Numerical methods for computing logarithmic capacity

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Tiivistelmä — Referat — Abstract

Logaritmisk kapacitet är viktigt inom flera områden av tillämpad matematik och kan ha olika benämningar beroende på forskningsområdet. T.ex. inom talteorin kallas den logaritmiska kapaciteten för transfinit diameter och inom approximering av polynom är den känd som Chebyshevs konstant. Inom potentialteorin definieras den logaritmiska kapaciteten som måttet på storleken av en kompakt mängd i $\mathbb{C}$.

Men trots att den logaritmiska kapaciteten är så viktig inom många forskningsområden, så är den ytterst svår att beräkna. Tack vare dess samband till Greens funktioner går det att beräkna den logaritmiska kapaciteten analytiskt för vissa enklare mängder, såsom ellipser och kvadrater, men när det gäller mer komplicerade mängder så kan man endast uppskatta övre och nedre gränser. På grund av detta har det utvecklats flera numeriska metoder för detta syfte.

I början av denna avhandling kommer vi att presentera nödvändig bakgrundsinformation för definiering och beräkning av logaritmisk kapacitet. I kapitel 4 presenterar vi definitionen av logaritmisk kapacitet och dess samband till Greens funktioner, samt hur man genom detta samband kan beräkna den logaritmiska kapaciteten analytiskt. Här presenterar vi även några gränser för den logaritmiska kapaciteten, samt definitionen för transfinit diameter och dess samband till den logaritmiska kapaciteten. I kapitel 5 kommer vi att presentera fyra olika numeriska metoder för approximering av logaritmisk kapacitet: Dijkstra-Hochstenbachs metod, Rostands metod, Ransford-Rostands metod, samt hur man kan använda Schwarz-Christoffel avbildningar för beräkning av logaritmisk kapacitet. Vi tillämpar även Rostands metod som ett MATLAB-program.

Avainsanat — Nyckelord — Keywords

logaritmisk kapacitet, potentialteori, Greens funktioner, numerisk approximering

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Chapter 1

Introduction

The logarithmic capacity of compact sets in $\mathbb{R}^2 = \mathbb{C}$ plays an important role in many fields of applied mathematics. In general it is defined as a measure for the capability of a set to support a unit amount of charge, but the formal definition depends on which field of mathematics it is being used in. For example, in potential theory logarithmic capacity is considered the measure of the size of a compact set in $\mathbb{C}$.

The term "potential theory" arose in the 19th century physics, when the belief was that the fundamental forces of nature could be derived from potentials satisfying Laplace's equation. These days, we know that nature is a bit more complicated then that, but the term "potential theory" has remained to describe the study of functions which satisfy Laplace's equation. The study of capacities is an important field of study in modern potential theory.

The notion of the capacity of a set, which refers to the measure of the "size" of a set in Euclidean space, was introduced by Gustave Choquet (1915-2006) in 1950. He was a French mathematician born in Solesmes in northern France, and he did his postgraduate work at the École normale supérieure in Paris, where he received his doctorate in 1946. Choquet became interested in potential theory in 1944, and it served as a constant source of inspiration for him. For a historical account on the development of the theory of capacity written by Choquet himself, see *La naissance de la théorie des capacités: réflexion sur une expérience personelle* [2].

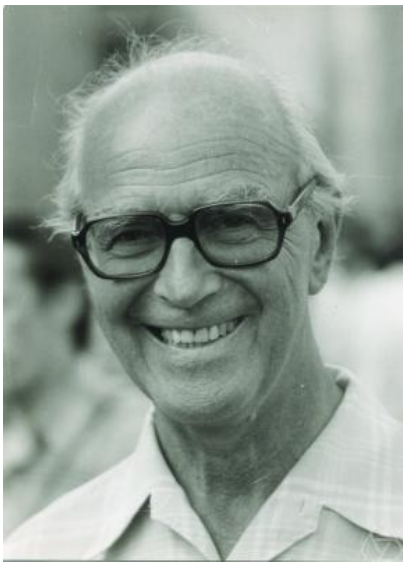

Figure 1.1: Gustave Choquet

In the field of number theory, the logarithmic capacity is known as the transfinite diameter. One of the key people in the subject of transfinite diameter was the hungarian mathematician Michael Fekete (1886-1957).

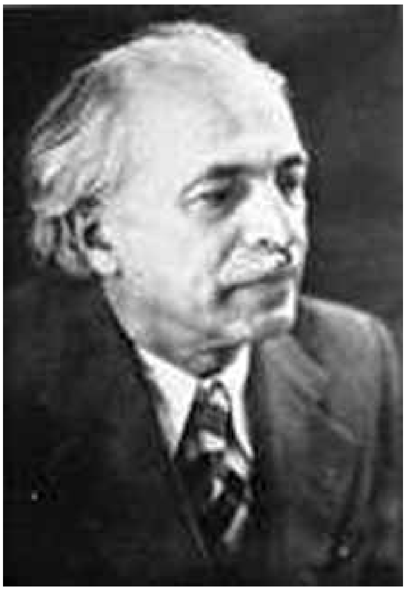

Figure 1.2: Michael Fekete

Fekete was born into a Jewish-Hungarian family in Zenta which, at the time of his birth, was a part of the Austro-Hungarian Empire. Today Zenta is located in Serbia. After having graduated from the local Gymnasium he entered the University of Budapest to study mathematics. He studied undet Lipót Fejér who he, as many hungarian mathematicians in his generation, was greatly influenced by. He was awarded a doctorate by the University of Budapest in 1909. In 1922 Fekete and John von Neumann published a joint paper, *Über die Lage der Nullstellen gewisser Minimum Polynome* [7]. This paper looked at the concept of the transfinite diameter of a set, a subject Fekete worked on throughout the rest of his career.

In the field of polynomial approximation, the logarithmic capacity is known as the Chebyshev constant. Fekete proved in 1923 that with the help of Chebyshev polynomials, named after the russian mathematician Pafnuty Chebyshev (1821-1894), one can construct the Chebyshev constant, which is identical to the transfinite diameter and the logarithmic capacity.

The logarithmic capacity is also linked to the Robin constant, which is important in the field of conformal mapping, and is thus connected to Green's functions. This connection is particularly useful for calculating the capacity of some simple sets. The logarithmic capacity is also critical in the field of boundary integral equations, where it determines if the integral equation for a Dirichlet problem is singular.

The mathematical definitions of capacity are numerous, but one of the more notable ones is conformal capacity, which is defined using conformal mapping onto the exterior of the unit circle [15]. Using this definition, one can also see an association between capacity and the conformal radius of a domain.

Despite the fact that logarithmic capacity appears in so many different fields, it is notoriously hard to compute. Analytical computations are only possible for some simple sets, such as squares and ellipses. In the case of more complex sets the capacity can be numerically approximated with the help of boundaries, but accurate approximations are rare.

The focus of this thesis is on numerical methods for computing logarithmic capacity, but first we will introduce the neccessary background information for understanding logarithmic capacity. In Chapter 2 we will present some preliminary concepts needed in defining and calculating the logarithmic capacity, such as the extended complex plane, the basics of integral transforms and the concept of superharmonic functions, which is essential for defining the concept of potentials. In Chapter 3 we will move on to the concept of potentials and energy, which are essential when defining logarithmic capacity. This chapter will also introduce us to Green's functions, which can be used to determine logarithmic capacity. In Chapter 4 we will finally introduce the concept of logarithmic capacity and some methods for calculating it with the help of Green's functions. We will also introduce the concept of transfinite diameter and the Chebyshev constant, as well as some methods for approximating logarithmic capacity using boundaries. In Chapter 5 we will finally arrive to the heart of this thesis: the numerical methods for computing logarithmic capacity. We will introduce some methods constructed by Dijkstra and Hochstenbach [4], by Rostand [23] and by Ransford and Rostand [20]. This chapter will also contain a short introduction to Schwarz-Christoffel mapping, and how the Schwarz-Christoffel toolbox for MATLAB can be used for computing logarithmic capacity [5, 6].

Chapter 2

Preliminaries

This chapter will contain to some of the background information needed when explaining some concepts regarding logarithmic capacity.

2.1 The extended complex plane

The extended complex plane is the complex plane $\mathbb{C}$ with a point at infinity attached. This is denoted by $\mathbb{C}_\infty$ or $\mathbb{C} \cup \{\infty\}$.

Definition 2.1. The point of infinity satisfies the following algebraic properties:

(i) $\frac{z}{\infty} = 0$,

(ii) $z + \infty = \infty$ $(z \neq \infty)$,

(iii) $\frac{z}{0} = \infty$ $(z \neq 0)$,

(iv) $z \cdot \infty = \infty$ $(z \neq 0)$,

(v) $\frac{\infty}{z} = \infty$ $(z \neq \infty)$,

where $z \in \mathbb{C}$.

The extended complex plane is often equated with the *Riemann sphere*. A common way of constructing the Riemann sphere is to set up a correspondence between the points of $\mathbb{C}$ and those of a sphere of radius $1/2$ with the center at $(0, 0, 1/2)$.

Definition 2.2. Let $\mathbb{C}$ be the complex plane and construct a line perpendicular to $\mathbb{C}$. This would make $\mathbb{C}$ the (x, y)-plane in $\mathbb{R}^3$, so that any $x + iy \in \mathbb{C}$ is identified with $(x, y, 0) \in \mathbb{R}^3$. Let

$$S = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \xi^2 + \eta^2 + (\zeta - 1/2)^2 = 1/4\}. \tag{2.3}$$

Now, the plane $\zeta = 0$ coincides with the complex plane $\mathbb{C}$ and the ξ and η axes are the x and y axes, respectively. A line joining the *north pole* $(0, 0, 1)$ to $(x, y, 0)$ cuts S in a unique point (ξ, η, ζ), so we have a unique one-to-one correspondence between $\mathbb{C}$ and the points of S with the exception of the north pole itself. As (ξ, η, ζ) approaches $(0, 0, 1)$ it follows, that $|x + iy|$ becomes very large. Thus, it is not unreasonable to assign the north pole to correspond to the point at infinity.

