Numerical methods for nonsmooth mechanical systems

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Objectives

- Formulation of nonsmooth dynamical systems
  - Measure differential inclusions
- Basics on Mathematical properties
- Formulation of unilateral contact, Coulomb's friction and impacts.

Lagrange's equations

Definition (Lagrange's equations)

$$\frac{d}{dt}\left(\frac{\partial L(q,v)}{\partial v_i}\right) - \frac{\partial L(q,v)}{\partial q_i} = Q_i(q,t), \quad i \in \{1 \ldots n\}, \tag{1}$$

where

- $q(t) \in \mathbb{R}^n$ generalized coordinates,
- $v(t) = \dfrac{dq(t)}{dt} \in \mathbb{R}^n$ generalized velocities,
- $Q(q,t) \in \mathbb{R}^n$ generalized forces
- $L(q,v) \in \mathbb{R}$ Lagrangian of the system,

$$L(q,v) = T(q,v) - V(q),$$

together with

- $T(q,v) = \dfrac{1}{2} v^T M(q) v$, kinetic energy, $M(q) \in \mathbb{R}^{n\times n}$ mass matrix,
- $V(q)$ potential energy of the system,

Lagrange's equations

$$M(q)\frac{dv}{dt} + N(q,v) = Q(q,t) - \nabla_q V(q) \tag{2}$$

where

- $N(q,v) = \left[\frac{1}{2} \sum_{k,l} \frac{\partial M_{ik}}{\partial q_l} + \frac{\partial M_{il}}{\partial q_k} - \frac{\partial M_{kl}}{\partial q_i}, i = 1 \dots n \right]$ the nonlinear inertial terms i.e., the gyroscopic accelerations

Internal and external forces which do not derive from a potential

$$M(q)\frac{dv}{dt} + N(q,v) + F_{int}(t,q,v) = F_{ext}(t), \tag{3}$$

where

- $F_{int} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ non linear interactions between bodies,
- $F_{ext} : \mathbb{R} \to \mathbb{R}^n$ external applied loads.

Linear time invariant (LTI) case

- $M(q) = M \in \mathbb{R}^{n \times n}$ mass matrix
- $F_{int}(t,q,v) = Cv + Kq$, $C \in \mathbb{R}^{n \times n}$ is the viscosity matrix, $K \in \mathbb{R}^{n \times n}$ is the stiffness matrix.

Smooth multibody dynamics

Definition (Equations of motion)

$$\begin{cases} M(q)\dfrac{dv}{dt} + F(t,q,v) = 0, \\ v = \dot{q} \end{cases} \tag{4}$$

where

- $F(t,q,v) = N(q,v) + F_{int}(t,q,v) - F_{ext}(t)$

Definition (Boundary conditions)

- Initial Value Problem (IVP):

$$t_0 \in \mathbb{R}, \quad q(t_0) = q_0 \in \mathbb{R}^n, \quad v(t_0) = v_0 \in \mathbb{R}^n, \tag{5}$$

- Boundary Value Problem (BVP):

$$(t_0, T) \in \mathbb{R} \times \mathbb{R}, \quad \Gamma(q(t_0), v(t_0), q(T), v(T)) = 0 \tag{6}$$

Perfect bilateral constraints, joints, liaisons and spatial boundary conditions

Bilateral constraints

- Finite set of m bilateral constraints on the generalized coordinates :

$$h(q,t) = \left[h_j(q,t) = 0, \quad j \in \{1 \ldots m\}\right]^T. \tag{7}$$

where h_j are sufficiently smooth with regular gradients, $\nabla_q(h_j)$.

- Configuration manifold, $\mathcal{M}(t)$

$$\mathcal{M}(t) = \{q(t) \in \mathbb{R}^n, h(q,t) = 0\}, \tag{8}$$

Tangent and normal space

- Tangent space to the manifold $\mathcal{M}$ at q

$$T_{\mathcal{M}}(q) = \{\xi, \nabla h(q)^T \xi = 0\} \tag{9}$$

- Normal space as the orthogonal to the tangent space

$$N_{\mathcal{M}}(q) = \{\eta, \eta^T \xi = 0, \forall \xi \in T_{\mathcal{M}}\} \tag{10}$$

Bilateral constraints as inclusion

Definition (Perfect bilateral holonomic constraints on the smooth dynamics)

$$\begin{cases} \dot{q} = v \\ \\ M(q)\dfrac{dv}{dt} + F(t,q,v) = r \\ \\ -r \in N_{\mathcal{M}}(q) \end{cases} \tag{11}$$

where r is the generalized force or generalized reaction due to the constraints.

