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NUMERICAL METHODS FOR HYPERBOLIC AND PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. An analysis by energy methods is given for fully discrete numerical methods for time-dependent partial integro-differential equations. Stability and error estimates are derived in H^1 and L_2. The methods considered pay attention to the storage needs during time-stepping.

1. Introduction. The main purpose of this paper is to study numerical methods for the solution of the hyperbolic integro-differential equation

$$u_{tt} + A(t)u = \int_0^t B(t,s)u(s)\,ds + f(t), \quad \text{in } \Omega \times J, \tag{1.1a}$$

together with the initial and boundary conditions

$$\begin{gathered} u = 0, \quad \text{on } \partial\Omega \times J, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \end{gathered} \tag{1.1b}$$

and for analogous problems for equations of parabolic type. Here Ω is a bounded domain in R^d with smooth boundary $\partial\Omega$, J denotes the interval $[0,T]$ with a fixed upper limit T, $A(t)$ is a self-adjoint, uniformly positive definite uniformly elliptic second order differential operator, and $B(t,s)$ is a second order partial differential operator, both with smooth coefficients. Problems of this nature, and nonlinear versions thereof, occur, e.g., in visco-elasticity, cf. Renardy, Hrusa, and Nohel [**5**] and references therein.

The numerical methods considered in this paper will be obtained by discretizing in space by a Galerkin finite element method, followed by a finite difference and quadrature scheme for the time stepping. They

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will be sought, for discrete time levels $t_n = nk$, with k the time step, in a finite dimensional space $S_h \subset H_0^1 = H_0^1(\Omega)$ belonging to a family such that, for a fixed integer $r \geq 2$, we have

$$\min_{\chi \in S_h} \{||v-\chi|| + h||v-\chi||_1\} \leq Ch^i||v||_i, \quad \text{for } v \in H_0^1 \cap H^i, \quad 1 \leq i \leq r, \tag{1.2}$$

where $||\cdot||$ and $||\cdot||_i$ denote the norms in $L_2 = L_2(\Omega)$ and $H^i = H^i(\Omega)$, respectively. Such special methods have been applied and analyzed for both hyperbolic and parabolic integro-differential equations, but with a first order operator B in the memory term, in Yanik and Fairweather [**10**].

As a starting point for the discretization of (1.1), we formulate its semi-discrete analogue, based on a weak form of the initial boundary value problem. Letting $(\cdot,\cdot)$, $A(t;\cdot,\cdot)$ and $B(t,s;\cdot,\cdot)$ denote the inner product in L_2 and the bilinear forms on $H_0^1 \times H_0^1$ defined by the differential operators $A(t)$ and $B(t,s)$, we define the semi-discrete solution of (1.1) as the function $u_h : J \to S_h$ such that

$$\begin{gathered}(u_{h,tt},\chi) + A(t;u_h,\chi) = \int_0^t B(t,s;u_h(s),\chi)\,ds + (f(t),\chi), \\ \text{for } \chi \in S_h,\ t \in J, \\ u_h(0) = u_{0h}, \qquad u_{h,t}(0) = u_{1h},\end{gathered} \tag{1.3}$$

where u_{0h} and u_{1h} are appropriate approximations of u_0 and u_1 in S_h.

It was shown in Cannon, Lin, and Xie [**2**], see Lin, Thomée, and Wahlbini [**3**] (cf. also [**10**] in the case that B is of first order) that

$$||u_h(t) - u(t)|| + h||u_h(t) - u(t)||_1 \leq C(u)h^r.$$

A principal tool used in [**2, 3**] was a generalization of the elliptic, or Ritz, projection called the Ritz-Volterra projection $W : C(J;H_0^1) \to C(J;S_h)$, defined by

$$A(t;(W-u)(t),\chi) = \int_0^t B(t,s;(W-u)(s),\chi)\,ds, \quad \text{for } \chi \in S_h,\ t \in J. \tag{1.4}$$

The completely discrete methods we shall consider in this paper will be derived, as is common for the purely hyperbolic equations, essentially by replacing the time derivative in (1.3) by a difference quotient, and using a quadrature rule for the integral of type

$$\sum_{j=0}^{n} \omega_{nj} B(t_n, t_j; U^j, \chi) \approx \int_0^{t_n} B(t_n, s; u_h(s), \chi)\, ds,$$

where ω_{nj} are quadrature coefficients and U^j is the approximation of $u_h(t_j)$.

One of the difficulties involved in such a time-stepping scheme is that if $\omega_{nj} \neq 0$ for $j \leq n$, then all the values of U^j have to be retained, causing great demands for data storage. This is in contrast to the situation for a purely hyperbolic equation where only a fixed low number of time levels is involved at each time step, and the data can be discarded as the computation goes along. As a way around this difficulty, in the case of a parabolic integro-differential equation, it was proposed in Sloan and Thomée [**6**] that the quadrature be based on fewer points, thus reducing the number of time levels at which the data need to be saved. We shall thus consider, in particular, several quadrature methods for which many of the ω_{nj} vanish, and for which the accuracy of the scheme equals that of the scheme for the pure differential equation. We shall assume that the quadrature formulas have persistent dominated weights, i.e., that the quadrature weights ω_{nj} are such that for some sequence of nonnegative numbers ω_j, $j = 0, 1, 2, \ldots$, independent of n,

$$(1.5) \qquad |\omega_{nj}| \leq \omega_j, \quad \text{for } 0 \leq j < n, \quad \text{with } \sum_{j=0}^{n-1} \omega_j \leq C, \quad \text{for } t_n \in J,$$

and

$$(1.6) \qquad \sum_{i=j+1}^{n-1} |\omega_{i+1,j} - \omega_{ij}| \leq \omega_j, \quad \text{for } 0 \leq j < n-1,\ t_n \in J.$$

The latter condition means that the quadrature coefficients associated with a particular time point t_j do not change too much as the upper limit of the integral progresses. In the schemes considered below, this will happen only a fixed finite number of times. For brevity, we shall refer to schemes satisfying (1.5) and (1.6) as ω-stable below.

The program just described has been carried out for integro-differential equations of parabolic type in Thomée and Zhang [**9**] and Zhang [**11, 12**] (cf. also Thomée [**8**] for a survey). In these papers the elliptic operator was assumed time independent, which permitted the use of a spectral argument at a crucial point in the proofs. In contrast, in the present paper we rely on energy arguments, which are not correspondingly restrictive. For this reason, the present approach will improve the results also in the parabolic case as will be indicated below. Also, the earlier work did not use the Ritz-Volterra projection, which made the proofs somewhat more cumbersome (however, see Cannon and Lin [**1**] for a special case).

Fully discrete methods for the hyperbolic problem, without the aspect of data storage, were treated in [**10**], for B of first order, and in [**2**]. The analysis in the latter paper appears incomplete.

After presenting some preliminary material in Section 2, we shall begin by considering in Section 3 a class of time stepping schemes for (1.3) based on a first order, three level, backward difference approximation of $u_{h,tt}$, and show first stability and error estimates in the natural energy norm associated with hyperbolic equations. The error estimate will contain an as yet undetermined term which depends on the choice of the quadrature rule used. This will also yield an L_2-norm error bound which, modulo the quadrature related term, is of optimal order $O(h^r + k)$ relative to (1.2). This analysis, however, will impose somewhat artificial restrictions in the choice of discrete initial data, and we therefore develop an analysis which shows optimal order results in L_2 also in this regard. The regularity assumptions will also be somewhat reduced in the latter approach.

In Section 4 we complete the above error analysis by bounding the quadrature error for three different choices of quadrature formulas which are consistent with the order of accuracy of the difference approximation, thus showing a total error bound of $O(h^r + k)$. They are based on the rectangle rule with time step k, the trapezoidal rule with time step of order $O(k^{1/2})$ and Simpson's rule with time step of order $O(k^{1/4})$; they require storage of the solution at a number of time levels of order $O(k^{-1})$, $O(k^{-1/2})$, and $O(k^{-1/4})$, respectively.

In Section 5 we derive similar results to those of Sections 3 and 4, now of order $O(h^r + k^2)$, for a symmetric approach to the differential equation part of (1.1), combined with quadrature rules based on midpoint and Simpson's type rules, with storage requirements of order $O(k^{-1})$ and $O(k^{-1/2})$, respectively. In this case the stability result will be expressed in terms of averages $(U^n + U^{n+1})/2$, which will slightly complicate the construction of the quadrature rules. Although the symmetric method has higher accuracy than the backward differencing method, and hence may be the more interesting one, we have treated the latter method first because it is, in some regards, the more straightforward. In return, we are able to draw on the techniques developed in Sections 3 and 4 in showing our results for the symmetric method.

In Section 6 we finally study corresponding results for a parabolic equation. Here we consider time stepping methods based on the backward Euler method, the Crank-Nicolson method, and a second order three level backward difference method, and combined with the appropriate quadrature rules. As mentioned above, the results in this section generalize those of [**11, 12**].

2. Preliminaries. In this section we shall review some material which will be used repeatedly below. Throughout this paper C will denote a constant independent of h, k, and u unless explicitly indicated. It is allowed to depend on T without explicit mention and is not necessarily the same at each occurrence.

We shall first consider the Ritz-Volterra projection $W : [0,T] \to S_h$ defined by (1.4). With u given, the existence of a unique such function follows from the theory of ordinary Volterra integral equations, cf. [**3**]. For the convenience of the reader we shall give a short proof of those results from [**3**] that we shall need in the sequel. For this purpose, we first recall the following lemma from Nitsche and Schatz [**4**]. Given a linear functional F on H_0^1, let

$$||F||_{-j} = \sup_{0 \neq \eta \in H_0^1 \cap H^j} \frac{|F(\eta)|}{||\eta||_j}, \quad \text{for } j = 1, 2.$$

Lemma 2.1. *Let $F : H_0^1 \to R$ be a linear functional and let $\Phi \in H_0^1$ satisfy*

$$A(t; \Phi, \chi) = F(\chi), \quad \text{for } \chi \in S_h.$$

Then

$$||\Phi||_1 \le C(||F||_{-1} + \inf_{\chi\in S_h} ||\Phi - \chi||_1)$$

and

$$||\Phi|| \le C\{||F||_{-2} + h(||F||_{-1} + \inf_{\chi\in S_h} ||\Phi - \chi||_1)\}.$$

Expressions such as $||\cdot||_{-2} + h||\cdot||_{-1}$ above will occur repeatedly in this paper. For brevity, we therefore introduce the notation

$$||\cdot||_{i,h} = ||\cdot||_i + h||\cdot||_{i+1}, \quad \text{for } i = -2, -1, 0. \tag{2.1}$$

Setting also

$$||u||_{(i,j)} = \sum_{l=0}^{j} ||D_s^l u||_i \quad \text{and} \quad ||u||_{(i,j),h} = \sum_{l=0}^{j} ||D_s^l u||_{i,h}, \tag{2.2}$$

we have the following estimates for the error in the Ritz-Volterra projection.

Proposition 2.2. *For any $j \ge 0$, we have*

$$||D_t^j(W-u)(t)||_{0,h} \le Ch^i\{||u(t)||_{(i,j)} + \int_0^t ||u||_{(i,j)}\, ds\},$$
$$\text{for } 1 \le i \le r,\ t \in J.$$

Proof. Setting $\rho = W - u$, by (1.4) we may apply Lemma 2.1 with $\Phi = \rho$, and

$$F(\eta) = \int_0^t B(t,s;\rho(s),\eta)\, ds.$$

It is easy to see that

$$||F||_{-l} \le C \int_0^t ||\rho||_{2-l}\, ds, \quad \text{for } l = 1, 2.$$

Using (1.2) we therefore obtain that

$$||\rho(t)||_1 \le C(h^{i-1}||u(t)||_i + \int_0^t ||\rho||_1\, ds)$$

and

$$||\rho(t)|| \le Ch\bigg(\int_0^t ||\rho||_1\,ds + h^{i-1}||u(t)||_i\bigg) + C\int_0^t ||\rho||\,ds.$$

Hence, by Gronwall's lemma, the desired estimate for $j = 0$ follows.

For $j = 1$, we differentiate (1.4) to get

$$A(t;\rho_t,\chi) = -A_t(t;\rho,\chi) + B(t,t;\rho,\chi) + \int_0^t B_t(t,s;\rho(s),\chi)\,ds.$$

Let $F(\chi)$ now be given by the right hand side of this equation. Then

$$||F||_{-l} \le C(||\rho(t)||_{2-l} + \int_0^t ||\rho||_{2-l}\,ds), \quad l = 1, 2,$$

which shows the estimates for ρ_t. The proof is completed by treating higher order derivatives in a similar way. □

We shall often use the following discrete version of Gronwall's lemma.

