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NUMERICAL METHODS FOR THE VARIABLE-ORDER FRACTIONAL ADVECTION-DIFFUSION EQUATION WITH A NONLINEAR SOURCE TERM*

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Abstract. In this paper, we consider a variable-order fractional advection-diffusion equation with a nonlinear source term on a finite domain. Explicit and implicit Euler approximations for the equation are proposed. Stability and convergence of the methods are discussed. Moveover, we also present a fractional method of lines, a matrix transfer technique, and an extrapolation method for the equation. Some numerical examples are given, and the results demonstrate the effectiveness of theoretical analysis.

Key words. fractional derivative of variable order, nonlinear fractional advection-diffusion equation, finite difference methods, method of lines, extrapolation method, stability and convergence

AMS subject classifications. 26A33, 45K05, 35K37, 65M12

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1. Introduction. The fractional advection-dispersion equation is a generalization of the classical advection-dispersion equation. Schumer et al. [30] gave an Eulerian derivation of this equation and demonstrated that highly skewed, non-Gaussian contaminant plumes with heavy leading edges can be a result of the infinite-variance particle jump distributions that arise during transport in a disordered porous medium. Zhang et al. [33] discussed the impact of the boundary conditions that are commonly used in hydrology to simulate solute movement. G. Huang, Q. Huang, and Zhan [6] developed a semi-analytical inverse method and a corresponding program for parameter estimation under the condition of steady-state flow and input of solute.

Some authors have discussed the numerical approximation for the fractional advection-dispersion equation. Liu, Anh, and Turner [16] considered the space fractional Fokker–Planck equation with instantaneous source and presented a fractional method of lines. Meerschaert and Tadjeran [22] developed numerical methods to solve the one-dimensional equation with variable coefficients on a finite domain. Roop [28] investigated the numerical approximation of the variational solution on bounded domains in $\mathbb{R}^2$ and presented a method for approximating the solution in two spatial dimensions using the finite element method. Yong et al. [34] examined the random walk particle tracking approach to solve the one-dimensional equation. Liu et al. [18] presented a random walk model for approximating a Lévy–Feller advection-dispersion process and proposed an explicit finite difference approximation. Liu et al. [17] con-

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sidered a space-time fractional advection-dispersion equation on a finite domain and proposed implicit and explicit difference methods to solve this equation. Momani [24] considered the following fractional convection-diffusion equation with nonlinear source term

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x} + \Psi(u) + f(x,t), \quad 0 < x < 1, \quad 0 < \alpha \le 1, \quad t > 0 \tag{1.1}$$

and proposed an algorithm based on the Adomian decomposition method.

The theory of pseudodifferential operators and equations has also received much attention [4, 29]. The behavior of some diffusion processes in response to temperature changes may be better described using variable-order exponents in a pseudodifferential operator than time-varying coefficients [19, 20]. Lorenzo and Hartley [19, 20] presented the concept of variable-order fractional integration and differentiation. Multifractional pseudodifferential models have been considered in the representation of heterogeneous local behaviors. The solutions to such models are defined in fractional Besov spaces of variable order on $\mathbb{R}^n$ (Leopold [14]). Gaussian processes defined by elliptic pseudodifferential equations have been studied in Ruiz-Medina, Anh, and Angulo [29]. The covariance function of these random processes defines the inner product of a fractional Sobolev space of variable order. An interesting special case is multifractional Brownian motion introduced in Peltier and Lévy Véhel [25] and Benassi, Jaffard, and Roux [1]. Several classes of Markov processes with multifractional transition probability densities on unbounded domains have been studied in Jacob and Leopold [10], Komatsu [13], Kikuchi and Negoro [11], and Kolokoltsov [12]. In particular, Kikuchi and Negoro [11] found the conditions for which general pseudodifferential operators on fractional Sobolev spaces of variable order on $\mathbb{R}^n$ form a Feller semigroup which has a transition density.

The research on variable-order fractional partial differential equations is relatively new, and numerical approximation of these equations is still at an early stage of development. Lin, Liu, Anh, and Turner [15] established an equality between the variable-order Riemann–Liouville fractional derivative and its Grünwald–Letnikov expansion. Using this relationship, they defined and obtained some properties of the operator $(-\frac{d^2}{dx^2})^{\alpha(x,t)}$ and devised an explicit finite difference approximation scheme for a corresponding variable-order nonlinear fractional diffusion equation. Ilic et al. [7, 8] proposed a new matrix method for a fractional-in-space diffusion equation with homogeneous and nonhomogeneous boundary conditions on a bounded domain.

In this paper, we consider numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. In section 2, we give the definitions of the variable-order fractional integral and derivative and introduce the equation. In section 3, the explicit Euler approximation and the implicit Euler approximation solutions are proposed. Stability and convergence of both methods are given in sections 4 and 5, respectively. We also present another three computationally effective numerical methods in section 6. Finally, some numerical examples are given in section 7.