Remark

- The formulation as an inclusion is very useful in practice
- The constraints are said to be perfect due to the normality condition.
- When $\mathcal{M} = \{q(t) \in \mathbb{R}^n, h(q,t) = 0\}$, the multipliers $\mu \in \mathbb{R}^m$ can be intoduced and we get

$$r = \nabla_q h(q,t)\,\mu$$

Perfect unilateral constraints

Unilateral constraints

- Finite set of ν unilateral constraints on the generalized coordinates :

$$g(q,t) = [g_\alpha(q,t) \geqslant 0, \quad \alpha \in \{1 \dots \nu\}]^T . \tag{12}$$

- Admissible set $\mathcal{C}(t)$

$$\mathcal{C}(t) = \{q \in \mathbb{R}^n, g_\alpha(q,t) \geqslant 0, \alpha \in \{1 \dots \nu\}\} . \tag{13}$$

Normal cone to $\mathcal{C}(t)$

$$N_{\mathcal{C}(t)}(q(t)) = \left\{ y \in \mathbb{R}^n, y = -\sum_\alpha \lambda_\alpha \nabla g_\alpha(q,t),\ \lambda_\alpha \geqslant 0,\ \lambda_\alpha g_\alpha(q,t) = 0 \right\} \tag{14}$$

Unilateral constraints as an inclusion

Definition (Perfect unilateral constraints on the smooth dynamics)

$$\begin{cases} \dot{q} = v \\ M(q)\dfrac{dv}{dt} + F(t,q,v) = r \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \tag{15}$$

where r it the generalized force or generalized reaction due to the constraints.

Remark

- The unilateral constraints are said to be perfect due to the normality condition.
- Notion of normal cones can be extended to more general sets. see (Clarke, 1975, 1983 ; Mordukhovich, 1994)
- When $\mathcal{C}(t) = \{q \in \mathbb{R}^n, g_\alpha(q,t) \geqslant 0, \alpha \in \{1 \ldots \nu\}\}$, the multipliers $\lambda \in \mathbb{R}^m$ such that $r = \nabla_q^T g(q,t)\,\lambda$.

Smooth dynamics as a DI

Differential Inclusion

$$-\left[M(q)\frac{dv}{dt} + F(t,q,v)\right] \in N_{\mathcal{C}(t)}(q(t)), \qquad (16)$$

with

$$\dot{q} = v.$$

Remark

- The right hand side is neither bounded (and then nor compact).
- The inclusion and the constraints concern the second order time derivative of q.

➜ Standard Analysis of DI does no longer apply.

Nonsmooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that

$$v^+ = \dot{q}^+ \tag{17}$$

- q is related to this velocity by

$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\, dt \tag{18}$$

- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that

$$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{19}$$

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{20}$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- The nonsmooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

(Schatzman, 1973, 1978 ; Moreau, 1983, 1988)

Nonsmooth Lagrangian Dynamics

Measures Decomposition (for dummies)

$$\left\{ \begin{array}{rlrr} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ di = & f\, dt + & p\, d\nu + & di_s \end{array} \right. \tag{21}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and di_S are singular measures with the respect to $dt + d\eta$.

Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the nonsmooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{22}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{23}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t, q, v)dt = fdt \tag{24}$$

or

$$M(q)\gamma^+ + F(t, q, v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{25}$$

The Moreau's sweeping process of second order

Definition (Moreau (1983, 1988))

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (15) is "replaced" by the following inclusion

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ v^+ = \dot{q}^+ \\ -di \in N_{T_C(q)}(v^+) \end{cases} \tag{26}$$

Comments

This formulation provides a common framework for the nonsmooth dynamics containing inelastic impacts without decomposition.

➜ Foundation for the time–stepping approaches.

The Moreau's sweeping process of second order

Comments

- *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- *The inclusion in terms of velocity* v^+ rather than of the coordinates q.