Lemma 2.3. *Let $\{\eta_n\}$ be a sequence of nonnegative numbers satisfying*

$$\eta_n \le \alpha_n + \sum_{j=0}^{n-1}\beta_j\eta_j, \quad \text{for } n \ge 0, \tag{2.3}$$

where $\{\alpha_j\}$ is a nondecreasing sequence and β_j are nonnegative. Then

$$\eta_n \le \alpha_n \exp\left(\sum_{j=0}^{n-1}\beta_j\right), \quad \text{for } n \ge 0.$$

Proof. The solution of (2.3) with $\alpha_n \equiv 1$ and equality is easily checked to be the following sum of the fundamental symmetric polynomials, namely

$$S_j = 1 + \sum_{l=0}^{j-1}\bigg(\sum_{0\le i_0<i_1<\cdots<i_l\le j-1}\beta_{i_0}\beta_{i_1}\cdots\beta_{i_l}\bigg).$$

By monotonicity of the α_n we hence have $\eta_n \le \alpha_n S_n$ and since $S_{n+1} = S_n + \beta_n S_n \le e^{\beta_n} S_n$, the desired result follows. □

In our proofs of L_2 estimates we shall utilize the operator $T_n = A_{n,h}^{-1} : S_h \to S_h$ where $A_{n,h} : S_h \to S_h$ is defined by

$$(A_{n,h}\phi, \chi) = A(t_n; \phi, \chi), \quad \text{for } \phi, \chi \in S_h.$$

Letting $\tilde{T}_n = A(t_n)^{-1}$ be its continuous analogue, we have, as is well known, cf. [**7**, p. 23],

$$||(T_n - \tilde{T}_n)\chi||_{0,h} \le Ch^2||\chi||, \quad \text{for } \chi \in S_h. \tag{2.4}$$

We shall also occasionally use the time dependent Ritz projection $R_h(t)$ onto S_h defined by

$$A(t; R_h(t)v - v, \chi) = 0, \quad \text{for } \chi \in S_h,\ t \in J.$$

From (1.2) and Lemma 2.1 (with $F \equiv 0$) it is immediate that

$$||R_h(t)v - v||_{0,h} \le Ch^i||v||_i, \quad \text{for } v \in H_0^1 \cap H^i,\ 1 \le i \le r. \tag{2.5}$$

3. Schemes for the hyperbolic equation based on backward differencing. In this section we shall study time-stepping methods for the semi-discrete equation (1.3) based on a three-level backward differencing method for the pure partial differential equation. The quadrature rules will be of the form

$$\sigma^n(g) = \sum_{j=0}^{n-1} \omega_{nj} g(t_j) \approx \int_0^{t_n} g(s)\, ds, \quad t_j = jk. \tag{3.1}$$

Thus, we define $U^n \in S_h$ by

$$(\bar{\partial}^2 U^n, \chi) + A_n(U^n, \chi) = \sigma^n(B_n(U, \chi)) + (f^n, \chi), \quad \text{for } \chi \in S_h,\ n \ge 2, \tag{3.2}$$

with U^0 and U^1 given. Here $\bar{\partial}$ denotes backward differencing, $A_n(\psi, \chi) = A(t_n; \psi, \chi)$, and $\sigma^n(B_n(U, \chi))$ is a shorthand notation for $\sum_{j=0}^{n-1} \omega_{nj} B(t_n, t_j; U^j, \chi)$.

We shall first derive a stability estimate in an energy norm which is a discrete analogue of the standard such norm for the wave equation. We then use this to derive a preliminary H^1 error estimate in which the initial data and the quadrature rule are yet to be specified, but which is otherwise of optimal order. Finally, we carry out a similar program for estimates in L_2.

For the purpose of later error estimates, we shall first derive a stability estimate for a slight generalization of (3.2), namely

$$(\bar{\partial}^2 U^n, \chi) + A_n(U^n, \chi) = \sigma^n(B_n(U, \chi)) + (f^n, \chi) + F^n(\chi), \quad \text{for } \chi \in S_h,\ n \geq 2, \tag{3.3}$$

where F^n is a linear functional on H_0^1. For ease of exposition we shall assume that σ^0 and σ^1, i.e., the quadrature formulae for $t = 0$ and k, are also given, although they do not appear in (3.2) or (3.3). Here, $\sigma^0 \equiv 0$ will always be taken and, as we shall see in our examples in Section 4, σ^1 will also have natural definitions.

The stability estimate will be stated in terms of a discrete energy norm defined by

$$|||\phi^n|||_1^2 = ||\bar{\partial}\phi^n||^2 + ||\phi^n||_1^2, \quad \text{for } n \geq 1.$$

Theorem 3.1. *Under the assumptions* (1.5) *and* (1.6) *that the quadrature rule* (3.1) *is ω-stable, we have for the solution of* (3.3)*, for $n \geq 2$, $t_n \leq T$,*

$$|||U^n|||_1 \leq C\bigg(||U^0||_1 + |||U^1|||_1 + k\sum_{m=2}^{n} ||f^m|| + ||F^2||_{-1} + k\sum_{m=3}^{n} ||\bar{\partial}F^m||_{-1}\bigg).$$

Proof. We choose $\chi = \bar{\partial}U^n$ in (3.3) to obtain

$$\begin{aligned}(\bar{\partial}^2 U^n, \bar{\partial}U^n) + A_n(U^n, \bar{\partial}U^n) &= \sigma^n(B_n(U, \bar{\partial}U^n)) \\ &\quad + (f^n, \bar{\partial}U^n) + F^n(\bar{\partial}U^n) \\ &= I_1^n + I_2^n + I_3^n.\end{aligned}$$

Note that

$$(\bar\partial^2 U^n, \bar\partial U^n) = \bar\partial ||\bar\partial U^n||^2/2 + k||\bar\partial^2 U^n||^2/2$$

and

$$A_n(U^n, \bar\partial U^n) = \bar\partial(A_n(U^n, U^n))/2 - (\bar\partial A_n)(U^{n-1}, U^{n-1})/2 + kA_n(\bar\partial U^n, \bar\partial U^n)/2 \tag{3.4}$$

where $\bar\partial A$ denotes the backward difference quotient of A with respect to its first argument.

Multiplying both sides of (3.3) by $2k$ and summing from $n = 2$ to N, we obtain, with c a positive constant,

$$||\bar\partial U^N||^2 + c||U^N||_1^2 \le ||\bar\partial U^1||^2 + C||U^1||_1^2 + k|2\sum_{n=2}^{N}(I_1^n + I_2^n + I_3^n) + (\bar\partial A_n)(U^{n-1}, U^{n-1})|,$$

whence

$$|||U^N|||_1^2 \le C\{|||U^1|||_1^2 + k|\sum_{n=2}^{N}(I_1^n + I_2^n + I_3^n)| + k\sum_{n=2}^{N}|||U^{n-1}|||_1^2\}.$$

Now letting

$$|||U|||_{1;N} = \max_{1\le n\le N}|||U^n|||_1, \tag{3.5}$$

we obtain

$$k\left|\sum_{n=2}^{N} I_2^n\right| \le k\sum_{n=2}^{N}||f^n||\,|||U|||_{1;N}.$$

In order to estimate the sum in I_1^n, we write

$$\begin{aligned} kI_1^n = k\bar\partial\sigma^n(B_n(U, U^n)) &- \omega_{n,n-1}B(t_n, t_{n-1}; U^{n-1}, U^{n-1}) \\ &- \sum_{j=0}^{n-2}[(\omega_{nj} - \omega_{n-1,j})B(t_n, t_j; U^j, U^{n-1}) \\ &+ k\omega_{n-1,j}(\bar\partial_1 B)(t_n, t_j; U^j, U^{n-1})], \end{aligned}$$

where $\bar{\partial}_1 B$ denotes the difference quotient with respect to the first argument. By (1.5) it follows that

$$
\begin{aligned}
k\Big|\sum_{n=2}^{N} I_1^n\Big| &\le |\sigma^N(B_N(U,U^N)) - \sigma^1(B_1(U,U^1))| \\
&\quad + \Big|\sum_{n=2}^{N} \omega_{n,n-1} B(t_n, t_{n-1}; U^{n-1}, U^{n-1})\Big| \\
&\quad + \Big|\sum_{n=2}^{N}\sum_{j=0}^{n-2} [(\omega_{nj} - \omega_{n-1,j}) B(t_n, t_j; U^j, U^{n-1}) \\
&\quad + k\omega_{n-1,j} (\bar{\partial}_1 B)(t_n; t_j; U^j, U^{n-1})]\Big| \\
&\le C\Big\{\sum_{n=2}^{N-1} \omega_n ||U^n||_1 + ||U^0||_1 + ||U^1||_1 \\
&\quad + \sum_{j=0}^{N-2}\sum_{n=j+2}^{N} |\omega_{nj} - \omega_{n-1,j}|\, ||U^j||_1 \\
&\quad + k\sum_{j=0}^{N-2}\sum_{n=j+2}^{N} \omega_j ||U^j||_1\Big\} |||U|||_{1;N},
\end{aligned}
$$

and hence, using (1.6) and (1.5),

$$
k\Big|\sum_{n=2}^{N} I_1^n| \le C\Big\{\sum_{n=2}^{N-1} \omega_n |||U^n|||_1 + ||U^0||_1 + ||U^1||_1\Big\} |||U|||_{1;N}.
$$

Further, for $kI_3^n = F^n(U^n - U^{n-1})$ we have by a simple summation by parts technique,

$$
\begin{aligned}
k\Big|\sum_{n=2}^{N} I_3^n\Big| &\le |F^N(U^N) - F^2(U^1)| + k\sum_{n=3}^{N} |(\bar{\partial} F^n)(U^{n-1})| \\
&\le (||F^N||_{-1} + ||F^2||_{-1} + k\sum_{n=3}^{N} ||\bar{\partial} F^N||_{-1}) |||U|||_{1;N} \\
&\le (2||F^2||_{-1} + 2k\sum_{n=3}^{N} ||\bar{\partial} F^n||_{-1}) |||U|||_{1;N},
\end{aligned}
$$

where in the last step we have used that $F^N = F^2 + k\sum_{n=3}^N \bar\partial F^n$.

Altogether we obtain

$$|||U^N|||_1^2 \le C|||U^1|||_1^2 + C\Big\{||U^0||_1 + ||U^1||_1 + k\sum_{n=2}^N ||f^n|| + ||F^2||_{-1} + k\sum_{n=3}^N ||\bar\partial F^n||_{-1} + \sum_{n=2}^{N-1} \omega_n |||U^n|||_1\Big\}|||U|||_{1;N} + Ck\sum_{n=1}^{N-1} |||U^n|||_1^2,$$

where we have shifted indices in the last sum. This estimate is clearly true also for $N = 1$. From this we easily conclude that

$$|||U|||_{1;N} \le C\Big\{||U^0||_1 + |||U^1|||_1 + k\sum_{n=2}^N ||f^n|| + ||F^2||_{-1} + k\sum_{n=3}^N ||\bar\partial F^n||_{-1}\Big\} + C\sum_{n=1}^{N-1} (\omega_n + k)|||U^n|||_1.$$

Replacing $|||U^n|||_1$ in the sum on the right by $|||U|||_{1;n}$, an application of the discrete Gronwall's lemma, Lemma 2.3, with (1.5), now completes the proof. □

We shall now turn to the preliminary error estimate, in which the choice of the initial values and the quadrature formula remain to be specified. For notational convenience, we first define a linear functional $q_B^n(W)$ on H_0^1 which is associated with the quadrature error

$$q^n(g) = \sigma^n(g) - \int_0^{t_n} g(s)\,ds$$

by

$$q_B^n(W)(\phi) = q^n(B_n(W,\phi)), \quad \text{for } \phi \in H_0^1,$$

cf. (3.1), (3.2) for notation. Note that $q^0 \equiv 0$.

Theorem 3.2. *Assume that the quadrature rule* (3.1) *is ω-stable. Then we have, with W the Ritz-Volterra projection defined by* (1.4),

$$||U^n - u(t_n)||_1 \le C\{||U^0 - W(0)||_1 + |||U^1 - W(k)|||_1 + k\sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-1}\} + C(u)(h^{r-1} + k),$$
$$\text{for } n \ge 2,\ t_n \le T.$$

Proof. We write the error as

$$U^n - u(t_n) = (U^n - W(t_n)) + (W(t_n) - u(t_n)) = \theta^n + \rho^n.$$

We have from Proposition 2.2 that

$$||\rho^n||_1 \le Ch^{r-1}\bigg(||u(t_n)||_r + \int_0^{t_n} ||u||_r\, ds\bigg) \le Ch^{r-1}\bigg(||u_0||_r + \int_0^{t_n} ||u_t||_r\, ds\bigg).$$

It thus remains to estimate θ. From the definitions (3.2) and (1.4), we obtain

$$(\bar{\partial}^2\theta^n, \chi) + A_n(\theta^n, \chi) = \sigma^n(B_n(\theta, \chi)) - (\bar{\partial}^2\rho^n + \tau^n, \chi) + q_B^n(W)(\chi),$$

where

$$\tau^n = \bar{\partial}^2 u(t_n) - u_{tt}(t_n).$$

We may now apply Theorem 3.1 to obtain

$$|||\theta^n|||_1 \le C\bigg\{||\theta^0||_1 + |||\theta^1|||_1 + k\sum_{m=2}^{n}(||\bar{\partial}^2\rho^m|| + ||\tau^m||) + k\sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-1}\bigg\},$$

where we have used that $q_B^0 \equiv 0$ to incorporate $||q_B^2(W)||_{-1}$ into the last term. By Proposition 2.2 again

$$k\sum_{m=2}^{n} ||\bar{\partial}^2\rho^m|| \le C\int_0^{t_n} ||\rho_{tt}||\, ds \le Ch^r \int_0^{t_n} ||u||_{(r,2)}\, ds,$$

cf. (2.2) for notation.

Also,

$$k\sum_{m=2}^{n}||\tau^m|| \le Ck\sum_{m=2}^{n}\int_{t_{m-2}}^{t_m}||u_{ttt}||\,ds \le Ck\int_0^{t_n}||u_{ttt}||\,ds.$$

Therefore, for $n \ge 2$,

$$|||\theta^n|||_1 \le C\Big\{||\theta^0||_1 + |||\theta^1|||_1 + h^r\int_0^{t_n}||u||_{(r,2)}\,ds + k\int_0^{t_n}||u_{ttt}||\,ds + k\sum_{m=1}^{n}||\bar\partial q_B^m(W)||_{-1}\Big\}.$$

This completes the proof. □

We remark that it is easy to trace the exact behavior of $C(u)$ in the proof above.