We have now obtained a one-to-one correspondence between the points of the Riemann sphere S and the points of the extended complex plane $\mathbb{C}_\infty$. [24, 17]

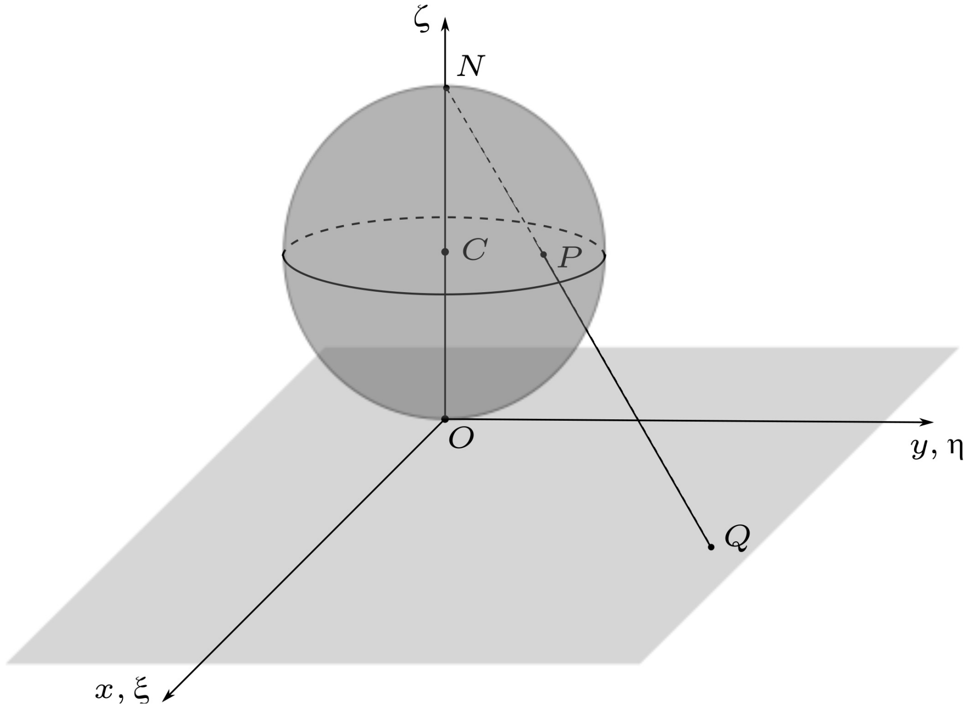

Figure 2.1: Projection of a point on the Riemann sphere.

2.2 Integral transforms

A general *integral transform* is defined by

$$Kf(x) = g(x) = \int_a^b K(x, y) f(y)\, dy,$$

where $K(x, y)$ is called the *kernel* or the *nucleus* of the transform. [22]

Integral representations play a central role in various fields of pure and applied mathematics, theoretical physics, and engineering. For example, many boundary value problems and initial boundary value problems can be solved using integral kernels. [16]

Example 2.4. Consider the homogeneous heat problem with the Dirichlet condition

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < L, t > 0 \\ u(0, t) = u(L, t) & t \geq 0 \\ u(x, 0) = f(x) & 0 \leq x \leq L. \end{cases} \tag{2.5}$$

This problem can be solved using a separation of variables technique, as described in [16, Ch. 5.2]. The basis for this method is finding a solution of the form $u(x, t) = X(x)T(t)$.

Using the initial conditions in (2.5), we arrive at

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} X_n(x)T_n(t),$$

where

$$X_n(x) = \sin \frac{n\pi x}{L} \quad \text{and} \quad T_n(t) = B_n e^{kn^2\pi^2 t/L},$$

with $n = 1, 2, 3 \ldots$.

The coefficients B_n are the Fourier coefficients

$$B_n = \frac{2}{L} \int_0^L \sin \frac{n\pi y}{L} f(y)\, dy,$$

with $n = 1, 2, 3 \ldots$.

For fixed $t > \varepsilon > 0$ and $0 < x < L$, the series

$$\frac{2}{L} \sum_{k=1}^{\infty} \left(e^{kn^2\pi^2 t/L} \sin \frac{n\pi x}{L} \right) \sin \frac{n\pi y}{L} f(y)$$

converges uniformly as a function of y. Thus, we can integrate term by term, and

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty} e^{kn^2\pi^2 t/L} \sin\frac{n\pi x}{L}\left(\frac{2}{L}\int_0^L \sin\frac{n\pi y}{L} f(y)\, d(y)\right) \\
&= \int_0^L \left(e^{kn^2\pi^2 t/L} \sin\frac{n\pi x}{L} \sin\frac{n\pi y}{L}\right) f(y)\, dy.
\end{aligned}
$$

We have now arrived at an integral representation

$$
u(x,t) = \int_0^L K(x,y,t) f(y)\, dy,
$$

where

$$
K(x,y,t) = e^{kn^2\pi^2 t/L} \sin\frac{n\pi x}{L} \sin\frac{n\pi y}{L}.
$$

The function $K(x,y,t)$ is called the *heat kernel* of the initial boundary condition (2.5).

2.3 Superharmonic functions

A function u is said to be *harmonic* if its *Laplacian*,

$$
\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2},
$$

satisfies the condition $\Delta u = 0$. In spirit, at least, a function u is called *superharmonic* if $\Delta u \leq 0$. But one of the advantages of superharmonic functions is their flexibility, which would be lost if we assumed that they were smooth. Therefore, we will define them in a different way.

But first, we will introduce the concept of *semicontinuity*. The following definition for *lower semicontinuous* functions is based on the definition of *upper semicontinuous* functions found in [19, Ch. 2.1].

Definition 2.6. Suppose X is a topological space. The function $u\colon X \to (-\infty, \infty]$ is said to be lower semicontinuous if $\{x \in X \colon u(x) > \alpha\}$ is an open set in X for all $\alpha \in \mathbb{R}$. The function $v\colon X \to [-\infty, \infty)$ is called upper semicontinuous if $-v$ is lower semicontinuous.

Thus a function u is continuous if and only if it is both upper and lower semicontinuous, and a straightforward check shows us that u is lower semicontinuous if and only if

$$
\liminf_{x \to y} u(x) \geq u(y)
$$

for each limit point y of X.

The following definition for *superharmonic* functions has been formulated with the help of [19, Ch. 2.2] and [1, Ch. 3].

Definition 2.7. Let U be an open subset of $\mathbb{C}$. The function $u\colon U \to (-\infty, +\infty]$ is called superharmonic if

(i) u is lower semicontinuous on U

(ii) for any $w \in U$ there is a $\phi > 0$ so that

$$u(w) \geq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{it})\, dt,$$

where $0 \leq r < \phi$.

A function $v\colon U \to [-\infty, +\infty)$ is considered *subharmonic* if $-v$ is superharmonic.

Superharmonic functions need not be smooth, and are indeed sometimes far from it, but they can always be approximated by others which are smooth. A standard way for doing this is through *convolutions*.

Definition 2.8. Let U be an open subset of $\mathbb{C}$, and define

$$U_r = \{z \in U : \operatorname{dist}(z, \partial U) > r\},$$

where $r > 0$.

Let $u\colon U \to [-\infty, \infty)$ be a function that is locally integrable, and let $\phi\colon \mathbb{C} \to \mathbb{R}$ be a continuous function with $\operatorname{supp} \phi \in \Delta(0, r)$. Then their *convolution* $u * \phi\colon U \to \mathbb{R}$ is given by

$$u * \phi(z) = \int_{\mathbb{C}} u(z - w)\phi(w)\, dA(w),$$

where $z \in U_r$. Here (dA denotes the two-dimensional Lebesgue measure)

After a change of variable, we also have

$$u * \phi(z) = \int_{\mathbb{C}} u(w)\phi(z - w)\, dA(w),$$

where $z \in U_r$, which shows that if $\phi \in C^\infty$, then also $u * \phi \in C^\infty$, since we can differentiate under the integral sign arbitrarily many times.

Theorem 2.9 (Weak Identity Principle). *Suppose that u and v are subharmonic functions on an open set U in $\mathbb{C}$ such that $u = v$ a.e. on U. Then $u \equiv v$ on U.*

Proof. First, suppose that u and v are bounded below on U. Let $\chi\colon \mathbb{C} \to \mathbb{R}$ be a function satisfying

$$\chi \in C^\infty, \quad \chi \geq 0, \quad \chi(z) = \chi(|z|), \quad \operatorname{supp} \chi \in \Delta(0,1), \quad \int_{\mathbb{C}} \chi \, dA = 1,$$

and for $r > 0$ define

$$\chi_r(z) = \frac{1}{r^2} \chi \left(\frac{z}{r} \right),$$

where $z \in \mathbb{C}$.

Now, $u * \chi_r = v * \chi_r$ on U_r, and letting $r \to 0$ we deduce that $u = v$ on U.

The general case follows from applying the one above to $u_n := \max(u, -n)$ and $v_n := \max(v, -n)$, and then letting $n \to \infty$. $\square$

Chapter 3

Results in Potential Theory

This chapter will introduce us to potentials and Green's functions, and results associated with these. These concepts will be necessary when we introduce the concept of capacity.

3.1 Potentials and energy

Potentials provide an important source of examples of superharmonic functions. They are almost as general as arbitrary superharmonic functions, and for many purposes the classes are equivalent.

Definition 3.1. Let E be a compact subset of $\mathbb{C}$. We write $\mathcal{P}(E)$ for the family of all Borel probability measures on E. Given $\mu \in \mathcal{P}(E)$, we define its *logarithmic potential* $p_\mu \colon \mathbb{C} \mapsto (-\infty, \infty]$ with the superharmonic function

$$p_\mu = \int_E \log \frac{1}{|z-w|}\, d\mu(w), \tag{3.2}$$

where $z \in \mathbb{C}$.

Potentials can also be defined as subharmonic functions, which will change the definition slightly (see [19, Ch. 3.1]). If $p_\mu \colon \mathbb{C} \mapsto (-\infty, \infty]$ is defined as a subharmonic function, then p_μ is defined as

$$p_\mu = \int_E \log |z-w|\, d\mu(w), \quad (z \in \mathbb{C}). \tag{3.3}$$

Potentials enjoy several properties outside of those displayed by general superharmonic functions. One of these is the Maximum principle. The proof for this principle can be found in [25, Theorem III.1].

Theorem 3.4 (**Maximum principle**). *Let μ be a finite Borel measure on $\mathbb{C}$ with compact support E. If $p_\mu \le M$ on E, then $p_\mu \le M$ on the whole of $\mathbb{C}$.*

The notion of potential induces a concept of energy. Here μ can be seen as a charge distribution on $\mathbb{C}$ and $p_\mu(z)$ as the potential energy at z. Thus, the following defines the total energy for μ.

Definition 3.5. Using the same definitions as in Definition 3.1, we define the *energy* of μ by

$$I(\mu) = \int_E p_\mu \, d\mu(z) = \int_E \int_E \log \frac{1}{|z-w|} \, d\mu(z) \, d\mu(w). \tag{3.6}$$

If $I(\mu) = \infty$ then E is said to be *polar*.

In physics, if a charge is placed on a conductor it distributes itself so that the energy is minimized. In our case, we consider at probability measures μ on a compact set E which minimizes $I(\mu)$.

