

Harold J. Kushner Paul Dupuis

Numerical Methods for Stochastic Control Problems in Continuous Time

Second Edition

With 40 Figures



Springer

Contents

Introduction	1
1 Review of Continuous Time Models	7
1.1 Martingales and Martingale Inequalities	8
1.2 Stochastic Integration	9
1.3 Stochastic Differential Equations: Diffusions	14
1.4 Reflected Diffusions	21
1.5 Processes with Jumps	28
2 Controlled Markov Chains	35
2.1 Recursive Equations for the Cost	36
2.1.1 Stopping on first exit from a given set	36
2.1.2 Discounted cost	38
2.1.3 Average cost per unit time	40
2.1.4 Stopping at a given terminal time	41
2.2 Optimal Stopping Problems	42
2.2.1 Discounted cost	43
2.2.2 Undiscounted cost	47
2.3 Discounted Cost	48
2.4 Control to a Target Set and Contraction Mappings	50
2.5 Finite Time Control Problems	52
3 Dynamic Programming Equations	53
3.1 Functionals of Uncontrolled Processes	54

3.1.1	Cost until a target set is reached	54
3.1.2	The discounted cost	56
3.1.3	A reflecting boundary	57
3.1.4	The average cost per unit time	58
3.1.5	The cost over a fixed finite time interval	59
3.1.6	A jump diffusion example	59
3.2	The Optimal Stopping Problem	60
3.3	Control Until a Target Set Is Reached	61
3.4	A Discounted Problem with a Target Set and Reflection . .	65
3.5	Average Cost Per Unit Time	65
4	Markov Chain Approximation Method: Introduction	67
4.1	Markov Chain Approximation	69
4.2	Continuous Time Interpolation	72
4.3	A Markov Chain Interpolation	74
4.4	A Random Walk Approximation	78
4.5	A Deterministic Discounted Problem	80
4.6	Deterministic Relaxed Controls	85
5	Construction of the Approximating Markov Chains	89
5.1	One Dimensional Examples	91
5.2	Numerical Simplifications	99
5.2.1	Eliminating the control dependence in the denominators of $p^h(x, y \alpha)$ and $\Delta t^h(x, \alpha)$	99
5.2.2	A useful normalization if $p^h(x, x \alpha) \neq 0$	100
5.2.3	Alternative Markov chain approximations for Example 4 of Section 5.1: Splitting the operator . .	103
5.3	The General Finite Difference Method	106
5.3.1	The general case	108
5.3.2	A two dimensional example: Splitting the operators	112
5.4	A Direct Construction	113
5.4.1	An introductory example	114
5.4.2	Example 2. A degenerate covariance matrix	117
5.4.3	Example 3	119
5.4.4	A general method	121
5.5	Variable Grids	122
5.6	Jump Diffusion Processes	127
5.6.1	The jump diffusion process model: Recapitulation .	127
5.6.2	Constructing the approximating Markov chain . . .	128
5.6.3	A convenient representation of $\{\xi_n^h, n < \infty\}$ and $\psi^h(\cdot)$	131
5.7	Reflecting Boundaries	132
5.7.1	General discussion	132
5.7.2	Locally consistent approximations on the boundary .	136

5.7.3	The continuous parameter Markov chain interpolation	138
5.7.4	Examples	138
5.7.5	The reflected jump diffusion	141
5.8	Dynamic Programming Equations	141
5.8.1	Optimal stopping	141
5.8.2	Control until exit from a compact set	144
5.8.3	Reflecting boundary	145
5.9	Controlled and State Dependent Variance	148
6	Computational Methods for Controlled Markov Chains	153
6.1	The Problem Formulation	154
6.2	Classical Iterative Methods	156
6.2.1	Approximation in policy space	156
6.2.2	Approximation in value space	158
6.2.3	Combined approximation in policy space and approximation in value space	160
6.2.4	The Gauss-Seidel method: Preferred orderings of the states	161
6.3	Error Bounds	164
6.3.1	The Jacobi iteration	164
6.3.2	The Gauss-Seidel procedure	165
6.4	Accelerated Jacobi and Gauss-Seidel Methods	166
6.4.1	The accelerated and weighted algorithms	166
6.4.2	Numerical comparisons between the basic and accelerated procedures	168
6.4.3	Example	170
6.5	Domain Decomposition	171
6.6	Coarse Grid-Fine Grid Solutions	174
6.7	A Multigrid Method	176
6.7.1	The smoothing properties of the Gauss-Seidel iteration	176
6.7.2	A multigrid method	179
6.8	Linear Programming	183
6.8.1	Linear programming	183
6.8.2	The LP formulation of the Markov chain control problem	186
7	The Ergodic Cost Problem: Formulation and Algorithms	191
7.1	Formulation of the Control Problem	192
7.2	A Jacobi Type Iteration	196
7.3	Approximation in Policy Space	197
7.4	Numerical Methods	199
7.5	The Control Problem	201
7.6	The Interpolated Process	206