We next address the question of how to choose initial data in Theorem 3.2. We shall assume that we have given suitable approximations U^0 and V in S_h to $u_0 = u(0)$ and $u_1 = u_t(0)$, respectively, and then construct U^1 as

$$U^1 = U^0 + kV. \tag{3.6}$$

More precisely, we assume that

$$||U^0 - u_0||_1 \le Ch^{r-1}, \qquad ||V - u_1|| + k||V - u_1||_1 \le Ch^{r-1}. \tag{3.7}$$

The appearance of the second condition is motivated by the proof below; note that it is weaker than requiring $||V - u_1||_1 \le Ch^{r-1}$.

Proposition 3.3. *Under the above hypotheses* (3.6) *and* (3.7) *we have*

$$||U^0 - W(0)||_1 + |||U^1 - W(k)|||_1 \le C(u)(h^{r-1} + k).$$

Proof. The first part of (3.7) and Proposition 2.2 give $||U^0-W(0)||_1 \le Ch^{r-1}$. With R_h the standard Ritz projection defined in Section 2, now let

$$\hat W(s) = R_h(s)(u_0 + su_1).$$

For $\eta(t) = W(t) - \hat{W}(t)$, we then have

$$A(t; \eta(t), \chi) = \int_0^t B(t, s; \rho(s), \chi)\, ds + A(t; u(t) - u_0 - tu_1, \chi),$$
$$\eta(0) = 0,$$

whence by Lemma 2.1 and Proposition 2.2

$$||\eta(k)||_1 \leq C(h^{r-1}k + k^2),$$

and

$$||\eta(k)|| \leq C(h^r k + k^2). \tag{3.8}$$

It follows that

$$|||\eta(k)|||_1 \leq C(u)(h^{r-1} + k).$$

It remains to estimate $|||U^1 - \hat{W}(k)|||_1$. The estimate for the H^1 part is obvious by (2.5) and (3.7). Further,

$$\bar{\partial}U^1 = V, \qquad \bar{\partial}\hat{W}(k) = R_h(k)u_1 + (\bar{\partial}R_h(k))u_0.$$

Since

$$A(k; (\bar{\partial}R_h(k))u_0, \chi) = (\bar{\partial}A(k))(R_h(0)u_0 - u_0, \chi)$$

the desired estimate for $\bar{\partial}(U^1 - \hat{W}(k))$ follows from (3.7), (2.5), Lemma 2.1, and (1.2). □

As a consequence of the proof of Theorem 3.2, we note the following.

Corollary 3.4. *Under the above hypotheses we have, for* $n \geq 1$, $t_n \leq T$,

$$||\bar{\partial}U^n - u_t(t_n)|| \leq C\{||U^0 - W(0)||_1 + |||U^1 - W(k)|||_1 + k\sum_{m=1}^{n} ||\bar{\partial}q_B^n(W)||_{-1}\} + C(u)(h^r + k).$$

The essential modification necessary in the proof of Theorem 3.2 simply consists of estimating ρ_t in L_2 instead of ρ in H^1 in the former part of the proof.

Choosing discrete initial data in a particular way, e.g.,

$$U^0 = R_h(0)u_0, \qquad U^1 = R_h(k)(u_0 + ku_1) \quad (\text{or } U^1 = R_h(0)(u_0 + ku_1)),$$

we see from (3.8), Corollary 3.4, and (2.5) that we obtain an optimal $O(h^r + k)$ order approximation to u_t in L_2, provided the quadrature error is of that order.

We shall next address the question of error estimates in L_2. Since the proof of Theorem 3.2 furnishes an estimate for $||\theta^n||_1$ of order $O(h^r+k)$, provided $||\theta^0||_1 + |||\theta^1|||_1$ is of that order, after estimating $||\rho(t)||$ we have an error estimate in L_2. However, for such an estimate to be of optimal order unnatural conditions are required of the discrete initial data. This blemish can be removed in a fashion that we now proceed to give.

Let

$$||\phi||_{-1} = \sup_{0\neq\psi\in H_0^1} \frac{(\phi, \psi)}{||\psi||_1}$$

and

$$|||\phi^n|||_0^2 = ||\bar\partial\phi^n||_{-1}^2 + ||\phi^n||^2.$$

Our result corresponding to Theorem 3.2 is the following. For notation, cf. (3.5) and (2.1).

Theorem 3.5. *If* (3.1) *is ω-stable, then for $n \geq 2$, $t_n \leq T$,*

$$|||U^n|||_0 \leq C\{||U^0|| + |||U^1|||_0 + h|||U|||_{1;N} + k\sum_{m=2}^{n} ||f^m||_{-1,h} + ||F^2||_{-2,h} + k\sum_{m=3}^{n} ||\bar\partial F^m||_{-2,h}\}.$$

Proof. With T_n as at the end of Section 2, we introduce

$$||\chi||_{-1,n} = (T_n\chi, \chi)^{1/2}, \quad \text{for } \chi \in S_h.$$

It is immediate that

$$||T_n\chi||_1 \approx ||\chi||_{-1,n}, \tag{3.9}$$

and it is well known, cf. [**7**, p. 82], that

$$c||\chi||_{-1,h} \le ||\chi||_{-1,n} + h||\chi|| \le C||\chi||_{-1,h}. \tag{3.10}$$

We shall further need the following results: With Q any bilinear form on $H_0^1 \times H_0^1$ associated with a second order partial differential operator,

$$|Q(\chi, T_n\psi)| \le C||\chi||_{0,h}||\psi||. \tag{3.11}$$

Also, with $\bar{\partial}T_n$ having the obvious meaning,

$$|(\chi, (\bar{\partial}T_n)\psi)| + |(\chi, (\bar{\partial}^2 T_n)\psi)| \le C||\chi||_{-1,h}||\psi||_{-1,h}. \tag{3.12}$$

For (3.11), with notation as in Section 2 and using (2.4),

$$\begin{aligned} |Q(\chi, T_n\psi)| &\le |Q(\chi, (T_n - \tilde{T}_n)\psi)| + |Q(\chi, \tilde{T}_n\psi)| \\ &\le C(||\chi||_1 h||\psi|| + ||\chi||\,||\psi||). \end{aligned}$$

For (3.12), we have $A_n(T_n\psi, \phi) = (\psi, \phi)$ whence

$$(\bar{\partial}A_n)(T_{n-1}\psi, \phi) + A_n((\bar{\partial}T_n)\psi, \phi) = 0. \tag{3.13}$$

Taking $\phi = T_n\chi$ and using (3.9) and (3.10), the appropriate bound for the first term on the left of (3.12) is obtained. For the second term, note first that $||(\bar{\partial}T_n)\chi||_1 \le C||\chi||_{-1,h}$, which follows by taking $\phi = (\bar{\partial}T_n)\chi$ in (3.13) and using (3.9) and (3.10). A further differencing of (3.13) then concludes the argument.

The proof will follow along the lines of that of Theorem 3.1 taking $\chi = \bar{\partial}(T_n U^n)$ in (3.3). We shall first consider the case when $f^m = F^m = 0$. It is easily checked that

$$\bar{\partial}(\chi^n, T_n\chi^n) = 2(\bar{\partial}\chi^n, T_n\chi^n) + (\chi^{n-1}, (\bar{\partial}T_n)\chi^{n-1}) - k(\bar{\partial}\chi^n, T_n(\bar{\partial}\chi^n)),$$

and that

$$(\psi^n, \bar{\partial}(T_n\Theta^n)) = (\psi^n, T_n(\bar{\partial}\Theta^n)) + (\psi^n, (\bar{\partial}T_n)\Theta^{n-1}).$$

Taking $\chi^n = \bar\partial U^n$, $\psi^n = \bar\partial \chi^n = \bar\partial^2 U^n$ and $\Theta^n = U^n$, we have
(3.14)

$$(\bar\partial^2 U^n, \bar\partial(T_n U^n)) = \frac{1}{2}\,\bar\partial ||\bar\partial U^n||^2_{-1,n} + \frac{1}{2} k ||\bar\partial^2 U^n||^2_{-1,n} + (\bar\partial^2 U^n, (\bar\partial T_n) U^{n-1}) - \frac{1}{2}(\bar\partial U^{n-1}, (\bar\partial T_n)\bar\partial U^{n-1}).$$

We also have that

$$(\bar\partial^2 U^n, (\bar\partial T_n) U^{n-1}) = \bar\partial(\bar\partial U^n, (\bar\partial T_n) U^{n-1}) - (\bar\partial U^{n-1}, (\bar\partial^2 T_n) U^{n-1}) - (\bar\partial U^{n-1}, (\bar\partial T_{n-1})\bar\partial U^{n-1}). \tag{3.15}$$

Thus, discarding the positive second term on the right of (3.14), we see that

$$(\bar\partial^2 U^n, \bar\partial(T_n U^n)) \ge \frac{1}{2}\,\bar\partial ||\bar\partial U^n||^2_{-1,n} + \bar\partial(\bar\partial U^n, (\bar\partial T_n) U^{n-1}) + J_1^n, \tag{3.16}$$

where

$$J_1^n = -(\bar\partial U^{n-1}, (\bar\partial^2 T_n + \frac{1}{2}\,\bar\partial T_n) U^{n-1}) - (\bar\partial U^{n-1}, (\bar\partial T_{n-1})\bar\partial U^{n-1}).$$

With the notation

$$L_n^2 = ||\bar\partial U^n||^2_{-1,n} + ||U^n||^2 + h^2(||\bar\partial U^n||^2 + ||U^n||_1^2), \quad \text{for } n \ge 1,$$

we have from (3.12) and (3.10) that $|J_1^n| \le C L_{n-1}^2$. Further, by Cauchy-Schwarz's inequality and the geometric-arithmetic mean inequality, and with $J_2^n = -(\bar\partial A_n)(U^n, T_{n-1}U^{n-1})$,

$$\begin{aligned} A_n(U^n, \bar\partial(T_n U^n)) &= k^{-1} A_n(U^n, T_n U^n) \\ &\quad - k^{-1} A_{n-1}(U^n, T_{n-1} U^{n-1}) + J_2^n \\ &= k^{-1}(||U^n||^2 - (U^n, U^{n-1})) + J_2^n \\ &\ge \frac{1}{2}\,\bar\partial ||U^n||^2 + J_2^n, \end{aligned} \tag{3.17}$$

where by (3.11), $|J_2^n| \le C||U^n||_{0,h}||U^{n-1}|| \le C L_n L_{n-1}$. Now take $\chi = \bar\partial(T_n U^n)$ in (3.3), multiply by $2k$ and sum from $n = 2$ to N. Since

$$L_n^2 \le ||\bar\partial U^n||^2_{-1,n} + ||U^n||^2 + h|||U^n|||_1 L_n$$

we have by (3.16), (3.17) and the estimates for J_1^n, J_2^n,

$$\begin{aligned}L_N^2 \le |||U^1|||_0 L_1 + h|||U^N|||_1 L_N - 2(\bar\partial U^N, (\bar\partial T_N)U^{N-1}) \\ + 2(\bar\partial U^1, (\bar\partial T_1)U^0) + 2\sum_{n=2}^{N} \sigma^n(B_n(U, \bar\partial(T_n U^n))) \\ + Ck\sum_{n=1}^{N-1}(L_n^2 + L_{n+1}L_n),\end{aligned} \tag{3.18}$$

where we have shifted the index of summation in the last sum. Here, using (3.12) and writing $U^{N-1} = U^0 + k\sum_{n=1}^{N-1}\bar\partial U^n$,

$$|(\bar\partial U^N, (\bar\partial T_N)U^{N-1})| \le CL_N||U^{N-1}||_{-1,h} \le CL_N(||U^0|| + k\sum_{n=1}^{N-1} L_n) \tag{3.19}$$

and

$$|(\bar\partial U^1, (\bar\partial T_1)U^0)| \le CL_1||U^0||_{-1,h} \le CL_1||U^0||.$$

For the quadrature sum above, we have, following the proof of Theorem 3.1 with obvious modifications, the bound

$$C\left\{\sum_{n=2}^{N-1}\omega_n L_n + ||U^0|| + |||U^1|||_0 + h(||U^0||_1 + |||U^1|||_1)\right\}\max_{1\le n\le N} L_n.$$

Combining the above with (3.18) and taking supremums, we thus have

$$\max_{1\le n\le N} L_n \le C\left\{||U^0|| + |||U^1|||_0 + h|||U|||_{1;N} + \sum_{n=1}^{N-1}(\omega_n + k)L_n\right\}. \tag{3.20}$$

(The bound is trivial for $N = 1$.)

As for the inhomogeneous terms, we have by use of (3.9), (3.10), and (3.12)

$$\begin{aligned}|(f^n, \bar\partial(T_n U^n))| &= |(f^n, T_n(\bar\partial U^n)) + (f^n, (\bar\partial T_n)U^{n-1})| \\ &\le C||f^n||_{-1,h}(L_n + L_{n-1}).\end{aligned}$$

Further, with notation as in Section 2 and using (2.4),

$$\begin{aligned}|(\bar\partial F^n)(T_n U^{n-1})| &= |(\bar\partial F^n)((T_n - \tilde T_n)U^{n-1})| + |(\bar\partial F^n)(\tilde T_n U^{n-1})| \\ &\le Ch||\bar\partial F^n||_{-1}||U^{n-1}|| + C||\bar\partial F^n||_{-2}||U^{n-1}|| \\ &\le C||\bar\partial F^n||_{-2,h}L_{n-1},\end{aligned}$$

and similarly for $F^N(T_N U^N)$ and $F^2(T_1 U^1)$, cf. the proof of Theorem 3.1. Adding corresponding terms on the right of (3.20), the proof is concluded by an appeal to the discrete Gronwall's lemma. $\square$

Proceeding now as in Theorem 3.2, with the appropriate changes, and choosing initial data as in (3.6), (3.7), with the additional requirement that

$$||U^0 - u_0|| \le Ch^r, \qquad ||V - u_1||_{-1} + k||V - u_1|| \le Ch^r,$$

we easily find the following.