Interpretation

- Inclusion of measure, $-di \in K$
  - Case $di = r' dt = f dt$.
$$-f \in K \tag{27}$$
  - Case $di = p_i \delta_i$.
$$-p_i \in K \tag{28}$$
- Inclusion in terms of the velocity. Viability Lemma
If $q(t_0) \in C(t_0)$, then
$$v^+ \in T_C(q), t \geqslant t_0 \Rightarrow q(t) \in C(t), t \geqslant t_0$$
➜ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

The Moreau's sweeping process of second order

The Newton-Moreau impact rule

$$-\,di \in N_{T_C(q(t))}(v^+(t) + ev^-(t)) \tag{29}$$

where e is a coefficient of restitution.

Velocity level formulation. Index reduction

$$\begin{array}{c} 0 \leqslant y \perp \lambda \geqslant 0 \\ \Updownarrow \\ -\lambda \in N_{\mathbb{R}^+}(y) \\ \Uparrow \\ -\lambda \in N_{T_{\mathbb{R}^+}(y)}(\dot{y}) \\ \Updownarrow \\ \text{if } y \leqslant 0 \text{ then } 0 \leqslant \dot{y} \perp \lambda \geqslant 0 \end{array} \tag{30}$$

The Moreau's sweeping process of second order

The case of C is finitely represented

$$\mathcal{C} = \{q \in \mathcal{M}(t), g_\alpha(q) \geqslant 0, \alpha \in \{1 \ldots \nu\}\}. \quad (31)$$

Decomposition of di and v^+ onto the tangent and the normal cone.

$$di = \sum_\alpha \nabla_q^T g_\alpha(q)\, d\lambda_\alpha \quad (32)$$

$$U_\alpha^+ = \nabla_q g_\alpha(q)\, v^+, \alpha \in \{1 \ldots \nu\} \quad (33)$$

Complementarity formulation (under constraints qualification condition)

$$-d\lambda_\alpha \in N_{T_{\mathbb{R}_+}(g_\alpha)}(U_\alpha^+) \Leftrightarrow \text{ if } g_\alpha(q) \leqslant 0, \text{ then } 0 \leqslant U_\alpha^+ \perp d\lambda_\alpha \geqslant 0 \quad (34)$$

The case of C is $\mathbb{R}_+$

$$-di \in N_C(q) \Leftrightarrow 0 \leqslant q \perp di \geqslant 0 \quad (35)$$

is replaced by

$$-di \in N_{T_C(q)}(v^+) \Leftrightarrow \text{ if } q \leqslant 0, \text{ then } 0 \leqslant v^+ \perp di \geqslant 0 \quad (36)$$

The Moreau's sweeping process of second order

Summary for perfect schleronomic constraints

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ v^+ = \dot{q}^+ \\ \\ di = H(q)d\lambda \\ U^+ = H(q)^T v^+ \\ \\ \text{ if } g_\alpha(q) \leqslant 0, \text{ then } 0 \leqslant U^+_\alpha \perp d\lambda_\alpha \geqslant 0 \end{cases} \tag{37}$$

where $H(q)$ is the transpose of the Jacobian matrix of the constraints,

$$H(q) = \nabla_q g(q)$$

.

The Moreau's sweeping process in Newton–Euler Formalism

Classical Newton-Euler formalism

The velocity of a rigid body is represented with

- $v_G \in \mathbb{R}^3$ the velocity of the center of mass expressed in a Galilean reference frame $\mathcal{R}_0$,
- $\Omega \in \mathbb{R}^3$ the angular velocity expressed in a frame attached to the solid $\mathcal{R}$, called the body frame .

Rotation matrix and angular velocity vector

By defining the rotation matrix $R \in SO^+(3)$ from $\mathcal{R}_0$ to $\mathcal{R}$, the angular velocity is given by

$$\tilde{\Omega} = R^T \dot{R} \text{ or equivalently } \dot{R} = R\tilde{\Omega}. \tag{38}$$

where the matrix $\tilde{\Omega}$ is defined by $\tilde{\Omega}x = \Omega \times x$, for all $x \in \mathbb{R}^3$.