Definition 3.7. If there exists $\mu \in \mathcal{P}(E)$ such that $I(\mu) < \infty$, then there exists a unique $\nu \in \mathcal{P}(E)$, called the *equilibrium measure*, such that

$$I(\nu) = \inf_{\mu \in \mathcal{P}(E)} I(\mu). \tag{3.8}$$

Physical intuition would tend to suggest that if ν is an equilibrium measure for E, then p_ν should be constant on E. Otherwise charge would flow from one part of E to another, and disturb the equilibrium. This idea is confirmed by the following theorem, called *Frostman's theorem*, which is considered important and is sometimes referred to as the "Fundamental Theorem of Potential Theory". The proof for the following theorem has been constructed based on the proofs in [19, Theorem 3.3.4] and in [25, Theorem III. 12].

Theorem 3.9 (**Frostman's theorem**). *Let E be a compact set in $\mathbb{C}$, and let ν be an equilibrium measure for E. Then:*

(a) *$p_\nu \le I(\nu)$ on $\mathbb{C}$,*

(b) *$p_\nu = I(\nu)$ on $E \setminus K$, where K is an F_σ[1] polar subset of ∂E.*

Proof. If $I(\nu)$ is polar, i.e. $I(\nu) = \infty$, then the result is obvious. Therefore we may assume that $I(\nu)$ is non-polar.

[1]With an F_σ set we mean a countable union of closed sets.

First, we will prove that

$$p_\nu(z) \geq I(\nu) \text{ on } E \setminus K, \text{ where } E \text{ is } F_\sigma \text{ polar.} \tag{3.10}$$

The first step in this is to prove that the set

$$K_n := \{z \in E : p_\nu(z) \leq I(\nu) - 1/n\}$$

is polar for each $n \geq 1$. This will be done through contradiction.

Suppose, if possible, that some K_n is non-polar. Choose $\mu \in \mathcal{P}(K_n)$ with $I(\mu) \leq \infty$. Since

$$I(\nu) = \int p_\nu(z)\, d\mu(z),$$

there exists such $z_0 \in \operatorname{supp}\nu$ that $p_\nu(z_0) \geq I(\nu)$. By lower semicontinuity, there exists $r > 0$ so that

$$p_\nu > I(\nu) - \frac{1}{2n}$$

on $\overline{B}(z_0, r)$. In particular, $\overline{B}(z_0, r) \cap K_n = \emptyset$.Since $z_0 \in \operatorname{supp}\nu$, $a := \nu(\overline{B}(z_0, r)) > 0$. Define a signed measure σ on E by

$$\sigma = \begin{cases} \mu & \text{on } K_n, \\ -\nu/a & \text{on } \overline{B}(z_0, r), \\ 0 & \text{otherwise.} \end{cases}$$

For each $t \in (0, a)$, $\nu_t := \nu + t\sigma \geq 0$, and therefore $\nu_t \in \mathcal{P}(E)$. As $I(\mu) < \infty$ implies that $I(|\sigma|) < \infty$, we have

$$\begin{aligned} I(\nu_t) - I(\nu) &= 2t \int p_\nu(z)\, d\sigma(z) + t^2 I(\sigma) \\ &= 2t \left(\int_{K_n} p_\nu(z)\, d\mu(z) - \int_{\overline{B}(z_0,r)} p_\nu(z)\, d\nu(z)/a + \frac{t}{2} I(\sigma) \right) \\ &\leq 2t \left((I(\nu) - \frac{1}{n}) - (I(\nu) - \frac{1}{2n}) + \frac{t}{2} I(\sigma) \right) \\ &= -t(\frac{1}{n} - tI(\sigma)). \end{aligned}$$

If t is sufficiently small, we have $I(\nu_t) < I(\nu)$. This contradicts the fact that ν is an equilibrium measure, and hence each K_n is polar.

Since every Borel polar set has a Lebesgue measure of zero, if we put $K = \cup_n K_n$ then this implies that K is an F_σ polar set, and $p_\nu \geq I(\nu)$ on $E \setminus K$.

Next, we shall prove that

$$p_\nu(z) \le I(\nu) \text{ on } \mathbb{C}. \tag{3.11}$$

The first part of this is to prove that the set the set

$$L_n := \{z \in \operatorname{supp}\nu : p_\nu(z) > I(\nu) + 1/n\}$$

is empty for each $n \ge 1$. This will also be done through contradiction.

Suppose, if possible, that some L_n is non-empty. Pick $Z_1 \in L_n$. By lower semicontinuity, there exists $s > 0$ so that

$$p_\nu > I(\nu) + \frac{1}{n}$$

on $\overline{B}(z_1, s)$. Since $z_1 \in \operatorname{supp}\nu$, $b := \nu(\overline{B}(z_1, s)) > 0$. By (3.10), we now have

$$\begin{aligned} I(\nu) = \int_E p_\nu \, d\nu &= \int_{\overline{B}(z_1,s)} p_\nu \, d\nu + \int_{E\setminus\overline{B}(z_1,s)} p_\nu \, d\nu \\ &\ge \left(I(\nu) + \frac{1}{n}\right) + I(\nu)(1-b) = I(\nu) + \frac{b}{n} \\ &> I(\nu), \end{aligned}$$

which is obviously a contradiction. Therefore, L_n is empty, which implies that $p_\nu \le I(\nu)$ on $\operatorname{supp}\nu$. By the maximum principle (Theorem 3.4) we get $p_\nu \le I(\nu)$ on $\mathbb{C}$, which proves part (a) of the theorem.

Using (3.10) in conjuction with (3.11) , we get that $p_\nu(z) = I(\nu)$ on $E \setminus K$. We observe that as K is polar, it must have a Lebesgue measure of zero. Therefore $p_\nu(z) = I(\nu)$ almost everywhere on E, and hence by the Weak Identity Principle (Theorem 2.9) $p_\nu(z) = I(\nu)$ everywhere on $\operatorname{int}(E)$. This proves part (b) of the theorem. □

3.2 Green's function

Green's function is the integral kernel for the Laplacian. In essence, a Green's function is a family of fundamental solutions to the Laplace equation, which are zero on the boundary.

The Green's function was discovered by, and named after, the English mathematician George Green (1793-1841). [16]

Definition 3.12. Let D be a proper subdomain of $\mathbb{C}_\infty$, that is, a connected open subset of the Riemann sphere. A *Green's function* for D is a map $g_D \colon D \times D \mapsto (-\infty, \infty]$, such that for each $w \in D$,

(a) $z \mapsto g_D(z,w)$ is harmonic on $D \setminus \{w\}$

(b) $g_D(w,w) = \infty$, and as $z \to w$,

$$g_D(z,w) = \begin{cases} \log|z| + O(1), & \text{if } w = \infty, \\ -\log|z-w| + O(1), & \text{if } w \neq \infty; \end{cases}$$

(c) $g_D(z,w) \to 0$ as $z \to \zeta$ for all $\zeta \in \partial_\infty D$. Here, $\partial_\infty D$ means the boundary of D taken with respect to the spherical topology.

In the case of the unit disk, the Möbius transformation

$$z \mapsto \frac{z-w}{1-z\overline{w}}$$

maps the unit disk onto itself. Thus, the Green's function for the unit disk $B = B(0,1)$ is

$$g_B(z,w) = -\log\left|\frac{z-w}{1-z\overline{w}}\right| = \log\left|\frac{1-z\overline{w}}{z-w}\right|.$$

Green's functions have several properties, among them the usual existence and uniqueness properties. The proofs for the properties listed below can be found in [19, Theorem 4.4.2, Theorem 4.4.3, Theorem 4.4.6, Theorem 4.4.8].

Theorem 3.13 (Properties of Green's functions). *Let D be a domain in $\mathbb{C}_\infty$ such that ∂D is non-polar. Then:*

(a) *The Green's function g_D on D exists and is unique.*

(b) $g_D(z,w) > 0$ *for all* $z, w \in D$.

(c) *If $(D_n)_{n\geq 1}$ are such subdomains of D that $D_1 \subset D_2 \subset \cdots$ and $\cup_n D_n = D$. Then*

$$\lim_{n\to\infty} g_{D_n}(z,w) = g_D(z,w)$$

for all $z, w \in D$.

(d) $g_D(z,w) = g_D(w,z)$ *for all* $z, w \in D$.

The following result will allow us to compute Green's functions for some elementary domains by using conformal mapping. The proof for this theorem can be found in [19, Theorem 4.4.4].

Theorem 3.14 (Subordination Principle). *Let D_1 and D_2 be domains in $\mathbb{C}_\infty$ with non-polar boundaries, and let $f\colon D_1 \to D_2$ be a meromorphic function. Then*

$$g_{D_2}(f(z), f(w)) \geq g_{D_1}(z, w)$$

for all $z, w \in D_1$, with equality if f is a conformal mapping of D_1 onto D_2.

Using the theorem above, and the Green's function for the unit disk, one can now calculate the Green's functions for some simple sets by constructing a conformal map from the set in question to the unit disk.

Example 3.15. (a) The Möbius transformation $z \mapsto z/r$ maps the open disk of radius r, denoted by $D_1 = \{z \in \mathbb{C} : |z| < r\}$, onto the unit disk, so the Green's function for D_1 is

$$g_{D_1}(z, w) = g_B(\frac{z}{r}, \frac{w}{r}) = \log \left| \frac{1 - \frac{z\overline{w}}{r^2}}{\frac{z}{r} - \frac{w}{r}} \right| = \log \left| \frac{r^2 - z\overline{w}}{r(z-w)} \right|.$$

(b) The Möbius transformation for the upper half plane $D_2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto the unit disk is the Möbius transformation

$$z \mapsto \frac{z - i}{z + i},$$

known as the *Cayley transform.* Therefore the Green's function for D_2 is

$$g_{D_2}(z, w) = g_B(\tfrac{z-i}{z+i}, \tfrac{w-i}{w+i}) = \log \left| \frac{z - \overline{w}}{z - w} \right|.$$

For further results, see Table 3.1.

D	$g_D(z,w)$
$\{\lvert z\rvert < r\}$	$\log\left\lvert\frac{r^2-z\overline{w}}{r(z-w)}\right\rvert$
$\{\operatorname{Im} z > 0\}$	$\log\left\lvert\frac{z-\overline{w}}{z-w}\right\rvert$
$\{\operatorname{Re} z > 0\}$	$\log\left\lvert\frac{z+\overline{w}}{z-w}\right\rvert$
$\{\lvert\arg z\rvert < \pi/(2a)\}$	$\log\left\lvert\frac{z^a+\overline{w}^a}{z^a-w^a}\right\rvert$
$\{\lvert\operatorname{Re} z\rvert < \pi/(2a)\}$	$\log\left\lvert\frac{e^{iaz}+e^{-ia\overline{w}}}{e^{iaz}-e^{iaw}}\right\rvert$

Table 3.1: Examples of Green's functions

Chapter 4

Logarithmic capacity

Logarithmic capacity is an important concept in several fields of applied mathematics, and it appears under several different guises. For example, in the field of polynomial approximation it is known as the Chebyshev constant. It is also called the transfinite diameter, which is a key ingredient in number theory. It can also be directly linked to the Robin constant and so, it is also linked to Green's functions and conformal mapping.