2. Preliminaries.

Firstly, we introduce some concepts of variable-order fractional derivatives.

Definition 2.1 (Riesz fractional derivative [5, 29]).

$$\frac{\partial^{\alpha(x)} f(x)}{\partial |x|^{\alpha(x)}} = -(-\Delta)^{\alpha(x)/2} f(x) = -\mathcal{F}^{-1}|\xi|^{\alpha(x)}\mathcal{F}f(\xi). \tag{2.1}$$

Considering that $m-1<\alpha(x)\le m$, where $m>0$ is a positive integer, from the above definition, we have

$$-(-\Delta)^{\alpha(x)/2}f(x)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\xi x}|\xi|^{\alpha(x)}\left[\int_{-\infty}^{\infty}f(\eta)e^{i\xi\eta}d\eta\right]d\xi.$$

Suppose that $f(x),f'(x),\ldots,f^{(m-1)}(x)$ vanish at $x=\pm\infty$, then we can perform integration by parts repeatedly to yield

$$\int_{-\infty}^{\infty}f(\eta)e^{i\xi\eta}d\eta=(-1)^m(i\xi)^{-m}\int_{-\infty}^{\infty}f^{(m)}(\eta)e^{i\xi\eta}d\eta.$$

Thus,

$$\begin{aligned}-(-\Delta)^{\alpha(x)/2}f(x)&=(-1)^{m+1}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\xi x}|\xi|^{\alpha(x)}(i\xi)^{-m}\left[\int_{-\infty}^{\infty}f^{(m)}(\eta)e^{i\xi\eta}d\eta\right]d\xi\\&=(-1)^{m+1}\frac{1}{2\pi}\int_{-\infty}^{\infty}f^{(m)}(\eta)\left[\int_{-\infty}^{\infty}e^{i\xi(\eta-x)}|\xi|^{\alpha(x)}(i\xi)^{-m}d\xi\right]d\eta.\end{aligned}$$

Let $I=\int_{-\infty}^{\infty}e^{i\xi(\eta-x)}|\xi|^{\alpha(x)}(i\xi)^{-m}d\xi$, then

$$\begin{aligned}I&=i^m\int_{-\infty}^{0}e^{i\xi(\eta-x)}(-\xi)^{\alpha(x)-m}d\xi+i^{-m}\int_{0}^{\infty}f(\eta)e^{i\xi(\eta-x)}\xi^{\alpha(x)-m}d\xi\\&=i^m\int_{0}^{\infty}e^{i\xi(x-\eta)}\xi^{\alpha(x)-m}d\xi+i^{-m}\int_{0}^{\infty}f(\eta)e^{i\xi(\eta-x)}\xi^{\alpha(x)-m}d\xi.\end{aligned}$$

Noting that

$$L(t^\nu)=\int_0^\infty t^\nu e^{-st}dt=\frac{\Gamma(\nu+1)}{s^{\nu+1}},\quad \mathrm{Re}\ \nu>-1,$$

and $-1<\alpha(x)-m\le 0$, we have

$$\begin{aligned}I&=i^m\frac{\Gamma(\alpha(x)-m+1)}{[(\eta-x)i]^{\alpha(x)-m+1}}+i^{-m}\frac{\Gamma(\alpha(x)-m+1)}{[(x-\eta)i]^{\alpha(x)-m+1}}\\&=\begin{cases}\dfrac{\Gamma(\alpha(x)-m+1)}{(\eta-x)^{\alpha(x)-m+1}}\left[\dfrac{i^m}{i^{\alpha(x)-m+1}}+\dfrac{i^{-m}}{(-i)^{\alpha(x)-m+1}}\right] & \text{if } \eta>x,\\ \dfrac{\Gamma(\alpha(x)-m+1)}{(x-\eta)^{\alpha(x)-m+1}}\left[\dfrac{i^m}{(-i)^{\alpha(x)-m+1}}+\dfrac{i^{-m}}{i^{\alpha(x)-m+1}}\right] & \text{if } x>\eta,\end{cases}\\&=\begin{cases}(-1)^m\dfrac{\Gamma(\alpha(x)-m+1)\Gamma(m-\alpha(x))}{(\eta-x)^{\alpha(x)-m+1}\Gamma(m-\alpha(x))}\left[i^{-(\alpha(x)+1)}+i^{\alpha(x)+1}\right] & \text{if } \eta>x,\\ \dfrac{\Gamma(\alpha(x)-m+1)\Gamma(m-\alpha(x))}{(x-\eta)^{\alpha(x)-m+1}\Gamma(m-\alpha(x))}\left[i^{\alpha(x)+1}+i^{-(\alpha(x)+1)}\right] & \text{if } x>\eta.\end{cases}\end{aligned}$$