The Moreau's sweeping process in Newton–Euler Formalism

Smooth Newton-Euler Equations

From the Fundamental Principle of Dynamics, the Newton–Euler equations are obtained as

$$\begin{cases} M\dot{v}_G &= F_{ext}(x_G, v_G, \Omega, R), \\ I\dot{\Omega} + \Omega \times I\Omega &= M_{ext}(x_G, v_G, \Omega, R), \\ \dot{x}_G &= v_G, \\ \dot{R} &= R\tilde{\Omega}, \quad R^{-1} = R^T, \quad \det(R) = 1. \end{cases} \tag{39}$$

where

- x_G is the position of the center of mass,
- $M = mI_{3\times 3}$ is the mass matrix and I the constant inertia matrix,
- F_{ext} is the vector of external forces expressed in $\mathcal{R}_0$
- M_{ext} is the vector of external moments expressed in $\mathcal{R}$

Another angular velocity vector

The Newton–Euler equations can be also expressed in terms of the angular velocity

$$\omega = R\Omega$$

that is the expression of the angular velocity in $\mathcal{R}_0$.

The Moreau's sweeping process in Newton–Euler Formalism

General Formulation

Choosing $q = [x_G, R]^T$ and $v = [v_G, \Omega]$, the Newton–Euler equations fits within the general framework

$$\begin{cases} \dot{q} = T(t,q)v, & (40a) \\ M(q)\dot{v} + F(t,q,v) = T^T(t,q)r = T^T(t,q)H(q)\mu & (40b) \\ h(q) = 0 & (40c) \end{cases}$$

where

- $H(q) = \nabla_q^T h(q)$
- $T(t,q)$ is the operator that links the velocity to the time–derivative of the parameters,
- $h(q) = 0$ are the constraints for the configuration manifold $R \in SO^+(3)$

The Moreau's sweeping process in Newton–Euler Formalism

Parametrization of rotations

The choice $q = [x_G, R]^T \in \mathbb{R}^{12}$ is not well–suited for numerical computation. Generally, the rotation matrix is parametrized by a set of parameters, Θ such that

$$R = R(\Theta)$$

and we get

$$\omega = P(\Theta)\dot{\Theta} \quad \text{or} \quad \Omega = Q(\Theta)\dot{\Theta}.$$

Examples of parametrization:

- geometrical description angles : Euler angles, Cardan/Bryant Angles,
- Rodrigues parameters,
- direct cosines,
- unitary quaternions,
- Cartesian oration vector
- Conformal rotation vector,
- linear parameters, ...

The Moreau's sweeping process in Newton–Euler Formalism

Smooth DI for Newton–Euler Formalism

$$\begin{cases} -\left[M(q)\dfrac{dv}{dt} + F(t,q,v)\right] \in T^T(q,t)N_{\mathcal{C}(t)}(q(t)) \\ \dot{q} = T(q,t)v \end{cases} \tag{41}$$

The case of C is finitely represented

$$\mathcal{C} = \{q \in \mathbb{R}^n, g_\alpha(q) \geqslant 0, \alpha \in \mathcal{I}, g_\alpha(q) = 0, \alpha \in \mathcal{E}\}\,. \tag{42}$$

we get

$$\begin{cases} \dot{q} = T(q,t)v \\ -\left[M(q)\dfrac{dv}{dt} + F(t,q,v)\right] \in T^T(q,t)r \\ \\ r = H(q)\lambda \\ U = H^T(q)T(q,t)v \\ \\ g_\alpha(q) = 0, \alpha \in \mathcal{E} \\ 0 \leqslant g_\alpha(q) \perp \lambda_\alpha \geqslant 0, \alpha \in \mathcal{E} \end{cases} \tag{43}$$

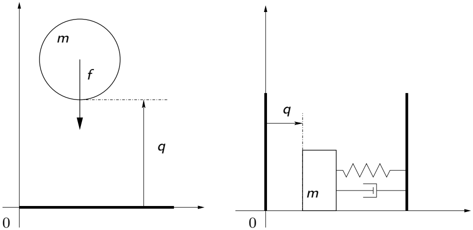
The Moreau's sweeping process in Newton–Euler Formalism

Measure DI for Newton–Euler Formalism

$$\begin{cases} [M(q)dv + F(t,q,v)dt] = T^T(q,t)di \\ -di \in N_{T_C(q)}(v^+) \\ \dot{q}^+ = T(q,t)v^+ \end{cases} \tag{44}$$