From the point of view of potential theory, the capacity measures the size of a set in $\mathbb{R}^n$. Logarithmic capacity, specifically, measures the size of a compact set in $\mathbb{R}^2$, which we will identify with the complex plane $\mathbb{C}$.

Logarithmic capacity is notoriously hard to compute. Analytically, it can only be computed for a few simple sets, like ellipses and squares. For slightly more complex sets it can be bounded, but accurate approximations are rare.

Definition 4.1. The *logarithmic capacity* of a non-polar subset E of $\mathbb{C}$ is given by

$$c(E) = e^{-I(\nu)}, \tag{4.2}$$

where $\nu \in \mathcal{P}(E)$ is the equilibrium measure.

If E is polar, that is, if $I(\mu) = \infty$ for all $\mu \in \mathcal{P}(E)$, then we define $c(E) := 0$.

Proofs for the following properties can be found in [19, Theorems 5.1.2, 5.1.3 and 5.2.5]

Theorem 4.3 (Properties of the logarithmic capacity). *Let $E, E_1, E_2, \dots$ be compact subsets of $\mathbb{C}$. Then:*

(a) *If $E_1 \subset E_2$ then $c(E_1) \leq c(E_2)$.*

(b) *If $\alpha, \beta \in \mathbb{C}$, then $c(\alpha E + \beta) = |\alpha| c(E)$.*

(c) *$c(E) = c(\partial_e E)$, where $\partial_e E$ is the exterior boundary of the unbounded component of $\mathbb{C} \setminus E$.*

(d) *If* $(E_n)_{n\geq 1}$ *is a decreasing sequence, then*

$$c\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} c(E_n).$$

(e) *If* $(E_n)_{n\geq 1}$ *is a n increasing sequence such that* $E = \cup_{n=1}^{\infty} E_n$ *is compact, then*

$$c(E) = \lim_{n\to\infty} c(E_n).$$

(f) *If* q *is a polynomial of the form* $q(z) = \sum_{k=0}^{n} a_k z^k$*, then*

$$c(q^{-1}(E)) = \left(\frac{c(E)}{|a_n|}\right)^{1/n}.$$

4.1 Calculating logarithmic capacity

While Definition 4.1 is well suited for deriving theoretical properties of logarithmic capacity, it is not well suited for actually computing the capacity. Even the simplest sets require a lot of work, while other sets are virtually impossible.

An easier alternative for calculating logarithmic capacity for compact sets is based on the following relation between the Green's function and logarithmic capacity.

Theorem 4.4. *Let* E *be a compact non-polar set, and let* D *be the unbounded component of* $\mathbb{C}_\infty \setminus E$*. Then*

$$g_D(z,\infty) = \log|z| - \log c(E) + O\left(\frac{1}{|z|}\right) \quad \text{as } z \to \infty. \tag{4.5}$$

Proof. Let ν be the equilibrium measure for E. First of all, as $|z| \to \infty$

$$\begin{aligned} p_\nu(z) &= \int_E \log\frac{1}{|z-w|}\, d\nu(w) = \log\frac{1}{|z|} + \int_E \log\frac{1}{|1-\frac{w}{z}|}\, d\nu(w) \\ &= \log\frac{1}{|z|} + O\left(\frac{1}{|z|}\right). \end{aligned}$$

Define the Green's function for D as

$$g_D(z,\infty) = \begin{cases} I(\nu) - p_\nu(z) & \text{if } z \in D\backslash\{\infty\}, \\ \infty & \text{if } z = \infty. \end{cases}$$

With Frostman's theorem (Theorem 3.9) one can easily check that $g_D(\cdot, \infty)$ satisfies the conditions in 3.12 with $w = \infty$.

Now, we have

$$g_D(z, \infty) = I(\nu) - p_\nu(z) = \log|z| - \log c(E) + O\left(\frac{1}{|z|}\right),$$

when $z \in D \backslash \{\infty\}$. □

As mentioned in the beginning of this chapter, logarithmic capacity can be directly linked to the Robin's constant. It can be done in the following way. [14]

Definition 4.6. The value

$$\gamma := \lim_{z \to \infty} (g_D(z, \infty) - \log|z|) \tag{4.7}$$

is called the *Robin's constant* and, as the theorem above shows, $I(\nu) = \gamma$.

Example 4.8. Using Theorem 4.4 to determine the capacity of a disk with radius r is relatively simple. As noted in Table 3.1, the Green's function for a disk with radius r, denoted by E, is

$$g_E(z, w) = \log\left|\frac{r^2 - z\overline{w}}{r(z - w)}\right|.$$

Set $D = \mathbb{C}_\infty \setminus E$, then

$$g_D(z, \infty) = g_E(\tfrac{1}{z}, 0) = \log\left|\frac{z}{r}\right|.$$

Thus, the value for the Robin's constant is

$$\gamma = \lim_{z \to \infty} (\log\left|\frac{z}{r}\right| - \log|z|) = -\log r,$$

where it follows that the capacity is $c(E) = e^{-(-\log r)} = r$.

From the subordination principle for Green's functions (Theorem 3.14) the following useful theorem can be determined. The proof for this inequality can be found in [19, Theorem 5.2.3]

Theorem 4.9. *Let E_1 and E_2 be compact subsets of $\mathbb{C}$, and let D_1 and D_2 be the unbounded components of $\mathbb{C}_\infty \backslash E_1$ and $\mathbb{C}_\infty \backslash E_2$ respectively. If there is a meromorphic function $f \colon D_1 \to D_2$, so that*

$$f(z) = z + O(1) \quad \text{as } z \to \infty, \tag{4.10}$$

then

$$c(E_2) \leq c(E_1),$$

with equality if f is a conformal mapping of D_1 onto D_2.

In practice, it is only possible to compute the conformal map f explicitly for relatively simple sets. One such set is the line segment, where the previous inequality will be used to calculate the capacity.

Example 4.11. The previous theorem can be used when determining the capacity of a line segment $[a, b]$, where $b \geq a$. The function $z \mapsto z + 1/z$ maps $\mathbb{C}_\infty \backslash \overline{B}(0, 1)$ conformally onto $\mathbb{C}_\infty \backslash [-2, 2]$ and thus,

$$c([-2, 2]) = c(\overline{B}(0, 1) = 1.$$

By translating and scaling, we can determine that the Möbius transformation

$$z \mapsto \frac{b-a}{4} z + \frac{b+a}{2}$$

"stretches" $[-2, 2]$ to $[a, b]$. From part (b) in Theorem 4.3 it then follows, that

$$c([a, b]) = \frac{b-a}{4}.$$

Some more values of the capacity for some relatively simple sets can be found in Table 4.1.

4.2 Transfinite diameter

A more direct approach for estimating logarithmic capacity is through the *transfinite diameter*. Consider n points z_i $(i = 1, \ldots, n)$ on the compact subset E of $\mathbb{C}$ and the function

$$\binom{n}{2}^{-1} \sum_{1 \leq j < k \leq n} \log \frac{1}{|z_j - z_k|}. \tag{4.12}$$

The minimum of the function is denoted by $\log(1/d_n)$, and for $n \geq 2$ the nth transfinite diameter d_n of E is

$$\begin{aligned} d_n(E) &= \sup \left\{ \exp \left(- \binom{n}{2}^{-1} \sum_{1 \leq j < k \leq n} \log \frac{1}{|z_j - z_k|} \right) : z_1, \ldots, z_n \in E \right\} \\ &= \sup \left\{ \prod_{1 \leq j < k \leq n} |z_j - z_k|^{2/n(n-1)} : z_1, \ldots, z_n \in E \right\}. \end{aligned}$$

The points $z_k = \xi_k$, for which the supremum is attained, are called the nth *Fekete points* of E. [11, 19]

E	$c(E)$
disc of radius r	r
ellipse with semi-axes a and b	$(a+b)/2$
line segment of length h	$h/4$
square with side h	$\frac{\Gamma^2(1/4)}{4\pi^{3/2}}h \approx 0.59017h$
equilateral triangle of height h	$\frac{3^{1/2}\Gamma^3(1/3)}{4\pi^2}h \approx 0.42175h$
regular n-gon with side h	$\frac{\Gamma(1/n)}{2^{1+2/n}\pi^{1/2}\Gamma(1/2+1/n)}$
half-disc of radius r	$\frac{4}{3^{3/2}}r \approx 0.76980r$
lemniscate $\{z : \lvert a_d z^d + \ldots + a_0\rvert \leq r\}$	$\left(\frac{r}{\lvert a_d\rvert}\right)^{1/d}$

Table 4.1: Examples of capacities, compiled from [12, 19]

As E is compact, this n-tuple of Fekete points always exists. The points are distinct and "as far apart from each other as possible", though, for fixed n, the set of points does not need to be unique.

Note that $d_2(E)$ is the diameter of E, while $d_3(E)$ measures its "spread".

The following theorem will show that the sequence $(d_n(E))_{n\geq 2}$ is decreasing and that its limit $d(E)$, the transfinite diameter of E, can be equated to the logarithmic capacity of E.The proof has been constructed using [19, Theorem 5.5.2], [12, Ch. 2 §4] and [11, Part 2.1].

Theorem 4.13 (**Fekete-Szegő Theorem**). *Let E be a compact subset of $\mathbb{C}$. Then the sequence $(d_n(E))_{n\geq 2}$ is decreasing, and*

$$\lim_{n\to\infty} d_n(E) = c(E).$$

Proof. The first part of the proof will show that $(d_n)_{n\geq 2}$ is decreasing. Let $\xi_1, \ldots \xi_{n+1} \in E$ be the points where the supremum for the $n+1$th transfinite diameter of E is attained.