Theorem 3.6. *Under the hypotheses above,*

$$||U^n - u(t_n)|| \le Ck \sum_{m=1}^{n} ||\bar\partial q_B^m(W)||_{-2,h} + C(u)(h^r + k),$$
$$\text{for } n \ge 2,\ t_n \le T.$$

4. The global quadrature error. In this section we shall estimate the quadrature dependent term in $\bar\partial q_B^n(W)$, occurring in our preliminary error estimates above, for various particular quadrature rules.

The rectangle rule. The simplest quadrature rule of type (3.1) which is consistent with the order of accuracy of the backward Euler scheme is the rectangle rule

$$\sigma^n(g) = k \sum_{j=0}^{n-1} g(t_j),$$

which thus corresponds to choosing $\omega_{nj} = k$, for $0 \le j \le n-1$. Here we regard also $\sigma^0 = 0$ and σ^1 as defined by this formula, cf. the remark preceding Theorem 3.1. For this choice we have the following, where we use the special notation of (2.2).

Proposition 4.1. *The rectangle rule is ω-stable and satisfies*

$$k \sum_{m=1}^{n} ||\bar\partial q_B^m(W)||_{-i} \le Ck \begin{cases} \int_0^{t_n} ||u||_{(1,1)}\, ds, & i = 1, \\ \int_0^{t_n} ||u||_{(0,1),h}\, ds, & i = 2. \end{cases}$$

Proof. In this case the quadrature weights ω_{nj} are all equal to k for $0 \le j \le n-1$, and we have with $\omega_j = k$,

$$\sum_{j=0}^{n-1} \omega_j = nk = t_n \le T.$$

Also

$$\sum_{n=j+2}^{N} |\omega_{nj} - \omega_{n-1,j}| = 0, \quad \text{for } 0 \le j \le N-2,\ t_N \le T,$$

so that the rule is ω-stable, cf. (1.5), (1.6).

We may represent the quadrature error as

$$\begin{aligned} q^m(g) &= \sum_{j=0}^{m-1} \{kg(t_j) - \int_{t_j}^{t_{j+1}} g(s)\,ds\} \\ &= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} (s - t_{j+1}) D_s g(s)\,ds = \int_0^{t_m} \psi_0(s) D_s g(s)\,ds, \end{aligned}$$

where

$$\psi_0(s) = s - t_{j+1}, \quad \text{for } s \in [t_j, t_{j+1}]. \tag{4.1}$$

We therefore have, for $\phi \in H_0^1$,

$$\begin{aligned} &q^m(B_m(W,\phi)) - q^{m-1}(B_{m-1}(W,\phi)) \\ &\quad = k \int_0^{t_{m-1}} \psi_0(s) D_s(\bar{\partial}_1 B)(t_m, s; W(s), \phi)\,ds \\ &\qquad + \int_{t_{m-1}}^{t_m} \psi_0(s) D_s B(t_m, s; W(s), \phi)\,ds, \end{aligned}$$

whence

$$\begin{aligned} &|q^m(B_m(W,\phi)) - q^{m-1}(B_{m-1}(W,\phi))| \\ &\qquad \le C\Big\{ k^2 \int_0^{t_{m-1}} ||W||_{(1,1)}\,ds + k \int_{t_{m-1}}^{t_m} ||W||_{(1,1)}\,ds \Big\} ||\phi||_1. \end{aligned}$$

Hence,

$$||q_B^m(W) - q_B^{m-1}(W)||_{-1} \leq Ck^2 \int_0^{t_{m-1}} ||W||_{(1,1)}\, ds + Ck \int_{t_{m-1}}^{t_m} ||W||_{(1,1)}\, ds.$$

We conclude that

$$k \sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-1} = \sum_{m=1}^{n} ||q_B^m(W) - q_B^{m-1}(W)||_{-1} \leq Ck \int_0^{t_n} ||W||_{(1,1)}\, ds.$$

The case $i = 1$ of the proposition now follows by the estimates of Proposition 2.2 for the Ritz-Volterra projection. For $i = 2$, we obtain, with trivial modifications above, that

$$k \sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-2} \leq Ck \int_0^{t_n} ||W||_{(0,1)}\, ds.$$

Proposition 2.2 again concludes the argument. □

A modified trapezoidal rule. Let $m_0 = [k^{-1/2}]$, where $[x]$ denotes the integral part of x. Set $k_1 = m_0 k$ and $\bar{t}_j = jk_1$, and let j_n be the largest integer such $\bar{t}_{j_n} < t_n$. In approximating the integral term over $[0, t_n]$ we shall apply the trapezoidal rule with step size k_1 on $[0, \bar{t}_{j_n}]$ and the rectangle rule with step size k on the remaining part $[\bar{t}_{j_n}, t_n]$. More precisely, we introduce the quadrature rule

$$\begin{aligned} \sigma^n(g) &= \sum_{j=0}^{n-1} \omega_{nj} g(t_j) \\ &= \frac{k_1}{2} \sum_{j=1}^{j_n} (g(\bar{t}_j) + g(\bar{t}_{j-1})) + k \sum_{j=m_0 j_n}^{n-1} g(t_j) \\ &= \sigma_1^n(g) + \sigma_0^n(g). \end{aligned}$$

Note that this rule has a storage requirement of $O(k^{-1/2})$ as opposed to $O(k^{-1})$ for the rectangle rule. For this choice, we have the following.

Proposition 4.2. *The modified trapezoidal rule is ω-stable and satisfies*

$$k\sum_{m=1}^{n}||\bar{\partial}q_B^m(W)||_{-i} \le Ck\begin{cases}\int_0^{t_n}||u||_{(1,2)}\,ds, & i=1,\\ \int_0^{t_n}||u||_{(0,2),h}\,ds, & i=2.\end{cases}$$

Proof. Setting

$$\omega_j = \begin{cases} k_1, & \text{for } j\equiv 0 \pmod{m_0},\\ k, & \text{otherwise},\end{cases}$$

we have that $\omega_{nj} \le \omega_j$ for $j \le n-1$, and

$$\sum_{j=0}^{n}\omega_j \le j_n k_1 + nk \le 2T,$$

so that the quadrature formula has dominated weights. We further note that for fixed j, ω_{nj} only changes its value once as n increases, and this happens when n for the first time has passed a multiple of m_0. With $j \in [(q-1)m_0, qm_0)$, we therefore have

$$\sum_{n=j+2}^{N}|\omega_{nj}-\omega_{n-1,j}| \le |\omega_{qm_0+1,j}-\omega_{qm_0,j}| = \begin{cases}\frac{1}{2}k_1 - k, & \text{if } j\equiv 0 \pmod{m_0}\\ k, & \text{otherwise},\end{cases}$$

so that the dominating weights are persistent.

This time we have for the corresponding quadrature error

$$q^m(g) = \left[\sigma_1^m(g) - \int_0^{\bar{t}_{jm}} g(s)\,ds\right] + \left[\sigma_0^m(g) - \int_{\bar{t}_{jm}}^{t_m} g(s)\,ds\right].$$

Thus,

$$q^m(g) = \int_0^{\bar{t}_{jm}}\psi_1(s)D_s^2 g(s)\,ds + \int_{\bar{t}_{jm}}^{t_m}\psi_0(s)D_s g(s)\,ds,$$

where $\psi_0(s)$ is defined in (4.1) and

$$\psi_1(s) = \begin{cases} (s-\bar t_{j-1})(s-\bar t_{j-1/2}), & \text{for } s\in[\bar t_{j-1},\bar t_{j-1/2}],\\ (s-\bar t_{j})(s-\bar t_{j-1/2}), & \text{for } s\in[\bar t_{j-1/2},\bar t_{j}]. \end{cases}$$

We therefore obtain

$$\begin{aligned} &q^m(B_m(W,\phi)) - q^{m-1}(B_{m-1}(W,\phi))\\ &\quad = k\int_0^{\bar t_{j_{m-1}}} \psi_1(s) D_s^2(\bar\partial_1 B)(t_m,s;W(s),\phi)\,ds\\ &\qquad + k\int_{\bar t_{j_m}}^{t_{m-1}} \psi_0(s) D_s(\bar\partial_1 B)(t_m,s;W(s),\phi)\,ds\\ &\qquad + \int_{\bar t_{j_{m-1}}}^{\bar t_{j_m}} \psi_1(s) D_s^2 B(t_m,s;W(s),\phi)\,ds\\ &\qquad - \int_{\bar t_{j_{m-1}}}^{\bar t_{j_m}} \psi_0(s) D_s B(t_{m-1},s;W(s),\phi)\,ds\\ &\qquad + \int_{t_{m-1}}^{t_m} \psi_0(s) D_s B(t_m,s;W(s),\phi)\,ds. \end{aligned} \tag{4.2}$$

We remark that in the case $\bar t_{j_{m-1}} = \bar t_{j_m}$ the third and fourth terms will vanish. If $\bar t_{j_{m-1}} \neq \bar t_{j_m}$, which happens exactly when $\bar t_{j_m} = t_{m-1}$, then $\bar t_{j_{m-1}} = \bar t_{j_m-1}$, and the second term will be zero. We conclude easily from (4.2) that

$$\begin{aligned} ||q_B^m(W) - q_B^{m-1}(W)||_{-1} &\le Ckk_1^2\int_0^{\bar t_{j_{m-1}}} ||W||_{(1,2)}\,ds\\ &+ Ck^2\int_{\bar t_{j_m}}^{t_{m-1}} ||W||_{(1,1)}\,ds + Ck_1^2\int_{\bar t_{j_{m-1}}}^{\bar t_{j_m}} ||W||_{(1,2)}\,ds\\ &+ Ck\int_{\bar t_{j_{m-1}}}^{\bar t_{j_m}} ||W||_{(1,1)}\,ds + Ck\int_{t_{m-1}}^{t_m} ||W||_{(1,1)}\,ds. \end{aligned}$$

Since $k_1^2 \le k$ and $\bar{t}_{j_m}$ is increasing in m, we now have

$$\begin{aligned} k\sum_{m=1}^{n} ||\bar{\partial} q_B^n(W)||_{-1} &= \sum_{m=1}^{n} ||q_B^m(W) - q_B^{m-1}(W)||_{-1} \\ &\le Ck\int_0^{\bar{t}_{jn}} ||W||_{(1,2)}\, ds + Ck\int_0^{t_n} ||W||_{(1,1)}\, ds \\ &\le Ck\int_0^{t_n} ||W||_{(1,2)}\, ds. \end{aligned}$$

Using the obvious modifications for $i = 2$, the proposition now follows by the known estimates for the Ritz-Volterra projection. □

A modified Simpson's rule. In this section we shall discuss a quadrature rule introduced in [**11**] which is based on using Simpson's rule on subintervals of length $O(k^{1/4})$. For this rule the number of values of U^j which need to be stored in $[0, T]$ is of order $O(k^{-1/4})$, thus further reducing the storage requirement as compared to the above modified trapezoidal rule.

In order to define our rule, let $m_0 = [k^{-1/4}]$ and define $k_i = m_0^i k$, for $0 \le i \le 3$, and note that $k_i = O(k^{(4-i)/4})$. The rule σ^n then uses Simpson's rule on as many intervals of length $2k_3$ which fitted into $[0, t_{n-1}]$. In the remaining interval, which is of length at most $O(k^{1/4})$, we use the trapezoidal rule on as many intervals of length k_2 as can be fitted in, leaving an interval of length at most $O(k^{1/2})$. Here we apply the trapezoidal rule based on intervals of length $k_1 = O(k^{3/4})$, and finally, the rectangle rule on the remaining basic intervals of length $k_0 = k$.

For given n, we thus introduce the quadrature points $\bar{t}_j^n$ as follows: Let j_{3n} be the largest even integer with $j_{3n}k_3 < t_n$, and set $\bar{t}_j^n = jk_3$ for $0 \le j \le j_{3n}$. Now let $\bar{t}_j^n = \bar{t}_{j_{3n}}^n + (j - j_{3n})k_2$, for $j_{3n} < j \le j_{2n}$, where j_{2n} is the largest integer such that $\bar{t}_{j_{2n}}^n < t_n$, and set $\bar{t}_j^n = \bar{t}_{j_{2n}}^n + (j - j_{2n})k_1$, for $j_{2n} < j \le j_{1n}$, where j_{1n} is the largest integer such that $\bar{t}_{j_{1n}}^n < t_n$, and finally $\bar{t}_j^n = \bar{t}_{j_{1n}}^n + (j - j_{1n})k$, for $j_{1n} < j \le j_{0n}$, where $\bar{t}_{j_{0n}}^n = t_n$. We thus have

$$[0, t_n] = \bigcup_{j=1}^{j_{3n}} [\bar{t}_{j-1}^n, \bar{t}_j^n] \bigcup_{j=j_{3n}+1}^{j_{2n}} [\bar{t}_{j-1}^n, \bar{t}_j^n] \bigcup_{j=j_{2n}+1}^{j_{1n}} [\bar{t}_{j-1}^n, \bar{t}_j^n] \bigcup_{j=j_{1n}+1}^{j_{0n}} [\bar{t}_{j-1}^n, \bar{t}_j^n],$$

and shall refer to the intervals in this partition as our basic integration intervals. The modified Simpson's rule under study is thus

$$\tag{4.3} \begin{aligned} \sigma^n(g) = \frac{k_3}{3} \sum_{j=1}^{j_{3n}/2} [g(\bar t^n_{2j}) + 4g(\bar t^n_{2j-1}) + g(\bar t^n_{2j-2})] \\ + \frac{k_2}{2} \sum_{j=j_{3n}+1}^{j_{2n}} [g(\bar t^n_j) + g(\bar t^n_{j-1})] \\ + \frac{k_1}{2} \sum_{j=j_{2n}+1}^{j_{1n}} [g(\bar t^n_j) + g(\bar t^n_{j-1})] + k \sum_{j=j_{1n}}^{j_{0n}-1} g(\bar t^n_j). \end{aligned}$$

We now have the following proposition.