Academic examples

The bouncing Ball and the linear impacting oscillator



(a) Bouncing ball example (b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

NonSmooth Multibody Systems (NSMBS)

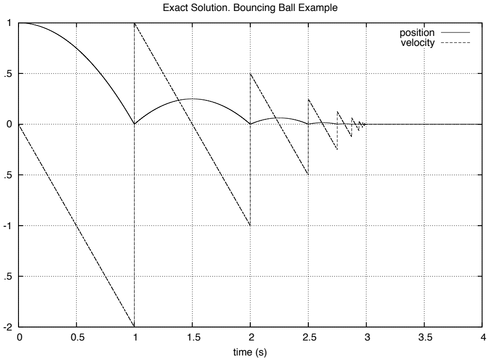


Figure: Analytical solution. Bouncing ball example

NonSmooth Multibody Systems (NSMBS)

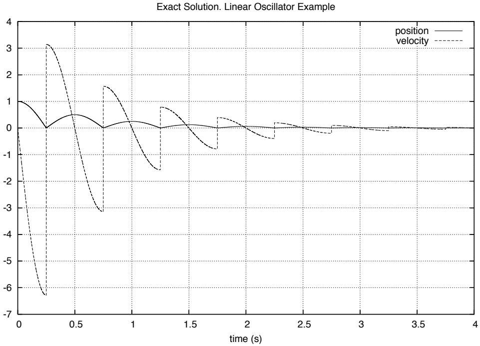


Figure: Analytical solution. Linear Oscillator

The Moreau's sweeping process of second order

Example (The Bouncing Ball)

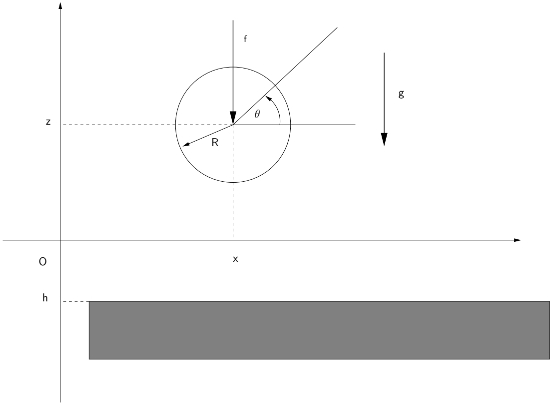


Figure: Two-dimensional bouncing ball on a rigid plane

The Moreau's sweeping process of second order

Example (The Bouncing Ball)

In our special case, the model is completely linear:

$$q = \begin{bmatrix} z \\ x \\ \theta \end{bmatrix} \tag{45}$$

$$M(q) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \text{ where } I = \frac{3}{5}mR^2 \tag{46}$$

$$N(q,\dot{q}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{47}$$

$$F_{int}(q,\dot{q},t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{48}$$

$$F_{ext}(t) = \begin{bmatrix} -mg \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} f(t) \\ 0 \\ 0 \end{bmatrix} \tag{49}$$

The Moreau's sweeping process of second order

Example (The Bouncing Ball)

Kinematics Relations The unilateral constraint requires that :

$$C = \{q, g(q) = z - R - h \geqslant 0\} \tag{45}$$

so we identify the terms of the equation the equation (32)

$$-di = [1, 0, 0]^T d\lambda_1, \tag{46}$$

$$U_1^+ = [1, 0, 0] \begin{bmatrix} \dot{z} \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \dot{z} \tag{47}$$

Nonsmooth laws The following contact laws can be written,

$$\begin{cases} \text{if } g(q) \leqslant 0, \text{ then } 0 \leqslant U^+ + eU^- \perp d\lambda_1 \geqslant 0 \\ \text{if } g(q) \geqslant 0, d\lambda_1 = 0 \end{cases} \tag{48}$$

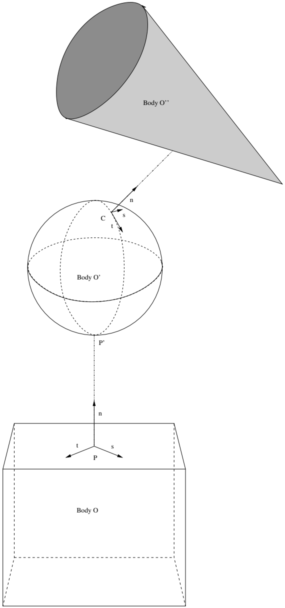
Local coordinates system at contact

Lagrangian approach of constraints is not sufficient.