Using $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$ $(0<z<1)$, we have

$$\Gamma(\alpha(x)-m+1)\Gamma(m-\alpha(x))=\frac{\pi}{\sin\pi(m-\alpha(x))}=(-1)^{m-1}\frac{\pi}{\sin\pi\alpha(x)}.$$

Again,

$$\begin{aligned} i^{-1-\alpha(x)} + i^{\alpha(x)+1} &= e^{\frac{\pi}{2}(-\alpha(x)-1)i} + e^{\frac{\pi}{2}(\alpha(x)+1)i} \\ &= 2\cos\frac{\pi}{2}(1+\alpha(x)) \\ &= -2\sin\frac{\pi}{2}(\alpha(x)). \end{aligned}$$

Hence, we have

$$I = \begin{cases} \dfrac{\pi}{\cos\frac{\alpha(x)\pi}{2}} \dfrac{1}{(\eta - x)^{\alpha(x)-m+1}\Gamma(m-\alpha(x))} & \text{if } \eta > x, \\ (-1)^m \dfrac{\pi}{\cos\frac{\alpha(x)\pi}{2}} \dfrac{1}{(x-\eta)^{\alpha(x)-m+1}\Gamma(m-\alpha(x))} & \text{if } x > \eta. \end{cases}$$

If $f(x)$ is defined on the finite interval $[a, b]$ and $f(a) = f(b) = 0$, then we can extend the function to have $f(x) = 0$ for all $x < a$ or $x > b$. Thus, we have

$$\begin{aligned} -(-\Delta)^{\alpha(x)/2} f(x) = -\frac{1}{2\cos\frac{\pi\alpha(x)}{2}} &\left[\frac{1}{\Gamma(m-\alpha(x))} \int_a^x \frac{f^{(m)}(\eta)d\eta}{(x-\eta)^{\alpha(x)-m+1}} \right. \\ &\left. + \frac{(-1)^m}{\Gamma(m-\alpha(x))} \int_x^b \frac{f^{(m)}(\eta)d\eta}{(\eta-x)^{\alpha(x)-m+1}} \right]. \end{aligned} \tag{2.2}$$

Definition 2.2 (Caputo fractional derivative).

$$\begin{aligned} {}_a^C D_x^{\alpha(x)} f(x) &= \frac{1}{\Gamma(m-\alpha(x))} \int_a^x \frac{f^{(m)}(\eta)d\eta}{(x-\eta)^{\alpha(x)-m+1}}, \\ {}_x^C D_b^{\alpha(x)} f(x) &= \frac{(-1)^m}{\Gamma(m-\alpha(x))} \int_x^b \frac{f^{(m)}(\eta)d\eta}{(\eta-x)^{\alpha(x)-m+1}}, \end{aligned}$$

where $m - 1 < \alpha(x) < m$.

Definition 2.3 (Riemann–Liouville fractional derivative).

$$\begin{aligned} {}_a D_x^{\alpha(x)} f(x) &= \left[\frac{1}{\Gamma(m-\alpha(x))} \frac{d^m}{d\xi^m} \int_a^\xi (\xi-\eta)^{m-\alpha(x)-1} f(\eta)d\eta \right]_{\xi=x}, \\ {}_x D_b^{\alpha(x)} f(x) &= \left[\frac{(-1)^m}{\Gamma(m-\alpha(x))} \frac{d^m}{d\xi^m} \int_\xi^b (\eta-\xi)^{m-\alpha(x)-1} f(\eta)d\eta \right]_{\xi=x}, \end{aligned} \tag{2.3}$$

where $m - 1 < \alpha(x) < m$.

Definition 2.4 (Grünwald–Letnikov fractional derivative [19]).

$$\begin{aligned} D_{a+}^{\alpha(x)} f(x) &= \lim_{h\to 0, nh=x-a} h^{-\alpha(x)} \sum_{j=0}^{n} (-1)^j \binom{\alpha(x)}{j} f(x-jh), \\ D_{b-}^{\alpha(x)} f(x) &= \lim_{h\to 0, nh=b-x} h^{-\alpha(x)} \sum_{j=0}^{n} (-1)^j \binom{\alpha(x)}{j} f(x+jh). \end{aligned} \tag{2.4}$$

For $m-1<\alpha(x)<m$, we obtain

$$D_{a+}^{\alpha(x)}f(x)=\sum_{j=0}^{m-1}\frac{f^{(j)}(a)(t-a)^{j-\alpha(x)}}{\Gamma(-\alpha(x)+j+1)}+\frac{1}{\Gamma(-\alpha(x)+m)}\int_a^x\frac{f^{(m)}(\eta)}{(x-\eta)^{\alpha(x)-m+1}}d\eta,$$

$$D_{b-}^{\alpha(x)}f(x)=\sum_{j=0}^{m-1}\frac{(-1)^{m-j}f^{(j)}(b)(b-\xi)^{-\alpha(x)+j}}{\Gamma(-\alpha(x)+j+1)}+\frac{1}{\Gamma(-\alpha(x)+m)}\int_\xi^b(\eta-\xi)^{m-\alpha(x)-1}f^{(m)}(\eta)d\eta$$