Then

$$\begin{aligned} d_{n+1}^{n(n+1)/2} &= \prod_{1\le j<k\le n+1} |\xi_j - \xi_k| \\ &= |(\xi_1-\xi_2)(\xi_1-\xi_3)\cdots(\xi_1-\xi_{n+1})| \prod_{2\le j<k\le n+1} |\xi_j-\xi_k| \\ &\le |(\xi_1-\xi_2)(\xi_1-\xi_3)\cdots(\xi_1-\xi_{n+1})| d_n^{n(n-1)/2}. \end{aligned}$$

Similarly

$$\begin{aligned} d_{n+1}^{n(n+1)/2} &\le |(\xi_2-\xi_1)(\xi_2-\xi_3)\cdots(\xi_2-\xi_{n+1})| d_n^{n(n-1)/2}, \\ &\vdots \\ d_{n+1}^{n(n+1)/2} &\le |(\xi_{n+1}-\xi_1)\cdots(\xi_{n+1}-\xi_n)| d_n^{n(n-1)/2}. \end{aligned}$$

Multiplying these $n+1$ inequalities together gives

$$(d_{n+1}^{n(n+1)/2})^{n+1} \le \big((d_n^{n(n-1)/2})^{n+1}\big)(d_{n+1}^{n(n+1)/2})^2 \iff (d_{n+1}^{n(n+1)/2})^{n-1} \le (d_n^{n(n-1)/2})^{n+1}.$$

Hence, $d_n \ge d_{n+1}$, as claimed.

The next part of the proof will show that $d_n \ge c(E)$ for all $n \ge 2$. In the beginning of this section we defined that $\log(1/d_n)$ is the minimum of function (4.12). In fact, we can define function (4.12) to be greater or equal to $\log(1/d_n)$. By intgrating this inequality with respect to $d\nu(z_1)\cdots d\nu(z_n)$, where ν is an equilibrium measure for E, we obtain

$$\binom{n}{2}^{-1} \sum_{1\le j<k\le n} \iint \log\frac{1}{|z_j-z_k|}\, d\nu(z_j)\, d\nu(z_k) \ge \log\frac{1}{d_n},$$

where $z_1,\ldots,z_n \in E$.

Hence $I(\nu) \ge \log(1/d_n)$, which gives $c(E) \le d_n$, as claimed.

The final part of the proof will show that $\limsup_{n\to\infty} d_n \le c(E)$. First, let $\varepsilon \ge 0$, and set

$$E^\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z,E) \le \varepsilon\}.$$

Let $n \ge 2$, and choose $\xi_1,\ldots,\xi_n \in E$ so that

$$d_n^{n(n-1)/2} = \prod_{1\le j<k\le n} |\xi_j-\xi_k|.$$

For each j, let μ_j be the normalized Lebesgue measure on the circle $\partial B(\xi_j, \varepsilon)$, and put $\mu = \sum_{j=1}^{n} \mu_j$. Then

$$\begin{aligned} I(\mu) &= \iint \log \frac{1}{|z-w|}\, d\mu(z)\, d\mu(w) \\ &= \frac{1}{n^2} \sum_{j=1}^{n} \iint \log \frac{1}{|z-w|}\, \mu_j(z)\, \mu_j(w) + \frac{2}{n^2} \sum_{1 \le j < k \le n} \iint \log \frac{1}{|z-w|}\, \mu_j(z)\, \mu_k(w). \end{aligned}$$

The equilibrium measure of a closed disc $\overline{B}(\xi_j, \varepsilon)$ is equal to the normalized Lebesgue measure of $\partial b(\xi_j, \varepsilon)$, and as the capacity of a closed disc is its radius, we get

$$\iint \log \frac{1}{|z-w|}\, \mu_j(z)\, \mu_j(w) = I(\mu_j) = \log \frac{1}{\varepsilon}$$

for each j. Because p_{μ_j} is superharmonic, we get

$$\iint \log \frac{1}{|z-w|}\, \mu_j(z)\, \mu_k(w) = \int p_{\mu_j}(w)\, d\mu_k(w) \le p_{\mu_j}(\xi_k)$$

for each pair $j \le k$. In a similar fashion, as $\log 1/|z - \xi_k|$ also is superharmonic, we get

$$p_{\mu_j}(\xi_k) = \int \log \frac{1}{|z-\xi_k|}\, d\mu_j(z) \le \frac{1}{|\xi_j - \xi_k|}.$$

Therefore,

$$I(\mu) \le \frac{1}{n^2} \sum_{j=1}^{n} \log \frac{1}{\varepsilon} + \frac{2}{n^2} \sum_{1 \le j < k \le n} \log \frac{1}{|\xi_j - \xi_k|} = \frac{1}{n} \log \frac{1}{\varepsilon} + \frac{n-1}{n} \log \frac{1}{d_n}.$$

Since μ is supported on E^{ε}, it follows that

$$C(E^{\varepsilon}) \ge \varepsilon^{1/n} d_n^{(n-1)/n}.$$

Hence $\limsup_{n \to \infty} d_n \le c(E^{\varepsilon})$. Since ε is arbitrary, the desired conclusion follows from property (d) in Theorem 4.3. $\square$

The following example will calculate the transfinite diameter, and thus, the logarithmic capacity of the unit circle. This example was presented in [3].

Example 4.14. Let $E = \{z \in \mathbb{C} \colon |z| = 1\}$. The point $e^{i\theta}$ on the unit circle E is the point that is rotated θ radians from he standard position.

The set of all Fekete points on E are the nth unit roots and its rotations, that is, the points $1, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \ldots, e^{i\frac{(n-1)\cdot 2\pi}{n}}$.

Because the points are equally spaced around the unit circle, the product of the distances between a specific Fekete point and the other Fekete points on E is the same no matter which point we choose. The product of the distances in $z = 1$, and thus for any Fekete point in E, is

$$\prod_{j=0}^{n-1} |1 - e^{i\frac{2j\pi}{n}}| = n.$$

Multiplying the distances for all the points will cause each distance to appear twice. Thus we have

$$\prod_{1 \le j < k \le n} |z_j - z_k| = \prod_{k=1}^{\frac{n}{2}} n = n^{\frac{n}{2}}.$$

The nth transfinite diameter for E is then

$$d_n = (n^{\frac{n}{2}})^{\frac{2}{n(n-1)}} = n^{\frac{1}{n-1}}$$

and so

$$d(E) = \lim_{n \to \infty} d_n = 1,$$

which is also the capacity for E.

The following properties of the transfinite diameter can easily be derived from the definition:

(i) If $E \subset F$, then $d(E) \le d(F)$.

(ii) If $z^* = \alpha z + \beta$ maps E onto E^*, then $d(E^*) = \alpha d(E)$.

(iii) Let $\phi\colon E \to \mathbb{C}$ be a map satisfying

$$|\phi(z) - \phi(w)| \le |z - w|$$

for $z, w \in E$. Then $d(\phi(E)) \le d(E)$.

The Chebyshev constant

There is a close connection between the trasfinite diameter of a compact set and polynomials. Consider polynomials of the form

$$p_n(z) = \prod_{k=1}^{n} (z - z_k), \tag{4.15}$$

where $z_k \in \mathbb{C}$ for all k.

Set

$$\tau_n(E) := \inf \max_{z \in E} |p_n(z)|,$$

where the infimum is taken over all polynomials of the form (4.15). Then there exists a unique polynomial t_n of the form (4.15), such that

$$\tau_n(z) = \max_{z \in E} |t_n(z)|.$$

The polynomial t_n is called the *Chebyshev polynomial* and from the definition of t_n it follows that all its zero points lie in the smallest convex set which contains E.

Fekete proved in 1923 that the limit

$$\tau(z) = \lim_{n \to \infty} \tau_n(z)^{1/n}$$

exists. The quantity $\tau(z)$ is called the *Chebyshev constant* of E.

A *Fekete polynomial* for E of degree n is a polynomial of the form

$$q_n(z) = \prod_{k=1}^{n} (z - \xi_k),$$

where $\xi_1, \ldots, \xi_n$ is a Fekete n-tuple for E.

Set $M_n(E) = \max_{z \in E} |q_n(z)|$. Then, for $n = 1, 2, \ldots$,

$$d(E) \leq \tau_n(E)^{1/n} \leq M_n(E)^{1/n} \to d(E)$$

when $n \to \infty$.

Hence, we get

$$c(E) = d(E) = \tau(E),$$

which remains valid as long as $\tau(E)$ is defined by polynomials of the form (4.15) having zeros in E only, as is the case with Fekete polynomials. [25, 11]

4.3 Bounds for capacity

As mentioned earlier, logarithmic capacity is very difficult to compute. Even for simple sets, like the square, the calculations require some effort, and for more complicated sets it is usually impossible. There are, however, methods for deriving upper and lower bounds for the capacity that are easier to compute, and many of these estimates rely on the following basic result. The proof for this theorem can be found in [19, Theorem 5.3.1]

Theorem 4.16. *Let E be a compact subset of $\mathbb{C}$, and let $T\colon E \to \mathbb{C}$ be a map that for all $z, w \in E$ satisfies*

$$|T(z) - T(w) \leq A|z - w|^{\alpha}, \tag{4.17}$$

where A and α are positive constants. Then

$$c(T(E)) \leq Ac(E)^{\alpha}.$$

Using this theorem and the knowledge that the capacity of a line segment of length h is $h/4$ (see Table 4.1), the following "1/4-estimates" for the capacity of a compact subset E of $\mathbb{C}$ can be derived [19, Theorem 5.3.2].

(i) If E is connected and has diameter d, then

$$c(E) \geq d/4.$$

(ii) If E is a rectifiable curve of length l, then

$$c(E) \leq l/4.$$

(iii) If E is a subset of the real axis of Lebesgue measure m, then

$$c(E) \geq m/4.$$

(iv) If E is a subset of the unit circle of arc-length measure a, then

$$c(E) \geq \sin(a/4).$$

An easy consequence of the definition of capacity is that $c(E) \leq \operatorname{diam}(E)$ for every compact set E. But, as the following theorem shows, this fact can be improved. The proof for this theorem can be found in [19, Theorem 5.3.4].

Theorem 4.18. *If E is a compact subset of $\mathbb{C}$, then*

$$c(E) \leq \frac{\operatorname{diam}(E)}{2}.$$

As there are sets with positive capacity but zero area, such as line segments, one cannot expect to find an upper bound in terms of area. However, a lower bound can be approximated.

Theorem 4.19. *If E is a compact subset of $\mathbb{C}$, then*

$$c(E) \geq \sqrt{\frac{\operatorname{area}(E)}{\pi}}$$

The proof for the theorem above follows from the proof for the Ahlfors-Beurling Inequality, found in [19, Lemma 5.3.6].

Chapter 5

Numerical methods for computing logarithmic capacity

As mentioned in previous chapters, the logarithmic capacity can only be computed analytically for a few simple sets. Accurate approximations are rare, and in this chapter we will introduce some numerical methods for achieving this. There are many methods for this, one of the more recent is a least-squares method developed by Malik Younsi and Thomas Ransford which produces rigorous upper and lower bounds which converge to the true value of capacity [21]. Due to the close relationship between conformal mapping and logarithmic capacity, numerical methods for approximating conformal maps, such as Schwarz-Christoffel mapping, can also be utilized. There are several methods for numerical approximation of conformal maps, two of these being Marshalls Zipper-algorithm [13] and Koebes algorithm [18, Part 6.1]. Another important method is the FEM-algorithm, which is a method based on the harmonic conjugate function and the properties of quadrilaterals [8, 9]. This method is also particularly useful in the case of unbounded areas [10].