7.7	Computations	207
7.7.1	Constant interpolation intervals	207
7.7.2	The equation for the cost (5.3) in centered form	209
7.8	Boundary Costs and Controls	213
8	Heavy Traffic and Singular Control	215
8.1	Motivating Examples	216
8.1.1	Example 1. A simple queueing problem	216
8.1.2	Example 2. A heuristic limit for Example 1	217
8.1.3	Example 3. Control of admission, a singular control problem	221
8.1.4	Example 4. A multidimensional queueing or production system under heavy traffic: No control	223
8.1.5	Example 5. A production system in heavy traffic with impulsive control	228
8.1.6	Example 6. A two dimensional routing control problem	229
8.1.7	Example 7	233
8.2	The Heavy Traffic Problem	234
8.2.1	The basic model	234
8.2.2	The numerical method	236
8.3	Singular Control	240
9	Weak Convergence and the Characterization of Processes	245
9.1	Weak Convergence	246
9.1.1	Definitions and motivation	246
9.1.2	Basic theorems of weak convergence	247
9.2	Criteria for Tightness in $D^k[0, \infty)$	250
9.3	Characterization of Processes	251
9.4	An Example	253
9.5	Relaxed Controls	262
10	Convergence Proofs	267
10.1	Limit Theorems	268
10.1.1	Limit of a sequence of controlled diffusions	268
10.1.2	An approximation theorem for relaxed controls	275
10.2	Existence of an Optimal Control	276
10.3	Approximating the Optimal Control	282
10.4	The Approximating Markov Chain	286
10.4.1	Approximations and representations for $\psi^h(\cdot)$	287
10.4.2	The convergence theorem for the interpolated chains	290
10.5	Convergence of the Costs	291
10.6	Optimal Stopping	296

11 Convergence for Reflecting Boundaries, Singular Control, and Ergodic Cost Problems	301
11.1 The Reflecting Boundary Problem	302
11.1.1 The system model and Markov chain approximation	302
11.1.2 Weak convergence of the approximating processes .	306
11.2 The Singular Control Problem	315
11.3 The Ergodic Cost Problem	320
12 Finite Time Problems and Nonlinear Filtering	325
12.1 Explicit Approximations: An Example	326
12.2 General Explicit Approximations	330
12.3 Implicit Approximations: An Example	331
12.4 General Implicit Approximations	333
12.5 Optimal Control Computations	335
12.6 Solution Methods	337
12.7 Nonlinear Filtering	340
12.7.1 Approximation to the solution of the Fokker-Planck equation	340
12.7.2 The nonlinear filtering problem: Introduction and representation	341
12.7.3 The approximation to the optimal filter for $x(\cdot), y(\cdot)$	345
13 Controlled Variance and Jumps	347
13.1 Controlled Variance: Introduction	348
13.1.1 Introduction	348
13.1.2 Martingale measures	351
13.1.3 Convergence	354
13.2 Controlled Jumps	357
13.2.1 Introduction	357
13.2.2 The relaxed Poisson measure	361
13.2.3 Optimal controls	364
13.2.4 Convergence of the numerical algorithm	365
14 Problems from the Calculus of Variations:	
Finite Time Horizon	367
14.1 Problems with a Continuous Running Cost	368
14.2 Numerical Schemes and Convergence	371
14.2.1 Descriptions of the numerical schemes	372
14.2.2 Approximations and properties of the value function	373
14.2.3 Convergence theorems	378
14.3 Problems with a Discontinuous Running Cost	384
14.3.1 Definition and interpretation of the cost on the interface	386
14.3.2 Numerical schemes and the proof of convergence . .	388

15 Problems from the Calculus of Variations:	
Infinite Time Horizon	401
15.1 Problems of Interest	403
15.2 Numerical Schemes for the Case $k(x, \alpha) \geq k_0 > 0$	404
15.2.1 The general approximation	404
15.2.2 Problems with quadratic cost in the control	405
15.3 Numerical Schemes for the Case $k(x, \alpha) \geq 0$	409
15.3.1 The general approximation	410
15.3.2 Proof of convergence	411
15.3.3 A shape from shading example	422
15.4 Remarks on Implementation and Examples	435
16 The Viscosity Solution Approach	443
16.1 Definitions and Some Properties of Viscosity Solutions . . .	444
16.2 Numerical Schemes	449
16.3 Proof of Convergence	453
References	455
Index	467
List of Symbols	473