Figure 3).

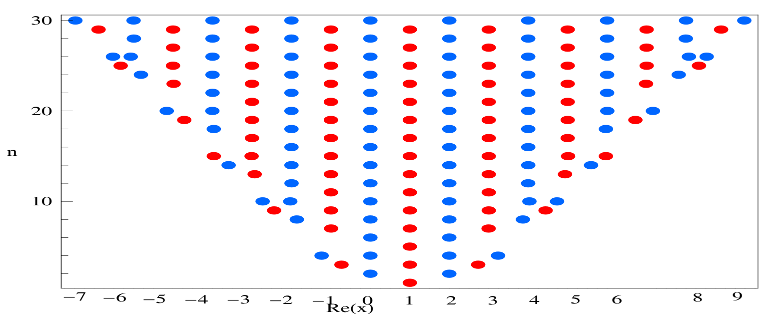

Figure 3: Real zeros of $T_{n,w}(x)$ for $w = e^{2\pi i}$ and $1 \leq n \leq 30$

We observe a remarkably regular structure of the complex roots of the twisted tangent polynomials $T_{n,w}(x)$. We hope to verify a remarkably regular structure of the complex roots of the twisted tangent polynomials $T_{n,w}(x)$(Table 1). Next, we calculated an approximate solution satisfying $T_{n,w}(x), x \in \mathbb{C}$. The results are given in Table 2.

Table 2. Approximate solutions of $T_{n,w}(x) = 0, w = e^{\frac{2\pi i}{7}}, x \in \mathbb{C}$

degree n	x
1	$1.00000 + 0.48157i$
2	$-0.10992 + 0.48157i, \quad 2.1099 + 0.4816i$
3	$-0.94142 + 0.32519i, \quad 1.0000 + 0.79435i, \quad 2.9414 + 0.3252i$
4	$-1.5808 + 0.0553i, \quad -0.04606 + 0.90788i$ $2.0461 + 0.9079i, \quad 3.5808 + 0.0553i$
5	$-2.1066 - 0.3317i, \quad -0.94452 + 0.96376i$ $1.000 + 1.1438i, \quad 2.9445 + 0.9638i, \quad 4.1066 - 0.3317i$

Finally, we shall consider the more general problems. How many zeros does $T_{n,w}(x)$ have? Prove or disprove: $T_{n,w}(x) = 0$ has n distinct solutions. Find the numbers of complex zeros $C_{T_{n,w}(x)}$ of $T_{n,w}(x), Im(x) \neq 0$. Since n is the degree of the polynomial $T_{n,w}(x)$, the number of real zeros $R_{T_{n,w}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{T_{n,w}(x)} = n - C_{T_{n,w}(x)}$, where $C_{T_{n,w}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{T_{n,w}(x)}$ and $C_{T_{n,w}(x)}$.

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