Proposition 4.3. *The modified Simpson's rule is ω-stable and satisfies*

$$\tag{4.4} k \sum_{m=1}^{n} ||\bar\partial q^m_B(W)||_{-i} \le Ck \begin{cases} \int_0^{t_n} ||u||_{(1,4)}\, ds, & i = 1, \\ \int_0^{t_n} ||u||_{(0,4),h}\, ds, & i = 2. \end{cases}$$

Proof. The proof will proceed along lines analogous to those in the case of the modified trapezoidal rule. However, a detailed reckoning of the changes in ω_{nj} and their differences as in that case would now be quite cumbersome, and is, fortunately, not needed.

Let us call a point t_j a point of type 3, 2, 1, or 0 according to the following: t_j is of

$$\text{type} \begin{cases} 3, & \text{if } j \equiv 0 \pmod{m_0^3} \\ 2, & \text{if } j \equiv 0 \pmod{m_0^2}, \quad j \not\equiv 0 \pmod{m_0^3}, \\ 1, & \text{if } j \equiv 0 \pmod{m_0}, \quad j \not\equiv 0 \pmod{m_0^2}, \\ 0, & \text{if } j \not\equiv 0 \pmod{m_0}. \end{cases}$$

This classification is thus not dependent on the quadrature interval $[0, t_n]$. A point of type 3 typically occurs in the first sum in (4.3) and also, occasionally, for some n, in the other sums. It is then clear by inspection that, for such a t_j, $\omega_{nj} \le Ck_3$. A point of type 2 can never occur in the first sum, and hence $\omega_{nj} \le Ck_2$. With similar arguments for points of types 1 and 0, we then have

$$\omega_{nj} \le Ck_i = \omega_j, \quad \text{for } t_j \text{ of type } i, \tag{4.5}$$

where the equality thus defines ω_j for points t_j of type i. The number of points t_j of given type i in $J = [0, T]$ equals $[T/k_i] + 1$, and hence we find at once that

$$\sum_{j=0}^{n} \omega_j \le C \sum_{i=0}^{3} k_i([T/k_i] + 1) \le CT.$$

We next want to bound the sums $S_j^N = \sum_{n=j+2}^{N} |\omega_{nj} - \omega_{n-1,j}|$. If t_j is of type 0, then $\omega_{j+1,j} = k_0 = k$ and ω_{nj} retains that value as n increases until the point is removed from the quadrature formula by introduction of a larger basic interval into σ^n. Hence $S_j^N = k$ for j of type 0. For type 1 points, $\omega_{j+1,j} = k_1/2 + k$ and then ω_{nj} may undergo at most two changes as n increases (e.g., changing first to k_1 and then to zero). Similarly, points of type 2 or 3 undergo a bounded number of changes in ω_{nj} only. It is now immediate from the triangle inequality and (4.5) that $S_j^N \le Ck_i \le C\omega_j$ for t_j of type i. We have thus verified that the scheme has persistent dominated weights.

We now turn to the proof of the inequality (4.4). We shall demonstrate that

$$\sum_{m=1}^{n} |q_B^m(W)(\phi) - q_B^{m-1}(W)(\phi)| \le Ck \int_0^{t_n} ||W||_{(1,4)}\, ds ||\phi||_1.$$

Using the known estimates for the Ritz-Volterra projection W, this will show the present proposition for $i = 1$.

We may write

$$\begin{aligned} q^m(g) &= \int_0^{t_m} \Psi_m(s, D_s) g(s)\, ds \\ &= \int_0^{t_m} (\chi_{m0}\psi_0 D_s + (\chi_{m1}\psi_1 + \chi_{m2}\psi_2) D_s^2 + \chi_{m3}\psi_3 D_s^4) g\, ds, \end{aligned}$$

where χ_{mj} is the characteristic function of the union of those subintervals of $[0, t_m]$ with length k_j which are used in the definition of σ^n, and where $|\psi_j(s)| \le k$ for $j = 0, 1, 2, 3$. We now appeal once more to the persistence of these intervals: When n increases by one unit a basic interval of length k_j, $j = 0, 1, 2$, can be discarded in favor of a larger interval, and this can only happen once for an individual such interval. Thus, comparing the quadrature errors q^m and q^{m-1} we find that there is one family of subintervals that is common to both formulas, and we begin by estimating its contribution. Here we may form a difference quotient of B_m, and, proceeding as in the first two terms on the right of (4.2), we arrive at a contribution bounded by

$$Ck^2 \int_0^{t_m} ||W||_{(1,4)}\, ds ||\phi||_1,$$

and the total contribution after summation in m is therefore bounded by the right-hand side of (4.4).

In the remaining integrals we bound the terms from q_B^m and q_B^{m-1} separately. Since, as argued above, each of the basic intervals of length k_j is only introduced and discarded at most once, their total contribution to the sum can be estimated by the right-hand side of (4.4). Indeed, these pieces correspond to the last three integrals in (4.2) and may be estimated by CkI_m where I_m is a sum of integrals over intervals $[\bar{t}^n_{j_{i,m-1}}, \bar{t}^n_{j_{i,m}}]$, $i = 3, 2, 1, 0$.

The proof of the proposition is now complete. □

5. Schemes for the hyperbolic equation based on a symmetric difference approximation. Because of the nonsymmetric choice of the discretization of u_{tt}, the backward differencing schemes discussed above are only first order accurate in time. We shall now discuss schemes that attain second order accuracy by symmetry around the point t_n. The method will be based on the standard symmetric discretization of the wave equation

$$\frac{U^{n+1} - 2U^n + U^{n-1}}{k^2} + A\left(\frac{U^{n+1} + 2U^n + U^{n-1}}{4}\right) = 0.$$

Our basic stability results will now principally involve the averages $(U^n + U^{n-1})/2$. Therefore, in order to apply these to the error analysis, we shall approximate the integral term in (1.1) by a quadrature formula

using only such averages. For this purpose, we set
(5.1)

$$\sigma^n(g) = \sum_{j=0}^{n-1} \omega_{nj} g(t_{j+1/2}) \approx \int_0^{t_n} g(s)\,ds, \quad \text{with} \quad t_{j+1/2} = (j+1/2)k,$$

where the quadrature weights ω_{nj} satisfy our above assumptions in (1.5) and (1.6). In order to apply this to our discrete function U^n, we introduce its continuous piecewise interpolant $\tilde{U}$ in time, so that, in particular,

$$\tilde{U}(t_{j+1/2}) = U^{j+1/2} = (U^j + U^{j+1})/2.$$

With $\bar{\partial}U^n$ as before, $\partial U^n = (U^{n+1} - U^n)/k$ the forward difference quotient, and

$$\hat{U}^n = (U^{n+1} + 2U^n + U^{n-1})/4 = (U^{n+1/2} + U^{n-1/2})/2,$$

we now define our completely discrete scheme by

$$(\partial\bar{\partial}U^n, \chi) + A_n(\hat{U}^n, \chi) = \sigma^n(B_n(\tilde{U}, \chi)) + (f^n, \chi), \quad \text{for } \chi \in S_h,\ n \geq 1,$$

with U^0 and U^1 given.

As earlier, we shall need a stability result for a somewhat more general equation, namely

$$(5.2) \qquad \begin{gathered}(\partial\bar{\partial}U^n, \chi) + A_n(\hat{U}^n, \chi) = \sigma^n(B_n(\tilde{U}, \chi)) + (f^n, \chi) + F^n(\chi),\\ \text{for } \chi \in S_h,\ n \geq 1,\end{gathered}$$

where F^n, $n = 1, 2, \dots$, are linear functionals on H_0^1. For notational convenience, we also set $F^0 = \sigma^0 = 0$. Our stability result will be expressed in terms of a discrete energy norm, now defined by

$$|||\phi^{n+1/2}|||_1^2 = ||\partial\phi^n||^2 + ||\phi^{n+1/2}||_1^2.$$

We have the following.

Theorem 5.1. *If the quadrature rule* (5.1) *is ω-stable, we have for the solution of* (5.2)*,*

$$|||U^{n+1/2}|||_1 \leq C\bigg\{|||U^{1/2}|||_1 + k\sum_{m=1}^{n} ||f^m|| + k\sum_{m=1}^{n} ||\bar{\partial}F^m||_{-1}\bigg\},$$
$$\text{for } n \geq 1,\ t_{n+1} \leq T.$$

Proof. We choose $\chi = \bar\partial U^{n+1/2} = (\partial U^n + \bar\partial U^n)/2$ in (5.2) to obtain

(5.3)
$$(\partial\bar\partial U^n, \bar\partial U^{n+1/2}) + A_n(\hat U^n, \bar\partial U^{n+1/2}) = \sigma^n(B_n(\tilde U, \bar\partial U^{n+1/2})) + (f^n, \bar\partial U^{n+1/2}) + F^n(\bar\partial U^{n+1/2}) = I_1^n + I_2^n + I_3^n.$$

Note that

$$(\partial\bar\partial U^n, \bar\partial U^{n+1/2}) = (\partial\bar\partial U^n, (\partial U^n + \bar\partial U^n)/2) = \frac{1}{2}\,\bar\partial ||\partial U^n||^2$$

and

$$A_n(\hat U^n, \bar\partial U^{n+1/2}) = \frac{1}{2}\,\bar\partial(A_n(U^{n+1/2}, U^{n+1/2})) - \frac{1}{2}\,(\bar\partial A_n)(U^{n-1/2}, U^{n-1/2}).$$

We now multiply both sides of (5.3) by $2k$ and sum from $n=1$ to N to obtain

$$||\partial U^N||^2 + c||U^{N+1/2}||_1^2 \le ||\partial U^0||^2 + C||U^{1/2}||_1^2 + 2k\Big|\sum_{n=1}^{N}(I_1^n + I_2^n + I_3^n + \frac{1}{2}(\bar\partial A_n)(U^{n-1/2}, U^{n-1/2}))\Big|.$$

Hence,

$$|||U^{N+1/2}|||_1^2 \le C\Big\{|||U^{1/2}|||_1^2 + k\Big|\sum_{n=1}^{N}(I_1^n + I_2^n + I_3^n)\Big| + k\sum_{n=1}^{N}|||U^{n-1/2}|||_1^2\Big\}.$$

We define

(5.4)
$$|||U|||_{1;N} = \max_{0\le n\le N} |||U^{n+1/2}|||_1,$$

and obtain

$$k\Big|\sum_{n=1}^{N} I_2^n\Big| \le k\sum_{n=1}^{N} ||f^n||\, |||U|||_{1;N}.$$

The terms I_1^n and I_3^n are treated as in the proof of Theorem 3.1, when we replace U^n by $U^{n+1/2}$ and t_j by $t_{j+1/2}$. Therefore, we have similarly to there, using now that $F^0 = 0$,

$$k\left|\sum_{n=1}^{N}(I_1^n + I_3^n)\right| \le C\left\{\sum_{n=1}^{N-1}\omega_n|||U^{n+1/2}|||_1 + |||U^{1/2}|||_1 + k\sum_{n=1}^{N}||\bar\partial F^n||_{-1}\right\}|||U|||_{1;N}.$$

The proof is then completed as in Theorem 3.1. □

The following is the corresponding preliminary H^1 error estimate.

Theorem 5.2. *If* (5.1) *is ω-stable, then for $n \ge 1$, $t_{n+1} \le T$,*

(5.5)
$$||U^{n+1/2} - u(t_{n+1/2})||_1 \le C\{||\partial(U^0 - W^0)|| + ||U^{1/2} - W(k/2)||_1 + k\sum_{m=1}^{n}||\bar\partial q_B^n(W)||_{-1}\} + C(u)(h^{r-1} + k^2).$$

Proof. The proof is analogous to that of Theorem 3.2. We let $\tilde W$ be the piecewise linear-in-time interpolant to W and $\tilde\theta = \tilde U - \tilde W$. Since $u - \tilde W = (u - W) + (W - \tilde W)$ and

(5.6)
$$||(W - \tilde W)(t)||_1 \le Ck^2 \sup_{t_n \le s \le t_{n+1}} ||D_s^2 W(s)||_1, \quad \text{for } t_n \le t \le t_{n+1},$$

we find using Proposition 2.2 that

$$||(u - \tilde W)(t_{n+1/2})||_1 \le C\left\{h^{r-1}\left(||u(t_{n+1/2})||_r + \int_0^{t_{n+1/2}} ||u||_r\, ds\right) + k^2 \sup_{0\le s\le t_{n+1}} ||u(s)||_{(1,2)}\right\}.$$

For $\tilde\theta$ we now have

$$(\partial\bar\partial\tilde\theta^n, \chi) + A_n(\hat{\tilde\theta}^n, \chi) = \sigma^n(B_n(\tilde\theta^n, \chi)) - (\partial\bar\partial\rho^n + \tau^n, \chi) + q_B^n(W)(\chi) - [A_n(\hat{\tilde W}^n - W^n, \chi) - \sigma^n(B_n(\tilde W - W, \chi))]$$

where $\tau^n = \partial\bar{\partial}u(t_n) - u_{tt}(t_n)$. The last extra term, compared to the corresponding expression in the proof of Theorem 3.2, in square brackets on the right, is caused by the deviation of $\tilde{W}$ from W at half-integer points.