(see [26], pp. 52–55). Using repeatedly integration by parts and differentiation, we obtain

$$\frac{1}{\Gamma(m-\alpha(x))}\frac{d^m}{d\xi^m}\int_a^\xi(\xi-\eta)^{m-\alpha(x)-1}f(\eta)d\eta$$
$$=\sum_{j=0}^{m-1}\frac{f^{(j)}(a)(\xi-a)^{-\alpha(x)+j}}{\Gamma(-\alpha(x)+j+1)}+\frac{1}{\Gamma(-\alpha(x)+m)}\int_a^\xi(\xi-\eta)^{m-\alpha(x)-1}f^{(m)}(\eta)d\eta.$$

Similarly,

$$\frac{(-1)^m}{\Gamma(m-\alpha(x))}\frac{d^m}{d\xi^m}\int_\xi^b(\eta-\xi)^{m-\alpha(x)-1}f(\eta)d\eta$$
$$=\sum_{j=0}^{m-1}\frac{(-1)^{m-j}f^{(j)}(b)(b-\xi)^{-\alpha(x)+j}}{\Gamma(-\alpha(x)+j+1)}+\frac{1}{\Gamma(-\alpha(x)+m)}\int_\xi^b(\eta-\xi)^{m-\alpha(x)-1}f^{(m)}(\eta)d\eta.$$

Hence, if the function $f(x)$ has $m+1$ continuous derivatives, then the Grünwald–Letnikov definition (2.4) is equivalent to the Riemann–Liouville definition (2.3).

From (2.2), the Riesz fractional derivative of order $\alpha(x)$ can be defined as

$$-(-\Delta)^{\alpha(x)/2}f(x)=-\frac{1}{2\cos\frac{\pi\alpha(x)}{2}}\left[{}_aD_x^{\alpha(x)}f(x)+{}_xD_b^{\alpha(x)}f(x)\right].$$

We consider the following variable-order fractional advection-diffusion equation with a nonlinear source term:

$$\frac{\partial u(x,t)}{\partial t}=\kappa(x,t)R_{\alpha(x,t)}u(x,t)-\nu(x,t)\frac{\partial u}{\partial x}+f(u,x,t),\quad (x,t)\in\Omega=[a,b]\times[0,T], \tag{2.5}$$

and the initial and boundary conditions

$$u(x,0)=\phi(x), \tag{2.6}$$

$$u(a,t)=0,u(b,t)=0, \tag{2.7}$$

where $1<\underline{\alpha}\le\alpha(x,t)\le\overline{\alpha}\le 2$; $\nu(x,t)$ $(0\le\nu(x,t)\le\overline{\nu})$ represents the average fluid velocity, $f(u,x,t)$ is a source term which satisfies the Lipschitz condition, i.e.,

$$\text{for all } u_1,u_2,\quad |f(u_1,x,t)-f(u_2,x,t)|\le L|u_1-u_2|, \tag{2.8}$$

and $0\le\kappa(x,t)\le\overline{\kappa}$.

In (2.5), $R_{\alpha(x,t)}u(x,t)$ is a variable-order fractional derivative defined by

$$R_{\alpha(x,t)}u(x,t) = c_+(x,t)_a D_x^{\alpha(x,t)} u(x,t) + c_-(x,t)_x D_b^{\alpha(x,t)} u(x,t), \tag{2.9}$$

where $0 < c_+(x,t) \le c_1, \quad 0 < c_-(x,t) \le c_2$.

If $c_+(x,t) = 1, c_-(x,t) \equiv 0$, $R_{\alpha(x,t)}$ represents the Riemann–Liouville left-handed spatial fractional derivative; if $c_+(x,t) \equiv 0, c_-(x,t) = 1$, $R_{\alpha(x,t)}u(x,t)$ represents the Riemann–Liouville right-handed spatial fractional derivative.

When $c_+(x,t) = c_-(x,t) = -\frac{1}{2\cos(\frac{\pi\alpha(x,t)}{2})}$,

$$\begin{aligned} R_{\alpha(x,t)}u(x,t) &= -(-\Delta)^{\alpha(x,t)}u(x,t) \\ &= -\frac{1}{2\cos\frac{\pi\alpha(x,t)}{2}} \left[{}_a D_x^{\alpha(x,t)} u(x,t) + {}_x D_b^{\alpha(x,t)} u(x,t) \right] \\ &= \frac{\partial^{\alpha(x,t)} u(x,t)}{\partial |x|^{\alpha(x,t)}} \end{aligned}$$

represents the Riesz fractional derivative.