The elegant Lagrangian approach of unilateral constraints and their associated multipliers is not sufficient for describing more complex behavior of the contact :

- The Lagrange multipliers have no physical dimensions
- The constraints can be multiplied by a positive constant.

For a mechanical description of the behaviour of the contact interface, a (set-valued) force laws needs to be introduced together with a coordinate systems at contact.



Definition of a contact frame

Assume that we have defined

- P and P' proximal points between O and O'
- **n** an outward unit normal vector along $P'P$
- **t** and **s** two unit tangent vectors
- $g(q)$ a gap function, i.e., the signed distance $\overline{P'P}$

Remark

This definition is not trivial for a nonsmooth or nonconvex surfaces.

Local coordinates system at contact

Relative local velocity

The relative local velocity U is defined by

$$U = V_P - V_{P'} \tag{49}$$

and is decomposed in the frame $(P', \mathbf{n}, \mathbf{t}, \mathbf{s})$ as

$$U = U_{\mathrm{N}}\mathbf{n} + U_{\mathrm{T}}, \quad U_{\mathrm{N}} \in \mathbb{R}, U_{\mathrm{T}} \in \mathbb{R}^2 \tag{50}$$

Link with the gap function

The derivative with respect to time of the gap function $t \to g(q(t))$ is the normal relative velocity U_{N}

$$\dot{g}(\cdot) = U_{\mathrm{N}}(\cdot) = \nabla g^T(q)v(\cdot) \tag{51}$$

Local reaction force at contact

The relative local velocity R acts from O' to O and is also decomposed as

$$U = R_{\mathrm{N}}\mathbf{n} + R_{\mathrm{T}}, \quad R_{\mathrm{N}} \in \mathbb{R}, R_{\mathrm{T}} \in \mathbb{R}^2 \tag{52}$$

Local coordinates system at contact

Relations with global/generalized coordinates

Is is assumed that there exists a relation between the local relative velocity U and the velocity of bodies v such that

$$U = H^T(q)v \tag{53}$$

By duality (expressed in terms of power) we get

$$r = H(q)R \tag{54}$$

Unilateral contact in terms of local variables

$$\text{if } g(q) \leqslant 0, \text{ then } 0 \leqslant U_{\mathrm{N}} \perp R_{\mathrm{N}} \geqslant 0 \tag{55}$$

Coulomb's friction

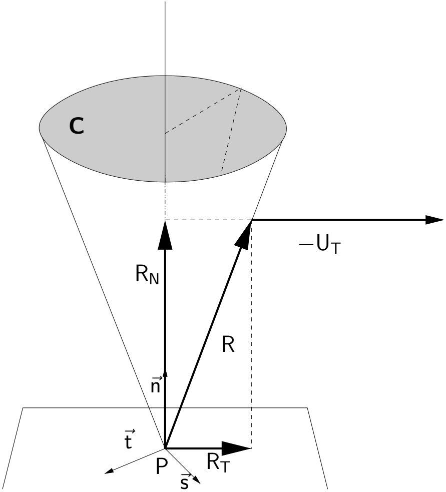


Figure: Coulomb's friction. The sliding case.

Coulomb's friction

Definition (Coulomb's friction)

Coulomb's friction says the following. If $g(q) = 0$ then:

$$\begin{cases} \text{If } U_{\mathrm{T}} = 0 & \text{then } R \in \mathbf{C} \\ \text{If } U_{\mathrm{T}} \neq 0 & \text{then } ||R_{\mathrm{T}}(t)|| = \mu |R_{\mathrm{N}}| \text{ and there exists a scalar } a \geqslant 0 \\ & \text{such that } R_{\mathrm{T}} = -aU_{\mathrm{T}} \end{cases} \tag{56}$$

where $C = \{R, ||R_{\mathrm{T}}|| \leqslant \mu |R_{\mathrm{N}}| \ \}$ is the Coulomb friction cone

Coulomb's friction

Definition (Coulomb's friction as an inclusion into a disk)