In this chapter, the following methods for calculating logarithmic capacity will be presented: the Dijkstra-Hochstenbach method, the Rostand method, the Ransford-Rostand method and a method using Schwarz-Christoffel mapping, which has been implemented as a toolbox for MATLAB by Tobin A. Driscoll.

5.1 Dijkstra-Hochstenbach method

W. Dijkstra and M.E. Hochstenbach present in their article [4] an algorithm for numerically estimating the logarithmic capacity of a set in $\mathbb{C}$ that is bounded by a finite set of Jordan curves. The algorithm requires the solution of a boundary integral equation with Dirichlet boundary data, and the solution can be achieved through a collocation approach or a Galerkin approach. For convenience sake, we will restrict us to simply connected sets, even though the algorithm is also valid for disconnected sets.

First, we will define the *single layer operator*.

Definition 5.1. Let Γ be the boundary of a closed bounded set E of $\mathbb{C}$. The single layer operator is defined as a boundary integral operator $\mathcal{V}\colon \mu \mapsto p_\mu$,

$$(\mathcal{V}\mu)(x) := \int_\Gamma \log \frac{1}{|x-y|} \mu(y)\, d\Gamma_y,$$

where $x \in \Gamma$.

$$\sum_{k=1}^{N} \mu_k \int_{\Gamma_k} \log \frac{1}{|x-y|}\, d\Gamma_y = f(x), \tag{5.2}$$

where $x \in \Gamma$.

5.1.1 Collocation approach

The collocation approach is the method most commonly used to discretize boundary integral equations like (5.2). The elements of the matrix it yields are evaluated by a single evaluation of an integral. A drawback to the method is that in general, the resulting matrices are asymmetric.

We begin by choosing nodes x^p, so that $x^p := (x_p + x_{p+1})/2$. We substitute x with x^p in (5.2), and get

$$\sum_{k=1}^{N} \mu_k \int_{\Gamma_k} \log \frac{1}{|x^p-y|}\, d\Gamma_y = f(x^p) := f_p, \tag{5.3}$$

where $p = 1, \ldots, N$.

We rewrite this in matrix-vector notation

$$A\mu = f,$$

where

$$\begin{aligned} A_{pk} &:= \int_{\Gamma_k} \log \frac{1}{|x^p-y|}\, d\Gamma_k, \\ \mu &:= [\mu_1, \ldots, \mu_N]^T, \\ f &:= [f_1, \ldots, f_N]^T. \end{aligned}$$

Algorithm 5.4. The following steps are needed in the collocation approach:

1. Compute the matrix A.
2. Construct a vector of ones with length N, $\mathbf{1} = 1, \dots, 1]^T$.
3. Construct a vector containing the lengths of the boundary elements, $\mathbf{l} = [|\Gamma_1|, \dots, |\Gamma_N|]^T$.
4. Compute
$$C = \exp\left(\frac{1}{\mathbf{l}^T(A\backslash\mathbf{1})}\right),$$
where C denotes the capacity.

5.1.2 Galerkin approach

The Galerkin approach is another well-known method to discretize boundary integral equations. In this case, the resulting matrices are symmetric, but the computation of a matrix element requires the evaluation of a double integral. Define the shape function ϕ_i by

$$\phi_i(x) = \begin{cases} 1 & \text{at } \Gamma_i, \\ 0 & \text{elsewhere,} \end{cases}$$

for $i = 1, \dots, N$.

$$\sum_{j=1}^{N}\sum_{k=1}^{N} \mu_k \int_{\Gamma_j} \phi_i(x) \int_{\Gamma_k} \log\frac{1}{|x-y|}\, d\Gamma_y\, d\Gamma_x = \sum_{j=1}^{N} \int_{\Gamma_j} f(x)\, d\Gamma_x \tag{5.5}$$

for $i = 1, \dots, N$.

$$\sum_{k=1}^{N} \mu_k \int_{\Gamma_i} \int_{\Gamma_k} \log\frac{1}{|x-y|}\, d\Gamma_y\, d\Gamma_x = f_i|\Gamma_i|, \tag{5.6}$$

for $i = 1, \dots, N$.

We rewrite this in matrix-vector notation

$$B\mu = g,$$

where

$$B_{ik} := \int_{\Gamma_i} \int_{\Gamma_k} \log\frac{1}{|x-y|}\, d\Gamma_y\, d\Gamma_x, \tag{5.7}$$

$$g := [f_1|\Gamma_1|, \dots, f_N|\Gamma_N|]^T. \tag{5.8}$$

Algorithm 5.9. The following steps are needed in the Galerkin approach:

1. Compute matrix B.
2. Construct a vector containing the lengths of the boundary elements, $\mathbf{l} = [|\Gamma_1|, \ldots, |\Gamma_N|]^T$.
3. Compute
$$C = \exp\left(-\frac{1}{\mathbf{l}^T(B\backslash\mathbf{l})}\right),$$
where C denotes the capacity.

5.2 Rostand method

J. Rostand describes in his article [23] the construction of an algorithm for estimating the capacity of nice compact subsets of the plane $\mathbb{C}$. The method is based on the relation between the capacity and Green's function (Theorem 4.4) together with Keldyš's theorem of uniform approximation. In order to get a good approximation, Rostand employs a least-square technique, which is easy to implement and has the advantage of giving a bound on the error.

Theorem 5.10 (Keldyš)**.** *Let E be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus E$ has a finite number of components. Let Λ be a subset of $\mathbb{C} \setminus E$ which has at least one point in each bounded component of $\mathbb{C} \setminus E$. Then each continuous function $\varphi \colon \partial E \to \mathbb{C}$ can be uniformly approximated on ∂E by functions of the form*

$$\operatorname{Re} q(z) + a \log |r(z)|,$$

where $a \in \mathbb{R}$, and q and r are rational functions such that the poles of q, r and $1/r$ all lie in $\Lambda \cup \{\infty\}$.

Keldyš's theorem says that it is possible to approximate a continuous function on the boundary of a compact set E by functions that are harmonic in the neighborhood of E. This will be used when constructing the numerical method.

5.2.1 Construction

To construct a numerical method, we will need the following hypothesis.

Hypothesis 5.11. *Let E be a compact subset of $\mathbb{C}$ with a finite number of components E_j ($j = 0, \ldots, n$) so that for each j, the interior $\operatorname{int}(E_j)$ is non-empty and has closure E_j.*

The construction of a numerical method proceeds in 4 steps.

1) The first step transforms the problem of estimating the capacity of E into a problem of uniform approximation of $\log|z|$ on ∂E by functions of the class $\mathfrak{F}^*_\Lambda$, where $\mathfrak{F}^*_\Lambda := \{z \mapsto u(1/z) : u \in \mathfrak{F}_\Lambda\}$. Here $\mathfrak{F}_\Lambda$ denotes the class of all functions of the form $\operatorname{Re} q(z) + a \log|r(z)|$, where $a \in \mathbb{R}$ and where q and r are rational functions such that the poles of q, r and $1/r$ are in $\Lambda := \{1/\lambda_1, \ldots, 1/\lambda_n\}$, $\lambda_1, \lambda_2, \ldots \lambda_n \in \mathbb{C}$, so that $\lambda_j \in \operatorname{int}(E_j)$, $j = 1, \ldots, n$. In conclusion, it shows that

$$c(E) = \tilde{\varepsilon}|r(0)|^a e^{\operatorname{Re} q(0)},$$

where $\tilde{\varepsilon} \in [e^{-\varepsilon}, e^{\varepsilon}]$, $\varepsilon > 0$.

Keldyš's theorem shows that this type of construction is possible, but in practice the theorem does not give a constructive way to obtain the desired approximation. The geometry of ∂E and the choice of λ_j will, in fact, play an important role in finding a good approximation of $\log|z|$.

2) In this step, we will restrict ourselves to a subspace of $\mathfrak{F}^*_\Lambda$ and a finite number of points of ∂E.

The class $\mathfrak{F}^*_\Lambda(N_0, N_1)$, with fixed integers N_0 and N_1, is defined as being the subclass of $\mathfrak{F}_\Lambda$. A typical function u^* of $\mathfrak{F}^*_\Lambda(N_0, N_1)$ can be written as

$$u^*(z) = b_0 + \sum_{j=1}^{N_1} \operatorname{Re} \frac{c_j + id_j}{z_j} + \sum_{k=1}^{n} \sum_{j=1}^{N_0} \operatorname{Re} \frac{(c_{jk} + id_{jk})z^j}{(z - \lambda_k)^j} + \sum_{k=1}^{n} b_k \log \left| \frac{z - \lambda_k}{z} \right|,$$

where all the constants b_k, c_j, d_j, c_{jk} and d_{jk} are real.

In order to approximate $\log|z|$ by functions from $\mathfrak{F}^*_\Lambda(N_0, N_1)$ we take an N-point discretization of the boundary of E, say $\{z_l \in \partial E : l = 1, \ldots, N\}$. N must be large enough, in fact $N \geq 1 + 2N_1 + n(2N_0 + 1) := M$. Using a least-square method we will determine the values for the constants b_k, c_j, d_j, c_{jk} and d_{jk}, for which $*u$ is as close as possible to $\log|z|$ on $\{z_1, \ldots, z_N\}$.

We define an $N \times M$ real matrix A by the following: for $k = 1, \ldots, N$, we have

$$\begin{aligned} A_l^1 &:= (1), \\ A_l^{2,N_1+1} &:= (\operatorname{Re} z_l^{-1}, \ldots, \operatorname{Re} z_l^{-N_1}), \\ A_l^{N_1+2,2N_1+1} &:= (-\operatorname{Im} z_l^{-1}, \ldots, -\operatorname{Im} z_l^{-N_1}); \end{aligned}$$

for $k = 1, \ldots, n$, we have

$$A_l^{2N_1+2+(k-1)N_0, 2N_1+1+kN_0} := \left(\mathrm{Re}\left(\frac{z_l}{z_l - z_k}\right)^1, \ldots, \mathrm{Re}\left(\frac{z_l}{z_l - z_k}\right)^{N_0} \right),$$

$$A_l^{2N_1+2+(n+k-1)N_0, 2N_1+1+(k-n)N_0} := \left(-\mathrm{Im}\left(\frac{z_l}{z_l - z_k}\right)^1, \ldots, \mathrm{Im}\left(\frac{z_l}{z_l - z_k}\right)^{N_0} \right),$$

and finally

$$A_l^{2N_1+2+2nN_0, 2N_1+1+n+2nN_0} := \left(\log\left|\frac{z_l - \lambda_1}{z_l}\right|, \ldots, \log\left|\frac{z_l - \lambda_n}{z_l}\right| \right).$$

We also define b as a column vector of dimension N whose lth component is the real number $\log|z_l|$. Finally, let x be an M-dimensional vector whose entries are

$$\begin{aligned}
&b_0, \\
&c_1, \ldots, c_{N_1}, \\
&d_1, \ldots, d_{N_1}, \\
&c_{11}, \ldots, c_{N_0 1}, \ldots, c_{1n}, \ldots, c_{N_0 n}, \\
&d_{11}, \ldots, d_{N_0 1}, \ldots, d_{1n}, \ldots, d_{N_0 n}, \\
&b_1, \ldots, b_n.
\end{aligned}$$

3) If we consider x to be a variable vector, then the goal is to minimize the function

$$g(x) := \|Ax - b\|_2,$$

where $\|\cdot\|_2$ is the standard Euclidean norm. The solutions are characterized by

$$A^T Ax = A^T b.$$

We can explicitly compute the unique solution $\tilde{x}$ with minimal norm. In fact, $\tilde{x} = A^+ b$, where A^+ is the *pseudoinverse*, or specifically, the *Moore-Penrose pseudoinverse* of A.