If $\alpha(x,t) = 2$, (2.5) becomes the following classical advection-diffusion equation with a nonlinear source term:

$$\frac{\partial u(x,t)}{\partial t} = \kappa(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} - \nu(x,t)\frac{\partial u(x,t)}{\partial x} + f(u,x,t).$$

3. Numerical approximations. In this section, we will derive a numerical approximation for (2.5). Let us suppose that the function $f(x)$ is $(m-1)$-continuously differentiable in the interval $[a,b]$ and that $f^{(m)}(x)$ is integrable in $[a,b]$. Then, for every α $(0 \le m-1 < \alpha(x) < m)$, the Riemann–Liouville fractional derivative exists and coincides with the Grünwald–Letnikov fractional derivative [16]. The relationship between the Riemann–Liouville and Grünwald–Letnikov definitions also have another consequence, which is important for the numerical approximation of fractional differential equations, manipulation with fractional derivatives, and formulation of physically meaningful initial- and boundary-value problems for fractional differential equations. This allows the use of the Riemann–Liouville definition during problem formulation and then the Grünwald–Letnikov definition for obtaining the numerical solution.

Using the relationship between Riemann–Liouville and Grünwald–Letnikov derivatives, a discrete approximation to the space fractional derivative terms $D_{a+}^{\alpha(x,t)}u(x,t)$ and $D_{b-}^{\alpha(x,t)}u(x,t)$ may be defined from the standard Grünwald formula:

$$\begin{aligned} D_{a+}^{\alpha(x,t)}u(x,t) &= \lim_{M_1\to\infty} (h_1)^{-\alpha(x,t)} \sum_{j=0}^{M_1} g_{\alpha(x,t)}^{(j)} u(x - jh_1, t), \\ D_{b-}^{\alpha(x,t)}u(x,t) &= \lim_{M_2\to\infty} (h_2)^{-\alpha(x,t)} \sum_{j=0}^{M_2} g_{\alpha(x,t)}^{(j)} u(x + jh_2, t), \end{aligned} \tag{3.1}$$

where M_1, M_2 are positive integers, $h_1 = (x-a)/M_1$, $h_2 = (b-x)/M_2$, and the normalized Grünwald weights are defined by

$$\begin{aligned} g_{\alpha(x,t)}^{(0)} &= 1, \\ g_{\alpha(x,t)}^{(j)} &= -\frac{\alpha(x,t) - j + 1}{j} g_{\alpha(x,t)}^{(j-1)} \quad \text{for} \quad j = 1, 2, 3, \ldots. \end{aligned} \tag{3.2}$$

Let $t_k = k\tau$, $k = 0, 1, 2, \ldots, n$, $x_i = a + ih$, $i = 0, 1, 2, \ldots, m$, where $0 \le t_k \le T$, $\tau = T/n$, and $h = (b-a)/m$ are time and space steps, respectively. We define u_i^k as the numerical approximation to $u(x_i, t_k)$. Similarly, we define $c_{+,i}^k = c_+(x_i, t_k), c_{-,i}^k = c_-(x_i, t_k)$, $\nu_i^k = \nu(x_i, t_k)$, $\kappa_i^k = \kappa(x_i, t_k)$, and $\alpha_i^k = \alpha(x_i, t_k)$.

From [22], if $u(x,t) \in L^1(\Omega)$, $D_{a+}^{\alpha(x,t)} u(x,t) \in \mathscr{C}(\Omega)$, and $D_{b-}^{\alpha(x,t)} u(x,t) \in \mathscr{C}(\Omega)$, we obtain

$$
\begin{aligned}
D_{a+}^{\alpha_i^k} u(x_i, t_k) &= h^{-\alpha_i^k} \sum_{j=0}^{i+1} g_{\alpha_i^k}^{(j)} u(x_{i-j}, t_k) + O(h), \\
D_{b-}^{\alpha_i^k} u(x_i, t_k) &= h^{-\alpha_i^k} \sum_{j=0}^{m-i+1} g_{\alpha_i^k}^{(j)} u(x_{i+j}, t_k) + O(h).
\end{aligned} \tag{3.3}
$$

It was shown in [22] that using the standard Grünwald formula to discretize the dispersion equation results in an unstable finite difference scheme. Hence, we adopt the shift Grünwald formula to approximate the space fractional derivatives ${}_xD_{a+}^{\alpha_i^k} u(x_i, t_k)$ and $D_{b-}^{\alpha_i^k} u(x_i, t_k)$. We now prove the following lemmas on consistency and convergence.