We now apply Theorem 5.1, letting $F^n = F_1^n + F_2^n$, where $F_1^n(\chi) = q_B^n(W)(\chi)$ and $F_2^n(\chi)$ is the square bracket. Using also the analogue of (5.6) for $||\hat{\tilde{W}}^n - W(t_n)||_1$, the terms corresponding to F_2 are easily bounded in the appropriate manner. The term $f^n = \partial\bar{\partial}\rho^n + \tau^n$ is estimated analogously to Theorem 3.2. One then obtains (5.5) with $\tilde{W}(k/2)$ instead of $W(k/2)$ in the second term on the right. Again using (5.6) completes the proof. □

We next consider the question of how to choose discrete initial data for the present scheme. Proceeding as in Section 3, let $u_0 = u(0)$, $u_1 = u_t(0)$ and $u_2 = u_{tt}(0) = -A(0)u_0 + f(0)$. With U^0, V_1 and V_2 approximations to these functions we set

$$U^1 = U^0 + kV_1 + k^2V_2/2.$$

Proceeding as in the proof of Proposition 3.3, now comparing with $R_h(0)(u_0 + su_1 + (s^2/2)u_2)$, we find:

Proposition 5.3. *Assume that*

$$||U^0 - u_0||_1 \le Ch^{r-1}, \qquad ||V_1 - u_1|| + k||V_1 - u_1||_1 \le Ch^{r-1},$$

and

$$k||V_2 - u_2|| \le Ch^{r-1}, \qquad ||V_2||_1 \le C.$$

Then

$$||\partial(U^0 - W^0)||_1 + ||U^{1/2} - W(k/2)||_1 \le C(u)(h^{r-1} + k^2). \tag{5.7}$$

A particular choice satisfying the above would be to take the Ritz projection at $t = 0$ of u_0, u_1 and u_2, respectively. Another choice for V_2 would be to define it via $(V_2, \chi) = A_0(u_0, \chi) + (f(0), \chi)$. This latter choice also satisfies (5.7) as can be seen by simple modifications in the arguments outlined.

The analogue of Corollary 3.4 also obtains.

We proceed to treat L_2-estimates. Letting

$$|||\phi^{n+1/2}|||_0^2 = ||\partial\phi^n||_{-1}^2 + ||\phi^{n+1/2}||^2,$$

we have the following analogue of Theorem 3.5. For notation, cf. (5.4).

Theorem 5.4. *If* (5.1) *is* ω*-stable, then for* $n \geq 1$, $t_{n+1} \leq T$,

$$(5.8)\quad |||U^{n+1/2}|||_0 \leq C\Big\{|||U^{1/2}|||_0 + ||U^0||_{-1,h} + h|||U|||_{1;n} + k\sum_{m=1}^{n}||f^m||_{-1,h} + k\sum_{m=1}^{n}||\bar{\partial}F^m||_{-2,h}\Big\}.$$

Proof. We take $\chi = \bar{\partial}(T_nU^{n+1/2})$ in (5.2). Corresponding to the development in Theorem 3.5, we have

$$A_n(\hat{U}^n, \bar{\partial}(T_nU^{n+1/2})) = \frac{1}{2}\,\bar{\partial}||U^{n+1/2}||^2 - \frac{1}{2}\,(\bar{\partial}A_n)(U^{n+1/2} + U^{n-1/2}, T_{n-1}U^{n-1/2})$$

and

$$(\partial\bar{\partial}U^n, \bar{\partial}(T_nU^{n+1/2})) = \frac{1}{2}\,\bar{\partial}||\partial U^n||_{-1,n}^2 - \frac{1}{2}(\partial U^n, (\bar{\partial}T_n)\partial U^{n-1}) + (\partial\bar{\partial}U^n, (\bar{\partial}T_n)U^n).$$

Analogously to (3.15), we write

$$(5.9)\quad (\partial\bar{\partial}U^n, (\bar{\partial}T_n)U^n) = \bar{\partial}(\partial U^n, (\bar{\partial}T_n)U^n) - (\partial U^{n-1}, (\bar{\partial}^2T_n)U^n) - (\partial U^{n-1}, (\bar{\partial}T_{n-1})\partial U^{n-1}).$$

The only essential modification compared to the proof of Theorem 3.5 is as follows: After summation, the first two terms on the right in (5.9) will give rise to terms involving ∂U^n and U^n, not $U^{n+1/2}$ which is what occurs in the norm under consideration. However, as indicated in the first step of (3.19), the norm involved in U^n will be $||U^n||_{-1,h}$ rather than the L_2-norm, and such terms can thus be bounded by $||U^0||_{-1,h}$ and $k\sum_{j=0}^{n-1}||\partial U^j||_{-1,h}$. (This accounts for the second term on the right of (5.8).) □

The analogue of Theorem 3.6 is now as follows: We choose initial data as in Theorem 5.3 and demand, in addition, that

$$||U^0 - u_0|| \le Ch^r, \qquad ||V_1 - u_1||_{-1} + k||V_1 - u_1|| \le Ch^r,$$

and

$$k||V_2 - u_2||_{-1} \le Ch^r.$$

Theorem 5.5. *Under the above hypotheses and if* (5.1) *is ω-stable, then for $n \ge 1$, $t_{n+1} \le T$,*

$$||U^{n+1/2} - u(t_{n+1/2})|| \le Ck\sum_{m=1}^{n}||\bar\partial q_B^m(W)||_{-2,h} + C(u)(h^r + k^2).$$

The global quadrature error. In the rest of this section we shall discuss two quadrature formulas with persistent quadrature weights and bound the corresponding error terms in the theorems above.

The midpoint rule. Let

$$\sigma^n(g) = k\sum_{j=0}^{n-1} g(t_{j+1/2}).$$

For this rule, the storage requirement is $O(k^{-1})$.

Proposition 5.6. *The midpoint rule is ω-stable and satisfies*

$$k\sum_{m=1}^{n}||\bar\partial q_B^m(W)||_{-i} \le Ck^2 \begin{cases} \int_0^{t_n}||u||_{(1,2)}\,ds, & i = 1,\\ \int_0^{t_n}||u||_{(0,2),h}\,ds, & i = 2.\end{cases}$$

Proof. That the present rule is ω-stable is by now obvious.

We can represent the quadrature error as

$$q^m(g) = \sum_{j=0}^{m-1} \left\{ kg(t_{j+1/2}) - \int_{t_j}^{t_{j+1}} g(s)\,ds \right\} = \int_0^{t_m} \psi(s) D_s^2 g(s)\,ds,$$

where $|\psi(s)| \le Ck^2$. We therefore have, for $\phi \in H_0^1$,

$$\begin{aligned}
&|q^m(B_m(W,\phi)) - q^{m-1}(B_{m-1}(W,\phi))| \\
&= \left| k \int_0^{t_{m-1}} \psi(s) D_s^2(\bar\partial_1 B)(t_m, s; W(s), \phi)\,ds \right. \\
&\quad \left. + \int_{t_{m-1}}^{t_m} \psi(s) D_s^2 B(t_m, s; W(s), \phi)\,ds \right| \\
&\le Ck^2 \left\{ k \int_0^{t_{m-1}} ||W||_{(1,2)}\,ds + \int_{t_{m-1}}^{t_m} ||W||_{(1,2)}\,ds \right\} ||\phi||_1.
\end{aligned}$$

After summation and use of Proposition 2.2 this completes the proof for $i = 1$. The case $i = 2$ involves only minor modifications. □

A modified Simpson's rule. As in the case of the schemes in Section 4 employing backward differencing in time, we shall now propose a quadrature formula which uses fewer time steps, thus reducing the storage requirements without sacrificing accuracy. This formula will be based on Simpson's formula, using larger time steps than k in the main part of the interval of integration.

Since we have assumed the quadrature formula to be expressed in terms of the averages $U^{n+1/2} = (U^n + U^{n+1})/2$, we shall need to shift by $k/2$ the intervals for which we use Simpson's rule based on the larger time step k_1. Thus let $m_1 = [k^{-1/2}]$ and $k_1 = m_1 k$. Choose j_n to be the largest even integer such that $j_n k_1 < t_n$. We shall then apply Simpson's rule with step k_1 on the interval $[k/2, j_n k_1 + k/2]$. On the remaining intervals $[0, k/2]$, $[j_n k_1 + k/2, j_n k_1 + k]$, and $[j_n k_1 + k, t_n]$ we use a rectangle rule on the two former intervals of length $k/2$ and the midpoint rule with step k on the latter. If we thus denote the quadrature points by

$$\bar t_j^n = \begin{cases} jk_1 + k/2, & \text{for } 0 \le j \le j_n, \\ j_n k_1 + (j - j_n + 1/2)k, & \text{for } j_n < j \le J_n = j_n + (n - 1 - j_n m_1), \end{cases}$$

we set

$$\sigma^n(g) = \frac{k}{2} g(\bar{t}_0^n) + \frac{k_1}{3} \sum_{j=1}^{j_n/2} \{g(\bar{t}_{2j}^n) + 4g(\bar{t}_{2j-1}^n) + g(\bar{t}_{2j-2}^n)\} + \frac{k}{2} g(\bar{t}_{j_n}^n) + k \sum_{j=j_n+1}^{n-1} g(\bar{t}_j^n).$$

The storage requirement for this rule is $O(k^{-1/2})$, and we have the following.

Proposition 5.7. *The modified Simpson's rule is ω-stable and satisfies*

(5.10)
$$k \sum_{m=1}^{n} \|\bar{\partial} q_B^m(W)\|_{-i} \le Ck^2 \begin{cases} \int_0^{t_n} \|u\|_{(1,4)}\, ds + \sup_{0\le s\le t_n} \|u(s)\|_{(1,2)}, & i=1, \\ \int_0^{t_n} \|u\|_{(0,4),h}\, ds + \sup_{0\le s\le t_n} \|u(s)\|_{(0,2),h}, & i=2. \end{cases}$$

Proof. We shall treat only the case $i = 1$. Corresponding to the proof of Proposition 4.3, we find that the quadrature coefficients are dominated by

$$\omega_j = \begin{cases} 2k_1, & \text{for } j \equiv 0 \pmod{m_1}, \\ k, & \text{otherwise,} \end{cases}$$

and persistence follows as there.

We proceed to estimate the quadrature error and set, for simplicity in writing, in this proof only, $T_m = \bar{t}_{j_m}^m$. The quantity $q^m(B_m(W,\phi)) - q^{m-1}(B_{m-1}(W,\phi))$ will involve portions where Simpson's rule or the midpoint rule is in effect, and as in Section 4 those contributions are bounded for each m by

$$C\bigg\{k^3 \int_0^{t_m} \|W\|_{(1,4)}\, ds + k^2 \int_{T_{m-1}}^{T_m} \|W\|_{(1,4)}\, ds + k^2 \int_{t_{m-1}}^{t_n} \|W\|_{(1,2)}\, ds\bigg\}.$$

Here the term involving k^3 comes from common quadrature intervals where difference quotients of B are formed and the other two terms come from changed intervals. The sum of these contributions is bounded by the right-hand side of (5.10), after use of Proposition 2.2.

For the two small intervals $I_0 = [0, k/2]$ and $I_1^m = [T_m, T_m + k/2]$, where the rectangle rule is in effect, a different argument will be made. For I_0, it occurs in the quadrature formula as soon as $j_m > 0$ and except for the first time it enters both in q^m and q^{m-1} and hence a difference quotient in B can be formed. Its total contribution is thus bounded by

$$(5.11)\quad C\bigg\{\sum_{m=1}^{n} k\int_0^{k/2} |\psi_0|\,||W||_{(1,1)}\,ds + \int_0^{k/2} |\psi_0|\,||W||_{(1,1)}\,ds\bigg\}||\phi||_1 \\ \le Ck^2 \sup_{0\le s\le t_{n+1}} ||W(s)||_{(1,1)}||\phi||_1.$$

For I_1^m, if it does not change between $m-1$ and m, we can again form a difference quotient in B and bound the total of such contributions by the right-hand side of (5.11). If it changes, i.e., if $T_{m-1} \ne T_m$, then $T_m = T_{m-1} + 2k_1$ and its contribution is

$$J_m = \bigg| \int_{T_{m-1}}^{T_{m-1}+k/2} \psi_0(s) D_s B(t_{m-1}, s; W(s), \phi)\,ds \\ - \int_{T_{m-1}+2k_1}^{T_{m-1}+2k_1+k/2} \psi_0(s) D_s B(t_m, s; W(s), \phi)\,ds \bigg|.$$

Since $2k_1$ is a multiple of $k/2$, we have $\psi_0(s + 2k_1) = \psi_0(s)$, cf. (4.1), and hence after a change of variable in the second integral,

$$J_m = \bigg| \int_{T_{m-1}}^{T_{m-1}+k/2} \psi_0(s) D_s[B(t_{m-1}, s; W(s), \phi) \\ - B(t_m, s + 2k_1; W(s + 2k_1), \phi)]\,ds \bigg| \\ \le Ck^2 k_1 \sup_{0\le s\le t_m} ||W(s)||_{(1,2)}||\phi||_1.$$

However, such changes in T_m occur at most $[t_n/(2k_1)]$ times for $0 \le m \le n$ and hence the sum total of those contributions is bounded by the right-hand side of (5.10), again after use of Proposition 2.2. This concludes the proof. □

We finally remark that the above results show error estimates of the form

$$||U^{n+1/2} - u(t_{n+1/2})||_j \le C(u)(h^{r-j} + k^2), \quad \text{for } j = 0, 1,$$

under various hypotheses. If one desires instead approximations to $u(t)$ at $t = t_n$, it is clear that $\hat{U}^n = (U^{n+1/2} + U^{n-1/2})/2$ furnishes such of the same orders of accuracy.