4) The final step in the construction is to combine the previous steps. First, we will compute the singular values decomposition of A. Next, we will apply step 3 to obtain a vector $\tilde{x}$ that satisfies

$$\|A\tilde{x} - b\|_2 \leq \|Ax - b\|_2$$

for all vectors $x \in \mathbb{R}^M$. Thus, the function

$$(5.12)\quad \tilde{u}^*(z) = \tilde{b}_0 + \sum_{j=1}^{N_1} \operatorname{Re} \frac{\tilde{c}_j + i\tilde{d}_j}{z^j} + \sum_{k=1}^{n}\sum_{j=1}^{N_0} \operatorname{Re} \frac{(\tilde{c}_{jk} + i\tilde{d}_{jk})z^j}{(z-\lambda_k)^j} + \sum_{k=1}^{n} \tilde{b_k} \log\left|\frac{z-\lambda_k}{z}\right|$$

of $\mathfrak{F}^*_\Lambda(N_0, N_1)$ is a good approximation of $\log|z|$ at each z_l of ∂E.

It has already been shown, that

$$c(E) = \tilde{\varepsilon} e^{\tilde{u}^*(\infty)},$$

where $\tilde{\varepsilon} \in [e^{-\varepsilon}, e^{\varepsilon}]$, and

$$\varepsilon = \left\|\tilde{u}^*(z) - \log|z|\right\|_{\partial E}.$$

From equation (5.12) follows that

$$c(E) = \tilde{\varepsilon} \exp\left(\tilde{b}_0 + \sum_{k=1}^{n}\sum_{j=1}^{N_0} \tilde{c}_{jk}\right).$$

Connected sets In the case where E is a connected set, several expressions in the method above are simplified. Since Λ is empty, the problem reduces to approximating $\log|z|$ by functions of the type

$$u^*(z) = b_0 + \sum_{j=1}^{N_1} \operatorname{Re} \frac{c_j + id_j}{z^j}.$$

The matrix A is now

$$A = \begin{pmatrix} 1 & \operatorname{Re} z_1^{-1} & \cdots & \operatorname{Re} z_1^{-N_1} & -\operatorname{Im} z_1^{-1} & \cdots - \operatorname{Im} z_1^{-N_1} \\ 1 & \operatorname{Re} z_2^{-1} & \cdots & \operatorname{Re} z_2^{-N_1} & -\operatorname{Im} z_2^{-1} & \cdots - \operatorname{Im} z_2^{-N_1} \\ \vdots & & & & & \vdots \\ 1 & \operatorname{Re} z_N^{-1} & \cdots & \operatorname{Re} z_N^{-N_1} & -\operatorname{Im} z_N^{-1} & \cdots - \operatorname{Im} z_N^{-N_1} \end{pmatrix},$$

and the vectors x and b look like

$$x^T = (b_0, c_1, \ldots, c_{N_1}, d_1, \ldots, d_{N_1}),$$
$$b^T = (\log|z_1|, \ldots, \log|z_N|).$$

Therefore, the capacity of E is given by

$$c(E) = \tilde{\varepsilon} e^{\tilde{b_0}}.$$

Theoretical bound for error A theoretical bound for the error can be estimated to

$$\begin{aligned}\varepsilon &= \left\| \tilde{u}^*(z) - \log|z| \right\|_{\partial E} \\ &\leq \max_{l=1,\dots,N} \left\{ \left| \tilde{u}^*(z_l) - \log|z_l| \right| + \delta \left(\sum_{j=1}^{N_1} \frac{j|\tilde{c}_j + i\tilde{d}_j|}{(|z_l| - \delta)^{j+1}} + \sum_{k=1}^{n} \sum_{j=1}^{N_0} \frac{j|\lambda_k||\tilde{c}_{jk} + i\tilde{d}_{jk}|(|z_l| + \delta)^{j-1}}{(|z_l - \lambda_k| - \delta)^{j+1}} \right.\right. \\ &\qquad \left.\left. + \sum_{k=1}^{n} \frac{|\tilde{b}_k||\lambda_k|}{(|z_l| - \lambda)(|z_l - \lambda_k| - \delta)} + \frac{1}{(|z_l| - \delta)} \right) \right\}.\end{aligned}$$

In the case of connected sets, the bound for the error is

$$\varepsilon \leq \max_{l=1,\dots,N} \left\{ \left| \tilde{u}^*(z_l) - \log|z_l| \right| + \delta \sum_{j=1}^{N_1} \frac{j|\tilde{c}_j + i\tilde{d}_j|}{(|z_l| - \delta)^{j+1}} \right\}.$$

5.2.2 Practical implementation

We will now use the method described to calculate the capacity of some sets.

Example 5.13 (The disk). Consider a closed disk of radius $r = 3$, that is, the compact set

$$E := z \in \mathbb{C} : |z| \leq 3.$$

We know from previous calculations that the capacity of a disk is its radius, so $c(E) = 3$. This value will allow us to verufy our results.

Let $f\colon [0,1] \to \mathbb{C}$ be the parametrization of ∂E given by $f(t) := 3e^{2\pi i t}$. Fix N and N_1, and consider the boundary points $z_l = f(l/N)$ for $l = 1, \dots, N$. As E is connected, we can use the reduced model for connected sets to compute each entry for the matrix A and the vector b. This has been implemented by the functions `makeA` and `makeb` respectively, found in Appendix A.1. Now, we can solve the problem $\min_x \|Ax - b\|_2$. We will do this with the MATLAB operator \, which will automatically give the least squares solution if the linear system

$$A^T A x = A^T b$$

does not have a direct solution. The capacity given by the algorithm, when $N_1 = 5$ and $N = 20$, is $c(E) = 2.999999999999998$, which has an error less than 1.8×10^{-15}. With $N_1 = 5$ and $N = 30$ the error is less than 8.9×10^{-16}.

Example 5.14 (The square). Consider E to be the square $[-1,1] \times [-1,1]$. According to Table 4.1, the capacity for this is

$$c(E) = \frac{\Gamma(1/4)^2}{4\pi^{3/2}} \cdot 2 \approx 1.180340599016096.$$

Let $f\colon [0,1] \to \mathbb{C}$ be the parametrization

$$f(t) := \begin{cases} -8it + 1 + i & \text{if } 0 \leq t \leq \frac{1}{4} \\ -8t + 3 - i & \text{if } \frac{1}{4} < t \leq \frac{1}{2} \\ 8it - 1 - 5i & \text{if } \frac{1}{2} < t \leq \frac{3}{4} \\ 8t - 7 + i & \text{if } \frac{3}{4} < t \leq 1 \end{cases}.$$

We choose as before the points $z_l = f(l/N)$, but we have to make sure that N is a multible of 4 if we want the corners of E to be in the discretization. The rest of the algorithm is exactly the same. The results have been compiled in Table 5.1.

The rate of convergence is noticeable slower in the case of the square due to the fact that, even though $\log|f(t)|$ is continuous on $[0,1]$, it is not differentiable in the corners of the square, that is, at $t = 0, 1/4, 1/2, 3/4, 1$. The approximation is more difficult in those points.

N_1	N	Capacity	Error
5	40	1.1616418	0.01870
5	80	1.1597769	0.02056
5	160	1.1593129	0.02103
10	80	1.1656674	0.01467
20	80	1.1719398	0.00840
30	120	1.1731882	0.00715
40	120	1.1747906	0.00555
50	160	1.1751579	0.00518

Table 5.1: Computation of capacity of a square with side $h = 2$.

5.3 Ransford-Rostand method

Ransford and Rostand introduce in [20] a method for computing upper and lower boundaries for the logarithmic capacity of a compact set. If the set has the *Hölder continuity property* then the bounds converge to the value of the capacity.

A compact set E set is said to have the Hölder continuity property if its Green's function g_E exists and satisfies

$$|g_E(z_1, \infty) - g_E(z_2, \infty)| \leq A|z_1 - z_2|^{\alpha},$$

where $z_1, z_2 \in \mathbb{C}$, and A and α are positive constants.

In order to compute the capacity, we will convert the problem to the calculation of certain matrix games.

Definition 5.15. Fix a postitive integer n, and set $\Delta_n = \{(t_1, \dots, t_n) : t_j \geq 0, \sum_j t_j = 1\}$. Given an $n \times n$ matrix $h = (h_{ij})$, write

$$M(h) = \min_{s \in \Delta_n} \max_{t \in \Delta_n} \sum_{i,j} h_{ij} s_i t_j = \max_{t \in \Delta_n} \min_{s \in \Delta_n} \sum_{i,j} h_{ij} s_i t_j.$$

Since a convex combination of numbers always lies between their maximum and minimum, we also have

$$M(h) \min_{s \in \Delta_n} \max_{j} \sum_{i} h_{ij} s_i = \max_{t \in \Delta_n} \min_{i} \sum_{j} h_{ij} t_j.$$

Let E be a compact subset of $\mathbb{C}$ whose capacity we wish to compute and suppose that we have the compact subsets $F_1, \dots, F_n$ of $\mathbb{C}$. Then

$$\begin{cases} E \subset F_1 \cup \cdots \cup F_n, \\ c(E \cap F_i) \geq \delta \, (i = 1, \dots n), \end{cases} \tag{5.16}$$

where $\delta \geq 0$ is a real number.

Define the symmetric $n \times n$-matrices a and b by

$$a_{ij} := \log \frac{1}{\operatorname{diam}(F_i \cup F_j)}, \quad b_{ij} := \log \frac{1}{\max(\delta, \operatorname{dist}(F_i, F_j))}, \tag{5.17}$$

where diam denotes the diameter of a set and dist the distance between two sets.