LEMMA 3.1. *Suppose that* $\psi(x,t) =_x D_{a+}^{\alpha(x,t)} u(x,t) \in C^1(\Omega)$*, then*

$$
\begin{aligned}
{}_xD_{a+}^{\alpha_i^k} u_i^k &= h^{-\alpha_{i+1}^k} \sum_{j=0}^{i+1} g_{\alpha_{i+1}^k}^{(j)} u_{i+1-j}^k + O(h), \\
{}_xD_{b-}^{\alpha_i^k} u_i^k &= h^{-\alpha_{i-1}^k} \sum_{j=0}^{m-i+1} g_{\alpha_{i-1}^k}^{(j)} u_{i-1+j}^k + O(h).
\end{aligned} \tag{3.4}
$$

Proof. From (3.3), we have

$$
{}_xD_{a+}^{\alpha_{i+1}^k} u_{i+1}^k = h^{-\alpha_{i+1}^k} \sum_{j=0}^{i+1} g_{\alpha_{i+1}^k}^{(j)} u_{i+1-j}^k + O(h).
$$

Using $\psi(x,t) \in C^1(\Omega)$, we obtain

$$
|\psi(x_i, t_k) - \psi(x_{i+1}, t_k)| \le C|x_{i+1} - x_i| \le O(h).
$$

Thus,

$$
\begin{aligned}
{}_xD_{a+}^{\alpha_i^k} u_i^k &= +{}_xD_{a+}^{\alpha_{i+1}^k} u_{i+1}^k + \psi(x_i, t_k) - \psi(x_{i+1}, t_k) \\
&= h^{-\alpha_{i+1}^k} \sum_{j=0}^{i+1} g_{\alpha_{i+1}^k}^{(j)} u_{i+1-j}^k + O(h).
\end{aligned}
$$

Similarly, we can obtain

$$
{}_xD_{b-}^{\alpha_i^k} u_i^k = h^{-\alpha_{i-1}^k} \sum_{j=0}^{m-i+1} g_{\alpha_{i-1}^k}^{(j)} u_{i-1+j}^k + O(h). \qquad \square
$$

Set $\mu_i^k = \nu_i^k \tau h^{-1}$, $r_{i,k}^{(1)} = \kappa_i^k c_{+,i}^k \tau h^{-\alpha_{i+1}^k}$, $r_{i,k}^{(2)} = \kappa_i^k c_{-,i}^k \tau h^{-\alpha_{i-1}^k}$, $g_{i,k}^{(j)} = g_{\alpha_i^k}^{(j)}$, and

$$
\begin{aligned}
L_{h,\tau} u(x_i, t_k) = r_{i,k}^{(1)} &\sum_{j=0}^{i+1} g_{i+1,k}^{(j)} u(x_{i+1-j}, t_k) \\
&+ r_{i,k}^{(2)} \sum_{j=0}^{m-i+1} g_{i-1,k}^{(j)} u(x_{i-1+j}, t_k).
\end{aligned}
$$

Applying Lemma 3.1 and the following formulas

$$
\begin{aligned}
\frac{\partial u(x_i,t_k)}{\partial t} &= \frac{u(x_i,t_{k+1})-u(x_i,t_k)}{\tau}+O(\tau),\\
\frac{\partial u(x_i,t_k)}{\partial t} &= \frac{u(x_i,t_k)-u(x_i,t_{k-1})}{\tau}+O(\tau),\\
\frac{\partial u(x_i,t_k)}{\partial x} &= \frac{u(x_i,t_k)-u(x_{i-1},t_k)}{h}+O(h),\\
f(u(x_i,t_k),x_i,t_k) &= f(u(x_i,t_{k-1}),x_i,t_{k-1})+O(\tau),
\end{aligned}
\tag{3.5}
$$

we get

$$
\begin{aligned}
u(x_i,t_k) &= u(x_i,t_{k-1})+\mu_i^{k-1}[u(x_i,t_{k-1})-u(x_{i-1},t_{k-1})]+L_{h,\tau}u(x_i,t_{k-1})\\
&\quad +\tau f(u(x_i,t_{k-1}),x_i,t_{k-1})+R_{i,k-1}^{(1)},\\
u(x_i,t_k) &= u(x_i,t_{k-1})+\mu_i^{k}[u(x_i,t_k)-u(x_{i-1},t_k)]+L_{h,\tau}u(x_i,t_k)\\
&\quad +\tau f(u(x_i,t_{k-1}),x_i,t_{k-1})+R_{i,k}^{(2)},
\end{aligned}
$$

where $|R_{i,k}^{(1)}|\le C_1(\tau^2+\tau h)$ and $|R_{i,k}^{(2)}|\le C_2(\tau^2+\tau h)$. Thus, we obtain the explicit Euler approximation

$$
\begin{aligned}
u_i^{k+1} &= u_i^k-\mu_i^k(u_i^k-u_{i-1}^k)\\
&\quad + r_{i,k}^{(1)}\sum_{j=0}^{i+1} g_{i+1,k}^{(j)}u_{i-j+1}^k + r_{i,k}^{(2)}\sum_{j=0}^{m-i+1} g_{i-1,k}^{(j)}u_{i+j-1}^k+\tau f(u_i^k,x_i,t_k),
\end{aligned}
\tag{3.6}
$$

where $k=0,1,2\ldots,n-1$, and the implicit Euler approximation

$$
\begin{aligned}
u_i^{k+1} &= u_i^k-\mu_i^{k+1}(u_i^{k+1}-u_{i-1}^{k+1})\\
&\quad + r_{i,k+1}^{(1)}\sum_{j=0}^{i+1} g_{i+1,k+1}^{(j)}u_{i-j+1}^{k+1} + r_{i,k+1}^{(2)}\sum_{j=0}^{m-i+1} g_{i-1,k+1}^{(j)}u_{i+j-1}^{k+1}+\tau f(u_i^k,x_i,t_k),
\end{aligned}
\tag{3.7}
$$

where $k=0,1,2,\ldots,n-1$.