6. The parabolic case. In this section, we shall briefly consider analogues of our above results in the case of the parabolic initial boundary value problem

$$\begin{aligned} u_t + A(t)u &= \int_0^t B(t,s)u(s)\,ds + f(t), \quad \text{in } \Omega \times J, \\ u &= 0, \quad \text{on } \partial\Omega \times J, \\ u(x,0) &= u_0(x), \quad \text{in } \Omega, \end{aligned} \tag{6.1}$$

with the same notation as in (1.1). We shall consider first the backward Euler, then the Crank-Nicolson method, and finally a second order three-level backward differencing method, all combined with quadrature rules of the same type as have been employed in the hyperbolic case above.

The backward Euler method. This method consists in approximating (6.1) by

$$\begin{aligned} (\bar{\partial}U^n, \chi) + A_n(U^n, \chi) &= \sigma^n(B_n(U,\chi)) + (f^n, \chi), \\ &\text{for } \chi \in S_h,\ n \ge 1, \\ U^0 &= u_{0h}, \end{aligned} \tag{6.2}$$

where A_n and σ^n are as in Section 3. We shall again first discuss the stability of a modified equation, namely,

$$\begin{aligned} (\bar{\partial}U^n, \chi) + A_n(U^n, \chi) &= \sigma^n(B_n(U,\chi)) + (f^n, \chi) + F^n(\chi), \\ &\text{for } \chi \in S_h,\ n \ge 1, \\ U^0 &= u_{0h}, \end{aligned} \tag{6.3}$$

where F^n is a linear functional on H_0^1, for convenience with $F^0 = 0$. We begin with a stability estimate in terms of the discrete norm

$$|||\phi^n|||_1^2 = k \sum_{m=1}^{n} ||\bar{\partial}\phi^m||^2 + ||\phi^n||_1^2, \quad \text{for } n \geq 1.$$

Theorem 6.1. *Assume that the quadrature rule σ^n is ω-stable. Then we have for the solution of* (6.3) *that, for $n \geq 1$, $t_n \leq T$,*

$$|||U^n|||_1 \leq C\bigg\{||U^0||_1 + \bigg(k \sum_{m=0}^{n} ||f^m||^2\bigg)^{1/2} + k \sum_{m=1}^{n} ||\bar{\partial}F^m||_{-1}\bigg\}.$$

Proof. We choose $\chi = \bar{\partial}U^n$ in (6.3) to obtain

(6.4)

$$\begin{aligned}(\bar{\partial}U^n, \bar{\partial}U^n) + A_n(U^n, \bar{\partial}U^n) &= \sigma^n(B_n(U, \bar{\partial}U^n)) + (f^n, \bar{\partial}U^n) + F^n(\bar{\partial}U^n) \\ &= I_1^n + I_2^n + I_3^n.\end{aligned}$$

Recalling (3.4) for the second term on the left, we find by multiplication of (6.4) by $2k$ and summation from $n = 1$ to N, that

$$\begin{aligned}2k \sum_{n=1}^{N} ||\bar{\partial}U^n||^2 &+ c||U^N||_1^2 \\ &\leq C\bigg\{||U^0||_1^2 + k\bigg|\sum_{n=1}^{N}(I_1^n + I_2^n + I_3^n)| + k \sum_{n=1}^{N} ||U^{n-1}||_1^2\bigg\}.\end{aligned}$$

Now letting

$$|||U|||_{1;N} = \max\bigg(\max_{1 \leq n \leq N} |||U^n|||_1, ||U^0||_1\bigg), \tag{6.5}$$

we obtain by Cauchy-Schwarz's inequality

$$\bigg|k \sum_{n=1}^{N} I_2^n\bigg| \leq \bigg(k \sum_{n=1}^{N} ||f^n||^2\bigg)^{1/2} |||U|||_{1;N}.$$

The terms I_1 and I_3 are treated exactly as in Theorem 3.1. Altogether, this yields

$$|||U|||_{1;N} \le C\Big\{||U^0||_1 + \Big(k\sum_{n=1}^{N}||f^n||^2\Big)^{1/2} + k\sum_{n=1}^{N}||\bar\partial F^n||_{-1}\Big\} + C\sum_{n=1}^{N-1}(\omega_n + k)|||U^n|||_1.$$

An application of the discrete Gronwall's lemma now completes the proof. □

For the purpose of an L^2 norm error estimate, we also show a stability result in a discrete norm now defined by

$$|||\phi^n|||_0^2 = k\sum_{m=1}^{n}||\bar\partial\phi^m||_{-1}^2 + ||\phi^n||^2.$$

For notation, cf. also (6.5).

Theorem 6.2. *Assume that the quadrature rule σ^n is ω-stable. Then, for $n \ge 1$, $t_n \le T$,*

$$|||U^n|||_0 \le C\Big\{||U^0||_{0,h} + h|||U|||_{1;n} + \Big(k\sum_{m=1}^{n}||f^m||_{-1,h}^2\Big)^{1/2} + k\sum_{m=1}^{n}||\bar\partial F^m||_{-2,h}\Big\}.$$

Proof. The proof parallels that of Theorem 3.5 with appropriate changes. We choose $\chi = \bar\partial(T_nU^n)$ in (6.3) to obtain

$$\begin{aligned}(6.6)\qquad &(\bar\partial U^n, \bar\partial(T_nU^n)) + A_n(U^n, \bar\partial(T_nU^n))\\ &\quad= \sigma^n(B_n(U, \bar\partial(T_nU^n)) + (f^n, \bar\partial(T_nU^n)) + F^n(\bar\partial(T_nU^n))\\ &\quad= I_1^n + I_2^n + I_3^n, \quad \text{for } n \ge 1.\end{aligned}$$

Corresponding to (3.16), we have by use of (3.12) that

$$\begin{aligned}(\bar\partial U^n, \bar\partial(T_nU^n)) &= ||\bar\partial U^n||_{-1,n}^2 + (\bar\partial U^n, (\bar\partial T^n)U^{n-1})\\ &\ge ||\bar\partial U^n||_{-1,n}^2 - C||\bar\partial U^n||_{-1,h}||U^{n-1}||.\end{aligned}$$

Further, recall (3.17). Now introduce L_n by

$$L_n^2 = k\sum_{j=1}^{n} ||\bar{\partial}U^j||_{-1,j}^2 + ||U^n||^2 + h^2|||U^n|||_1^2, \quad \text{for } n \geq 1,$$
$$L_0^2 = ||U^0||^2 + h^2||U^0||_1^2.$$

Multiplying both sides of (6.6) by $2k$ and summing from $n = 1$ to N, we get, similarly to (3.18),

$$L_N^2 \leq ||U^0||L_0 + h|||U^N|||_1 L_N + \left|2k\sum_{n=1}^{N}(I_1^n + I_2^n + I_3^n)\right| + C\left\{k\sum_{n=1}^{N}(||\bar{\partial}U^n||_{-1,h} + ||U^n||_{0,h})||U^{n-1}||\right\}.$$

Using Cauchy-Schwarz's inequality and shifting indices, the last term on the right above is bounded by $Ck(\sum_{n=1}^{N-1} L_n)\max_{0\leq n\leq N} L_n$. By (3.12),

$$\begin{aligned} k\left|\sum_{n=1}^{N} I_2^n\right| &\leq k\sum_{n=1}^{N}\{|(f^n, T_n(\bar{\partial}U^n)| + |(f^n, (\bar{\partial}T_n)U^{n-1})|\} \\ &\leq k\sum_{n=1}^{N}\{||f^n||_{-1,h}||\bar{\partial}U^n||_{-1,n} + ||f^n||_{-1,h}||U^{n-1}||_{-1,h}\} \\ &\leq \left(k\sum_{n=1}^{N}||f^n||_{-1,h}^2\right)^{1/2}\max_{0\leq n\leq N} L_n. \end{aligned}$$

The terms involving I_1 and I_3 are treated as in Theorem 3.5. Altogether we obtain, corresponding to (3.20),

$$\max_{0\leq n\leq N} L_n \leq C\left\{||U^0||_{0,h} + h|||U|||_{1;N} + \left(k\sum_{n=1}^{N}(||f^n||_{-1,h}^2\right)^{1/2} + k\sum_{n=1}^{N}||\bar{\partial}F^n||_{-2,h}\right\} + C\sum_{n=1}^{N-1}(\omega_n + k)L_n.$$

An application of the discrete Gronwall's lemma then completes the proof. □

We now state our preliminary, quadrature-dependent error estimates:

Theorem 6.3. *Assume that the quadrature rule σ^n is ω-stable, and let U^0 be such that*

$$||U^0 - u_0||_{0,h} \le Ch^r.$$

Then, with W the Ritz-Volterra projection of u, we have

$$||U^n - u(t_n)||_1 \le Ck \sum_{m=1}^{n} ||\bar\partial q_B^m(W)||_{-1} + C(u)(h^{r-1} + k),$$

and

$$||U^n - u(t_n)|| \le Ck \sum_{m=1}^{n} ||\bar\partial q_B^m(W)||_{-2,h} + C(u)(h^r + k), \quad \text{for } n \ge 1,\ t_n \le T.$$

Proof. We shall only carry out the proof of the H^1-estimate. With

$$U^n - u(t_n) = (U^n - W(t_n)) + (W(t_n) - u(t_n)) = \theta^n + \rho^n,$$

we have

$$(\bar\partial\theta^n, \chi) + A_n(\theta^n, \chi) = \sigma^n(B_n(\theta, \chi)) - (\bar\partial\rho^n + \tau^n, \chi) + q_B^n(W)(\chi), \quad \text{for } \chi \in S_h,\ n \ge 1,$$

where $\tau^n = \bar\partial u(t_n) - u_t(t_n)$. We now apply Theorem 6.1 to obtain

$$||\theta^n||_1 \le C\bigg\{||\theta^0||_1 + \bigg(k \sum_{m=1}^{n} (||\bar\partial\rho^m||^2 + ||\tau^m||^2)\bigg)^{1/2} + k \sum_{m=1}^{n} ||\bar\partial q_B^m(W)||_{-1}\bigg\}.$$

We have easily

$$k\sum_{m=1}^{n}(||\tau^m||^2+||\bar\partial\rho^m||^2)\le Ch^{2r}\int_0^{t_n}||u||^2_{(r,1)}\,ds+Ck^2\int_0^{t_n}||u_{tt}||^2\,ds,$$

and the desired estimate now follows using the triangle inequality from the known estimate for $||\rho^n||_1$. □

If one were to carry out the details of the above L_2 estimates, one would find that $C(u)$ contains a term $(\int_0^T ||u_t||_r^2\,ds)^{1/2}$ caused by the treatment of the term involving f^n. The corresponding term in [**11, 12**] would be the weaker $\int_0^T ||u_t||_r\,ds$, under an inverse assumption. For $r\ge 4$, it is possible to avoid the square integrated terms by treating the f^n differently: In essence, treat them like F^n. This leads to terms like $k\sum_{n=1}^N ||\bar\partial f^n||_{-2}$. Considering the part of f^n corresponding to $\bar\partial\rho^n$, we thus have to estimate essentially $\int_0^T ||D_s^2\rho||_{-2}\,ds$. For $r\ge 4$, one easily deduces (cf. the corresponding well-known result in negative norms for elliptic projections) that

$$\int_0^T ||D_2^s\rho||_{-2}\,ds\le Ch^3\int_0^T ||\rho||_{(1,2)}\,ds,$$

and here, using Proposition 2.2 and (6.1),

$$\begin{aligned}\int_0^T ||\rho||_{(1,2)}\,ds &\le Ch^{r-3}\int_0^T ||u||_{(r-2,2)}\,ds\\ &\le Ch^{r-3}\int_0^T (||u||_{(r,1)}\,ds+||f_t||_{r-2})\,ds.\end{aligned}$$

The argument sketched above does not work for $r=2,3$. We shall show next that our energy techniques do give the results of [**11, 12**] with the inverse assumption

$$||\chi||_1\le Ch^{-1}||\chi||,\quad \text{for } \chi\in S_h. \tag{6.7}$$

Using this, we obtain from (3.11)

$$|Q(\chi,T_n\psi)|\le C||\chi||\,||\psi||,\quad \text{for } \chi,\psi\in S_h. \tag{6.8}$$

Theorem 6.4. *Assume that the quadrature rule σ^n is ω-stable and that* (6.6) *holds. Then*

$$||U^n|| \le C\Big\{||U^0|| + k\sum_{m=1}^{n}||f^m|| + k\sum_{m=1}^{n}||\bar\partial F^m||_{-2,h}\Big\},$$
$$\text{for } n \ge 1,\ t_n \le T.$$

Proof. In view of Theorem 6.2, it suffices, by linearity, to consider the case $U^0 = 0$, $F^m = 0$, for $m \ge 0$. Let $I_1^n(\chi) = \sigma^n(B_n(U,\chi))$ and $I_2^n(\chi) = (f^n,\chi)$, and define $U_i^n \in S_h$, $i = 1, 2$, by

$$(\bar\partial U_i^n, \chi) + A_n(U_i^n, \chi) = I_i^n(\chi), \quad \text{for } \chi \in S_h,\ n \ge 1, \qquad U_i^0 = 0, \tag{6.9}$$

so that the solution of (6.3) may be written as $U^n = U_1^n + U_2^n$. For the estimate of U_2^n, we choose $\chi = U_2^n$ and obtain by a standard calculation

$$||U_2^n|| \le Ck\sum_{m=1}^{n}||f^m||. \tag{6.10}$$

We therefore only need to discuss the L_2-estimate for U_1^n. For this purpose, let $\chi = \bar\partial(T_nU_1^n)$ in (6.9) with $i = 1$. Using (3.10) and (3.12), we now proceed in the same way as in the proof of Theorem 6.2, with the obvious changes resulting from the inverse assumption (6.7) and the estimate (6.8), and obtain

$$|||U_1^n|||_0 \le C\Big\{h|||U_1|||_{1;N} + \sum_{n=0}^{N-1}(\omega_n + k)||U^n||\Big\}.$$

Combining with (6.10) gives

$$||U^N|| \le C\Big\{h|||U_1|||_{1;N} + k\sum_{n=1}^{N}||f^n||\Big\} + C\sum_{n=0}^{N-1}(\omega_n + k)||U^n||. \tag{6.11}$$

We now claim that

$$h|||U_1|||_{1;N} \le C\sum_{n=0}^{N-1}(\omega_n + k)||U^n||. \tag{6.12}$$

Inserted into (6.11), the proof of Theorem 6.4 is then completed by an application of Gronwall's lemma. In order to show (6.12), we set $\chi = \bar{\partial} U_1^n$ in (6.9) with $i = 1$ and proceed in the same way as in the proof of Theorem 6.1 to obtain

$$|||U_1^N|||_1 \leq C \sum_{n=0}^{N-1} (\omega_n + k)||U^n||_1.$$

Multiplication by h and use of the inverse assumption in the sum on the right completes the proof of the estimate (6.12). □

For the general scheme, we may use any of the three quadrature rules discussed in Section 4, with the global quadrature bounds obtained there. In the presence of the inverse assumption (6.7), we also obtain the following estimate which conforms with the L_2-estimate in [**11, 12**].