The method for computing the capacity of E using upper and lower bounds is based on the following theorem.

Theorem 5.18. *Let E be a compact subset of $\mathbb{C}$. Suppose that $F_1, \dots, F_n, \delta$ satisfy the conditions in* (5.16) *and that a and b are defined as in* (5.17). *Then*

$$M(a) \leq \log \frac{1}{c(E)} \leq M(b).$$

5.3.1 Computing $M(h)$

The method described depends on being able to compute $M(h)$ for symmetric $n \times n$ matrices h. There are several ways to do this, one such is by reformulating the complex non-linear programming problem stated in 5.15,

$$M(h) = \min_{s \in \Delta_n} \max_{j} \sum_{i} h_{ij} s_i,$$

into a linear programming problem by use of an auxiliary variable T.

However, solving large dense linear problems is time-consuming, and in some cases $M(h)$ can be obtained by solving a linear equation instead of a linear program. Another method for bypassing the linear program is to investigate the upper and lower bounds for $M(h)$, though while this method is faster it will yield worse bounds. For example, due to the fact that

$$\max_j \sum_i t_i h_{ij} \geq \sum_{i,j} t_i t_j h_{ij}$$

for all $t \in \Delta_n$, $M(h)$ is bounded below by the quadratic program

$$Q(h) := \min_{t \in \Delta_n} \sum_{i,j} t_i t_j h_{ij}.$$

Thus, $e^{-Q(a)}$ is an upper bound for the capacity $c(E)$, and a quadratic program can be solved faster than a linear program.

5.3.2 Other methods

In stead of calculating matrices a and b directly, it may be easier to compute some other symmetric matrix that satisfies $a \leq h \leq b$. Then $M(a) \leq M(h) \leq M(b)$.

The fact that $M(a)$ and $M(b)$ converge to $\log 1/c(E)$ means that $M(h)$ will converge to the same limit at least as quickly, and sometimes even quicker.

One method utilizing this idea, is *the midpoint method*. For each i, pick a point $x_i \in F_i$ so that $x_i \neq x_j$ for all $i \neq j$. Now, set

$$h_{ij} := \log \frac{1}{\max(\delta, |x_i - x_j|)}.$$

We know that

$$\operatorname{dist}(F_i, F_j) \leq |x_i - x_j| \leq \operatorname{diam}(F_i \cup F_j) \tag{5.19}$$

for all i, j.

Since

$$\operatorname{diam}(F_i \cup F_j) \geq \operatorname{diam}(F_i) \geq c(F_i) \geq c(F_i \cap E) \geq \delta$$

for all i, j, we get that $a \leq h \leq b$ when we take the maximum with δ in (5.19).

Thus, $e^{-M(h)}$ gives an approximation of $c(E)$.

5.4 Schwarz-Christoffel Mapping

The Schwarz-Christoffel transformation and its variations yield formulas for conformal maps from standard regions to the interiors or exteriors of possibly unbounded polygons.

The idea behind the Schwarz-Christoffel transformation is that a conformal transformation f may have a derivative that can be expressed as

$$f' = \prod f_k$$

for certain canonical functions f_k. Virtually all conformal transformations, whose analytical forms are known, are Schwarz-Christoffel maps, though sometimes disguised by an additional change of variables.

In this section, a generalized polygon Γ is defined by a collection of vertices $w_1, \ldots, w_n$ on the extended complex plane, and interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$. For indexing purposes, it is also convenient to define that $w_{n+1} = w_1$ and $w_0 = w_n$. The vertices are given in counterclockwise order with respect to the interior of the polygon. If $|w_k| < \infty$, then $o < \alpha_k \leq 2$. If $w_k = \infty$, then the definition of the interior angle is applied to the Riemann sphere and $-2 \leq \alpha_k \leq 0$. In addition,

$$\sum_{k=1}^{n}(1 - \alpha_k) = 2.$$

The following is a fundamental theorem of Schwarz-Christoffel mapping, which describes the mapping of the upper half-plane onto the interior of a polygon. The proof for this theorem can be found in [6, Theorem 2.1].

Theorem 5.20 (**Schwarz-Cristoffel formula for a half-plane**). *Let P be the interior of a polygon Γ with vertices $w_1, \ldots, w_n$ and interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$ in counderclockwise order. Let f be any type of conformal map from the upper half-plane H^+ to P, with $f(\infty) = w_n$. Then*

$$f(z) = a + c \int^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1}\, d\zeta \tag{5.21}$$

for some complex constants a and c, where $w_k = f(z_k)$ for $k = 1, \ldots, n-1$.

The Schwarz-Christoffel formula (5.21) for a half-plane can be adapted to maps for different regions, to exterior map, to maps with branch points, to doubly connected regions, to regions bounded by circular arcs, and even to piecewise analytic boundaries. One of the simplest adaptations has the unit disk as its domain.

Theorem 5.22 (**Schwarz-Cristoffel formula for a unit disk**)**.** *Let P be the interior of a polygon Γ with vertices $w_1, \ldots, w_n$ and interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$ in counterclockwise order. Let f be any type of conformal map from the unit disk B to P. Then*

$$f(z) = a + c\int^{z} \prod_{k=1}^{n} (1 - \frac{\zeta}{z_k})^{\alpha_k - 1}\, d\zeta \tag{5.23}$$

for some complex constants a and c, where $w_k = f(z_k)$ for $k = 1, \ldots, n-1$.

By modifying (5.23), one can also find a function which maps the unit disc into the exterior of a polygon:

$$f(z) = a + c\int^{z} \prod_{k=1}^{n} \zeta^{-2}(1 - \frac{\zeta}{z_k})^{1-\alpha_k}\, d\zeta. \tag{5.24}$$

This equation is important, because $|c|$ is in fact the value for the capacity. [6, Sections 4.4 and 5.8]

The Schwarz-Christoffel formula is mathematically appealing, but problematic in practice. In order to compute a map, one must find the prevertices by solving a system of nonlinear equations which is analytically intracable in most cases. In addition, the integral in (5.21) rarely has a simple closed form. Finally, it is usually impossible to invert f explicitly, and for these reasons, calculations with Schwarz-Christoffel maps must generally be done on a computer.

5.4.1 Schwarz-Christoffel Toolbox for MATLAB

The Schwarz-Christoffel Toolbox for MATLAB is an implementation of the Schwarz-Christoffel formulas for maps from the disk, half-plane, strip and rectangle domains to polygon interiors, and from the disk to polygon exteriors. Disk mapping using the cross-ratio formula has also been implemented in the program. By tinkering with the provided routines, one can also implement maps for gearlike regions and Riemann surfaces.

From an algorithmic standpoint, the variations of the Schwarz-Christoffel formula, some of which were discussed in the previous section, are pretty similar. The challenge comes from computing integrals of the form (5.21), solving the parameter problem and, if desired, computing the inverse of (5.21).

The toolbox defines polygons, and the maps to them, as named objects. Once the objects are created, they can be manipulated by common MATLAB functions, such as `plot` and `inv`. Polygons are created either by an interactive drawing or from a list of vertices. Once a polygon is given, one can construct a map to the region defined by the polygon. [5]

To calculate the capacity of an object with the help of the Schwarz-Christoffel Toolbox, one must first construct a polygon object with the command `polygon(W)`, where `W` are the vertices of the object in question. Using the `extermap` command, a Schwarz-Christoffel exterior map object will be constructed for the polygon. After this, one can simply use the `capacity` command to find out the logarithmic capacity of the object. To illustrate this process, consider the following example.

Example 5.25. A 2×2-square has its corners in $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$. In the following program, these points will be given using their complex representations.

```
>> format long
>> p=polygon([1+i -1+i -1-i 1-i]);
>> f=extermap(p);
>> capacity(f)

ans =

   1.180340599090706
```

The exact value of the capacity for this square, as defined in Table 4.1, is

$$\frac{\Gamma^2(1/4)}{4\pi^{3/2}} \cdot 2 \approx 1.180340599016096$$

Thus, the error for the value given by the Schwarz-Christoffel Toolbox is approximately $7.4610 \cdot 10^{-11}$. Comparing this value to the error values given by the Rostand method in Table 5.1, where the smallest error was 0.00518, the Schwarz-Christoffel method appears to be much more accurate in the case of a square.

For some more examples and comparisons, see Table 5.2.

Area	Exact	S-C	Error
disk with radius $r = 1$	1.0	0.999918054482060	$8.1946 \cdot 10^{-5}$
square with side $h = 2$	1.180340599016096	1.180340599090706	$7.4610 \cdot 10^{-11}$
line segment of length $h = 2$	0.5	0.500002729695464	$2.7297 \cdot 10^{-6}$
half-disk of radius $r = 1$	0.769800358919501	0.769740915989960	$5.9443 \cdot 10^{-5}$

Table 5.2: Values for capacities computed with the Schwarz-Christoffel toolbox.

Appendix A

Implementation of Rostand's method

The following programs have been implemented in MATLAB based on the program found in [23].

A.1 Connected sets

```
format long
N=10; M=3;
f=@(t) 3*exp(2*pi*1i*t);
A=makeA(f,N,M);
b=makeb(f,N);
x=(A'*A)\(A'*b');
cap=exp(x(1));
disp('Capacity:')
disp(cap)
disp('Error:')
disp(abs(3-cap))

function A=makeA(f,N,M)
A=ones(N,1+2*M);
for k=1:N
    A(k,2:(M+1))=real(f(k/N).^(-1:-1:-M));
    A(k,(M+2):(2*M+1))=-imag(f(k/N).^(-1:-1:-M));
end

function b=makeb(f,N)
b=zeros(1,N);
for k=1:N
```

```
    b(k)=log(abs(f(k/N)));
end
```

A.2 Disconnected sets

```
function A=makeA2(f,N,N0,N1,lam)
A=ones(N,1+2*N1);
n=length(lam);
for l=1:N
    zl=f(3*l/N);
    A(l,2:(N1+1))=real(zl.^(-1:-1:-N1));
    A(l,(N1+2):(2*N1+1))=-imag(zl.^(-1:-1:-N1));
    for k=1:n
        A(l,(2*N1+2+(k-1)*N0):(2*N1+1+k*N0))=real(zl/(zl-lam(k))).^(1:N0);
        A(l,(2*N1+2+(n+k-1)*N0):(2*N1+1+(k+n)*N0))=-imag(zl/(zl-lam(k))).^(1:N
    end
    A(l,(2*N1+2+2*n*N0):(2*N1+1+n+2*n*N0))=log(abs((zl-lam)./zl));
end

function b=makeb2(f,N)
b=zeros(1,N);
for l=1:N
    zl=f(3*l/N);
    b(l)=log(abs(zl));
end
```

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