Let us introduce the following inclusion (Moreau, 1988), using the indicator function $\psi_{\mathbf{D}}(\cdot)$:

$$-U_{\mathrm{T}} \in \partial\psi_{\mathbf{D}}(R_{\mathrm{T}}) \tag{57}$$

where $D = \{R_{\mathrm{T}}, ||R_{\mathrm{T}}(t)|| \leqslant \mu|R_{\mathrm{N}}|\ \}$ is the Coulomb friction disk

Definition (Coulomb's friction as a variational inequality (VI))

Then (57) appears to be equivalent to

$$\begin{cases} R_{\mathrm{T}} \in \mathbf{D} \\ \langle U_{\mathrm{T}}, z - R_{\mathrm{T}} \rangle \geqslant 0 \text{ for all } z \in \mathbf{D} \end{cases} \tag{58}$$

and to

$$R_{\mathrm{T}} = \mathrm{proj}_{\mathbf{D}}[R_{\mathrm{T}} - \rho U_{\mathrm{T}}], \text{ for all } \rho > 0 \tag{59}$$

Definition (Coulomb's Friction as a Second–Order Cone Complementarity Problem)

Let us introduce the modified velocity $\widehat{U}$ defined by

$$\widehat{U} = [U_{\mathsf{N}} + \mu \,||U_{\mathsf{T}}||, U_{\mathsf{T}}]^T. \tag{60}$$

This notation provides us with a synthetic form of the Coulomb friction as

$$-\widehat{U} \in \partial\psi_{\mathbf{C}}(R), \tag{61}$$

or

$$\mathbf{C}^* \ni \widehat{U} \perp R \in \mathbf{C}. \tag{62}$$

where $\mathbf{C}^* = \{v \in \mathbb{R}^n \mid r^T v \geqslant 0, \forall r \in \mathbf{C}\}$ is the dual cone.

Coulomb's friction

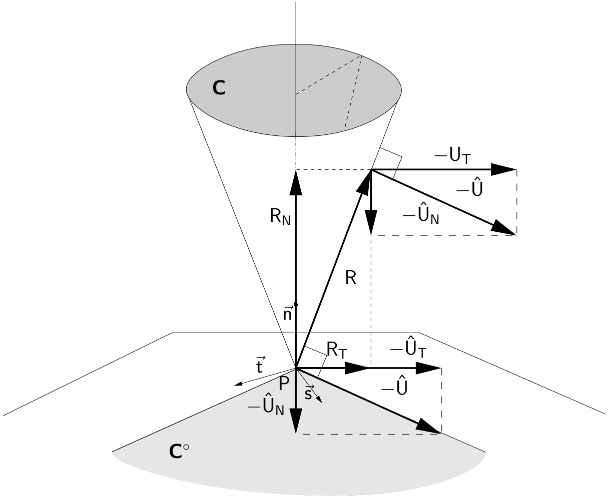


Figure: Coulomb's friction and the modified velocity $\widehat{U}$. The sliding case.

Coulomb's friction with impacts

It is for instance proposed in (Moreau, 1988) to extend (57) (**??**) to densities, i.e. to impulses with a tangential restitution

$$\begin{cases} -P_{\mathrm{N}} \in \partial\psi^*_{\mathbb{R}^-}(\dfrac{1}{1+\rho}U^+_{\mathrm{N}}(t) + \dfrac{\rho}{1+\rho}U^-_{\mathrm{N}}(t)) \\ -P_{\mathrm{T}} \in \partial\psi^*_{\mathbf{D}}(\dfrac{1}{1+\tau}U^+_{\mathrm{T}}(t) + \dfrac{\tau}{1+\tau}U^-_{\mathrm{T}}(t)). \end{cases} \tag{63}$$

with ρ and τ are constants with values in the interval $[0,1]$ or

$$\begin{cases} -P_{\mathrm{N}} \in \partial\psi^*_{\mathbb{R}^-}(U^+_{\mathrm{N}}(t) + e_{\mathrm{N}}U^-_{\mathrm{N}}(t)) \\ -P_{\mathrm{T}} \in \partial\psi^*_{\mathbf{D}}(U^+_{\mathrm{T}}(t) + e_{\mathrm{T}}U^-_{\mathrm{T}}(t)) \end{cases} \tag{64}$$

where $e_{\mathrm{N}} \in [0,1)$ and $e_{\mathrm{T}} \in (-1,1)$.

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