The boundary and initial conditions are discretized as

$$
u_i^0=\phi(ih),\quad u_0^k=0,\quad u_m^k=0,
\tag{3.8}
$$

where $k=0,1,2,\ldots,n$, and $i=0,1,2,\ldots,m$.

The explicit and implicit Euler approximations with boundary and initial conditions can be rewritten in the following matrix form:

$$
\begin{cases}
\mathbf{u}^{k+1}=\mathbf{A}\mathbf{u}^k+\mathbf{b}^k,\\
\quad \mathbf{u}^0=[u_1^0,u_2^0,\ldots,u_{m-1}^0]^T,
\end{cases}
\tag{3.9}
$$

where $\mathbf{u}^k=[u_1^k,u_2^k,\ldots,u_{m-1}^k]^T$, $\mathbf{b}^k$ includes a column vector of known boundary values and known source term values, and $\mathbf{A}$ is an $(m-1)\times(m-1)$ matrix of known elements. We suppose that $\widetilde{\mathbf{u}}_k$ is a vector of approximate solution of (3.9); the errors $\varepsilon_i^{(k)}=\widetilde{u}_i^{(k)}-u_i^{(k)}$, $(i=0,1,2,\ldots,m;k=0,1,2,\ldots)$ satisfy

$$
\begin{cases}
\mathbf{E}^{k+1} & =\mathbf{A}\mathbf{E}^k,\\
\mathbf{E}^0 & \quad\text{given},
\end{cases}
\tag{3.10}
$$

where $\mathbf{E}^k=[\varepsilon_1^k,\varepsilon_2^k,\ldots,\varepsilon_{m-1}^k]^T$.

DEFINITION 3.2 (see [31, 32]). *For any arbitrary initial rounding error $\boldsymbol{E}^0$, there exists a positive number K, independent of h and τ, such that*

$$\|\boldsymbol{E}^k\| \le K\|\boldsymbol{E}^0\| \tag{3.11}$$

or

$$\|\boldsymbol{A}^k\| \le K. \tag{3.12}$$

The difference approximation (3.9) *is then stable.*

LEMMA 3.3. *For $i = 1, 2, \ldots, m, k = 1, 2, \ldots, n$, the coefficients*

$$g_{i,k}^{(j)}, j = 1, 2, \ldots,$$

satisfy

(1) $g_{i,k}^{(0)} = 1,\ \ g_{i,k}^{(1)} = -\alpha_i^k < 0$, *and* $g_{i,k}^{(j)} > 0, (j \ne 1)$*;*

(2) $\sum_{j=0}^{\infty} g_{i,k}^{(j)} = 0$, *and for* $l = 1, 2, \ldots, \sum_{j=0}^{l} g_{i,k}^{(j)} < 0$.

Proof. (1) By Definition (3.2) of $g_{i,k}^{(j)}$ $(j = 0, 1, \ldots)$, we have $g_{i,k}^{(0)} = 1$ and $g_{i,k}^{(1)} = -\alpha_i^k < 0$.

Since $1 < \alpha_i^k < 2$, if $j > 2$, then $-\frac{\alpha_i^k - j + 1}{j} > 0$. Hence, from $g_{i,k}^{(2)} > 0$, we obtain $g_{i,k}^{(j)} > 0, (j > 3)$.

(2) By Definition (3.2) of $g_{i,k}^{(j)}$ $(j = 0, 1, \ldots)$, we have

$$g_{i,k}^{(j)} = (-1)^j \begin{pmatrix} \alpha_i^k \\ j \end{pmatrix}, \quad j = 0, 1, 2, \ldots.$$

Using $(1-x)^{\alpha_i^k} = \sum_{j=0}^{\infty} g_{i,k}^{(j)} x^j$, taking $x = 1$, then $\sum_{j=0}^{\infty} g_{i,k}^{(j)} = 0$. In view of (1), for $l = 1, 2, \ldots, \sum_{j=0}^{l} g_{i,k}^{(j)} < 0$. ◻