Proposition 6.5. *Assume that the quadrature rule σ^n is ω-stable, and that S_h satisfies the inverse assumption. Then we have, for $n \geq 1$, $t_n \leq T$,*

$$k \sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-2,h} \leq C \biggl\{ h^r \biggl(||u_0||_r + \int_0^{t_n} ||u_t||_r \, ds \biggr) + k \int_0^{t_n} ||u||_{(0,\nu)} \, ds \biggr\},$$

where $\nu = 1$ for the rectangle rule, $\nu = 2$ for the modified trapezoidal rule, and $\nu = 4$ for the modified Simpson's rule.

Proof. With P_h the L_2-projection onto S_h, write $W = \eta + P_h u$, where $\eta = W - P_h u$. As in Propositions 4.1, 4.2, and 4.3, we obtain, replacing W by $P_h u$, that

$$\begin{aligned} k \sum_{m=1}^{n} ||\bar{\partial} q_B^m(P_h u)||_{-2,h} &\leq Ck \int_0^{t_n} (||P_h u||_{(0,\nu)} + h||P_h u||_{(1,\nu)}) \, ds \\ &\leq Ck \int_0^{t_n} ||u||_{(0,\nu)} \, ds, \end{aligned}$$

where, in the last step, we have used the inverse assumption and the boundedness of P_h in L_2.

Next, we easily have, using (1.2),

$$k\sum_{m=1}^{n}||\bar{\partial}\int_0^{t_m} B(t_m,s;\eta(s),\cdot)\,ds||_{-2,h}\,ds \\ \le C\int_0^{t_n}||\eta||_{0,h}\,ds \le Ch^r\int_0^{t_n}||u||_r\,ds,$$

and similarly to the estimate of $\sum_n I_1^n$ in Theorem 3.1, in obvious notation,

$$k\sum_{m=1}^{n}||\bar{\partial}\sigma_B^m(\eta)||_{-2,h} \le C\max_{0\le m\le n}||\eta^m||_{0,h} \le Ch^r(||u_0||_r+\int_0^{t_n}||u_t||_r\,ds).$$

This completes the proof. $\square$

The Crank-Nicolson method. We shall now discuss a discretization of (6.1) which is based on the Crank-Nicolson scheme, and which is thus symmetric around the point $t_{n-1/2}$. For this purpose, we introduce the quadrature formula

$$\sigma^n(g) = \sum_{n=0}^{n-1}\omega_{nj}g(t_j) \approx \int_0^{t_{n-1/2}} g(s)\,ds,$$

and define, with $U^{n-1/2} = (U^n + U^{n-1})/2$, the fully discrete scheme by

(6.13)
$$(\bar{\partial}U^n,\chi) + A_{n-1/2}(U^{n-1/2},\chi) = \sigma^n(B_{n-1/2}(U,\chi)) + (f^{n-1/2},\chi), \\ \chi\in S_h,\ n\ge 1, \\ U^0 = u_{0h}.$$

As earlier, we shall need a stability result for an associated modified equation, namely

(6.14)
$$(\bar{\partial}U^n,\chi) + A_{n-1/2}(U^{n-1/2},\chi) \\ = \sigma^n(B_{n-1/2}(U,\chi)) + (f^{n-1/2},\chi) + F^n(\chi), \\ \text{for } \chi\in S_h,\ n\ge 1.$$

Theorem 6.6. *Assume that the quadrature rule σ^n is ω-stable. Then we have, for $n \geq 1$, $t_n \leq T$,*

$$|||U^n|||_1 \leq C\Big\{||U^0||_1 + \Big(k\sum_{m=1}^{n}||f^{m-1/2}||^2\Big)^{1/2} + k\sum_{m=1}^{n}||\bar{\partial}F^m||_{-1}\Big\},$$

and

$$|||U^n|||_0 \leq C\Big\{||U^0||_{0,h} + h|||U|||_{1;n} + \Big(k\sum_{m=1}^{n}||f^{m-1/2}||^2_{-1,h}\Big)^{1/2} + k\sum_{m=1}^{n}||\bar{\partial}F^m||_{-2,h}\Big\}.$$

Further, under the inverse assumption (6.7), *we have*

$$|||U^n|||_0 \leq C\Big\{||U^0|| + k\sum_{m=1}^{n}||f^{m-1/2}|| + k\sum_{m=1}^{n}||\bar{\partial}F^m||_{-2,h}\Big\}.$$

Proof. We assume that the coefficients of A and B are smoothly extended to small negative t so that $A_{-1/2}$ and $B_{-1/2}$ are defined. Noting that

$$A_{n-1/2}(U^{n-1/2}, \bar{\partial}U^n) = \frac{1}{2}\,\bar{\partial}(A_{n-1/2}(U^n, U^n)) - \frac{1}{2}\,(\bar{\partial}A_{n-1/2})(U^{n-1}, U^{n-1}),$$

the first estimate follows in a straightforward way as in Theorem 6.1. For the second estimate, choose $\chi = \bar{\partial}(T_{n-1/2}U^n)$ as in (6.14). The proof is then parallel to that of Theorem 6.2, with obvious changes caused by the form of the terms $A_{n-1/2}, B_{n-1/2}$ and $f^{n-1/2}$, and uses the estimate, cf. (3.17),

$$A_{n-1/2}(U^{n-1/2}, \bar{\partial}(T_{n-1/2}U^n)) \geq \frac{1}{2}\,\bar{\partial}||U^n||^2 - C||U^{n-1/2}||_{0,h}||U^{n-1}||.$$

Using the inverse assumption, the final estimate follows as in Theorem 6.4 with minor modifications. □

Next we state our preliminary error estimate. The proof is analogous to those of Theorems 5.2 and 5.5 and will not be given.

Theorem 6.7. *Assume that the quadrature rule σ^n is ω-stable and that U^0 is chosen so that*

$$||U^0 - u_0||_{0,h} \le Ch^r.$$

Then, with our standard notation, we have, for $n \ge 1$, $t_n \le T$,

$$||U^n - u(t_n)||_1 \le Ck \sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-1} + C(u)(h^{r-1} + k^2)$$

and

$$||U^n - u(t_n)|| \le Ck \sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-2,h} + C(u)(h^r + k^2).$$

We proceed to discuss two quadrature formulas for the Crank-Nicolson method. The first consists of applying the standard trapezoidal rule with stepsize k on $[0, t_{n-1}]$ and then the rectangle rule on $[t_{n-1}, t_{n-1/2}]$. Thus,

$$\sigma_T^n(g) = \frac{k}{2} \sum_{j=1}^{n-1} [g(t_j) + g(t_{j-1})] + \frac{k}{2} g(t_{n-1}).$$

The storage requirement for this modified trapezoidal rule is $O(k^{-1})$. Our second rule, with storage requirement $O(k^{-1/2})$, is the following modified Simpson's rule. Let $m_1 = [k^{-1/2}]$ and $k_1 = m_1 k$, let j_n be the largest even integer such that $j_n k_1 < t_n$ and introduce quadrature points

$$\bar{t}_j^n = \begin{cases} jk_1, & 0 \le j \le j_n, \\ \bar{t}_{j_n}^n + (j - j_n)k & j_n \le j \le J_n, \end{cases}$$

where $\bar{t}_{J_n}^n = t_{n-1}$. Then set

$$\sigma_S^n(g) = \frac{k_1}{3} \sum_{j=1}^{j_n/2} [g(\bar{t}_{2j}^n) + 4g(\bar{t}_{2j-1}^n) + g(\bar{t}_{2j-2}^n)] + \frac{k}{2} \sum_{j=j_n+1}^{J_n} [g(\bar{t}_j^n) + g(\bar{t}_{j-1}^n)] + \frac{k}{2} g(\bar{t}_{J_n}^n).$$

Proposition 6.8. *The quadrature rules above are ω-stable and*

$$k \sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-i}$$
$$\leq Ck^2 \begin{cases} \int_0^{t_n} ||u||_{(1,\nu)}\, ds + \sup_{0\leq s\leq t_n} ||u(s)||_{(1,2)} & i=1, \\ \int_0^{t_n} ||u||_{(0,\nu),h}\, ds + \sup_{0\leq s\leq t_n} ||u(s)||_{(0,2),h} & i=2, \end{cases}$$

where $\nu = 2$ or $\nu = 4$ for the modified trapezoidal and Simpson's rule, respectively.

Proof. By now, ω-stability is clear. The proof for the quadrature error follows standard lines with a slight modification to treat the intervals of length $k/2$ at the right ends. (The modification is essentially as that in the proof of Proposition 5.7.) The contribution to $k\bar{\partial} q_B^m(W)(\phi)$ from these ends equals

$$\int_{t_{m-1}}^{t_{m-1/2}} \psi_0(s) D_s B(t_{m-1/2}, s; W(s), \phi)\, ds$$
$$- \int_{t_{m-2}}^{t_{m-3/2}} \psi_0(s) D_s B(t_{m-3/2}, s; W(s), \phi)\, ds$$
$$= \int_{t_{m-2}}^{t_{m-3/2}} \psi_0(s) D_s (B(t_{m-1/2}, s+k; W(s+k), \phi)$$
$$- B(t_{m-3/2}, s; W(s), \phi))\, ds,$$

where we have used that $\psi_0(s+k) = \psi_0(s)$. Its $||\cdot||_{-i}$ norm is thus bounded by

$$Ck^3 \sup_{t_{m-2}\leq s\leq t_m} ||W(s)||_{(2-i,2)}$$

for $i = 1, 2$, respectively. After summation in m and use of Proposition 2.2, we obtain the correct $O(k^2)$ contribution from these terms. □

A second order backward difference method. We shall finally consider a three-step backward difference method. For this, let

$$D^{(2)} U^n = \bar{\partial} U^n + \frac{1}{2} k \bar{\partial}^2 U^n$$

and let U^n be defined by

$$(D^{(2)}U^n, \chi) + A_n(U^n, \chi) = \sigma^n(B_n(U, \chi)) + (f^n, \chi), \quad \chi \in S_h, \ n \geq 2,$$
$$(\bar{\partial} U^1, \chi) + A_1(U^1, \chi) = \sigma^1(B_1(U, \chi)) + (f^1, \chi), \quad \chi \in S_h,$$
$$U^0 = u_{0h}.$$

Here σ^n is a quadrature rule which uses values U^j with $j = 0, \dots, n-1$.

Following the development for the previous methods, we have:

Theorem 6.9. *Assume that the quadrature rule is ω-stable and that U^0 is chosen so that*

$$||U^0 - u_0||_{0,h} \leq Ch^r.$$

Then, for $n \geq 1$, $t_n \leq T$,

$$||U^n - u(t_n)||_1 \leq Ck \sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-1} + C(u)(h^{r-1} + k^2)$$

and

$$||U^n - u(t_n)|| \leq Ck \sum_{m=1}^{n} ||\bar{\partial} q_B^m(W)||_{-2,h} + C(u)(h^r + k^2).$$

Proof. A direct calculation shows that U^1 satisfies the above estimates. The argument then proceeds in the usual way with the following modifications. Now

$$(D^{(2)}U^n, \bar{\partial} U^n) = ||\bar{\partial} U^n||^2 + \frac{1}{2} k(\bar{\partial}^2 U^n, \bar{\partial} U^n) \geq ||\bar{\partial} U^n||^2 + \frac{1}{4} k \bar{\partial}(||\bar{\partial} U^n||^2).$$

Further,

$$(D^{(2)}U^n, \bar{\partial}(T_n U^n)) = (\bar{\partial} U^n, \bar{\partial}(T_n U^n)) + \frac{1}{2} k(\bar{\partial}^2 U^n, \bar{\partial}(T_n U^n)),$$

and here the first term is treated as in Theorem 6.2 and the second as in (3.16). □

As for quadrature rules for the present scheme, it is natural to use the rectangle rule when calculating U^1. For $n \geq 2$, a modified trapezoidal rule or a modified Simpson's rule may then be used.

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