LEMMA 3.4 (discrete Gronwall inequality). *Suppose that $f_k \ge 0, \eta_k \ge 0, k = 0, 1, 2, \ldots,$ and*

$$\eta_{k+1} \le \rho\eta_k + \tau f_k, \rho = 1 + C_0\tau, j = 0, 1, 2, \ldots, \eta_0 = 0, \tag{3.13}$$

where $C_0 \ge 0$ is constant, then

$$\eta_{k+1} \le e^{C_0 t_k} \sum_{j=0}^{k} \tau f_j.$$

Proof. From (3.13), we obtain

$$\eta_{k+1} \le \rho\eta_k + \tau f_k \le \rho^2\eta_{k-1} + \rho\tau f_{k-1} + \tau f_k \le \cdots \le \rho^{k+1}\eta_0 + \tau \sum_{j=0}^{k} \rho^{k-j} f_j.$$

Since $\rho > 1$ and $\eta_0 = 0$, we have $\eta_{k+1} \le \rho^k \sum_{j=0}^{k} \tau f_j$.

Note that $\rho^k = (1 + C_0\tau)^k \le e^{C_0\tau \cdot k} = e^{C_0 t_k}$, so that $\eta_{k+1} \le e^{C_0 t_k} \sum_{j=0}^{k} \tau f_j$. ◻

4. Stability and convergence of the explicit Euler approximation. We suppose that $\widetilde{u}_i^{(j)}, (i = 0, 1, 2, \ldots, m; j = 0, 1, 2, \ldots, n)$ is an approximate solution of (3.6) and (3.8); the errors $\varepsilon_i^{(j)} = \widetilde{u}_i^{(j)} - u_i^{(j)} \quad (i = 0, 1, 2, \ldots, m; j = 0, 1, 2, \ldots, n)$ satisfy

$$
\begin{aligned}
\varepsilon_i^{k+1} &= \varepsilon_i^k + \mu_i^k(\varepsilon_i^k - \varepsilon_{i-1}^k) + r_{i,k}^{(1)} \sum_{j=0}^{i+1} g_{i+1,k}^{(j)} \varepsilon_{i-j+1}^k \\
&\quad + r_{i,k}^{(2)} \sum_{j=0}^{m-i+1} g_{i-1,k}^{(j)} \varepsilon_{i+j-1}^k + \tau f(\widetilde{u}_i^k, x_i, t_k) - f(u_i^k, x_i, t_k).
\end{aligned}
\tag{4.1}
$$

Suppose that $\max_{1 \le i \le m-1} |\varepsilon_i^{k+1}| = \|\mathbf{E}^{k+1}\|_\infty$ and $S_c = \overline{\nu}\tau h^{-1} + \overline{\kappa}(c_1 + c_2)\overline{\alpha}\tau h^{-\underline{\alpha}}$, where the quatities $\overline{\nu}, \overline{\kappa}, \overline{\alpha}$, and c_1, c_2 are defined in section 2 after (2.7), then we obtain the following theorem.

THEOREM 4.1 (stability of the explicit Euler approximation). *If $S_c < 1$, then the explicit Euler approximation defined by* (3.6) *and* (3.8) *is stable, and we have*

$$\|\mathbf{E}^k\|_\infty \le K\|\mathbf{E}^0\|_\infty, k = 1, 2, \ldots, n,$$

where K is a positive number independent of k, h, and τ.

Proof. From $S_c < 1$, it follows immediately that $1 - \mu_i^k - \alpha_i^k r_{i,k}^{(1)} - \alpha_i^k r_{i,k}^{(2)} \ge 0$. For $k = 0, 1, 2, \ldots, n-1$; $i = 1, 2, \ldots, m-1$ using (4.1), we obtain

$$
\begin{aligned}
|\varepsilon_i^{k+1}| &\le (1 - \mu_i^k - \alpha_i^k r_{i,k}^{(1)} - \alpha_i^k r_{i,k}^{(2)})|\varepsilon_i^k| + \mu_i^k |\varepsilon_{i-1}^k| \\
&\quad + r_{i,k}^{(1)} \sum_{j=0, j\ne 1}^{i+1} g_{i+1,k}^{(j)} |\varepsilon_{i+1-j}^k| + r_{i,k}^{(2)} \sum_{j=0, j\ne 1}^{m-i+1} g_{i-1,k}^{(j)} |\varepsilon_{i-1+j}^k| + \tau L |\varepsilon_i^k| \\
&\le \|\mathbf{E}^k\|_\infty + \mu_i^k(\|\mathbf{E}^k\|_\infty - \|\mathbf{E}^k\|_\infty) \\
&\quad + r_{i,k}^{(1)} \sum_{j=0}^{i+1} g_{i+1,k}^{(j)} \|\mathbf{E}^k\|_\infty + r_{i,k}^{(2)} \sum_{j=0}^{m-i+1} g_{i-1,k}^{(j)} \|\mathbf{E}^k\|_\infty + \tau L \|\mathbf{E}^k\|_\infty.
\end{aligned}
$$

Since $\sum_{j=0}^{i+1} g_{i+1,k+1}^{(j)} < 0$ and $\sum_{j=0}^{m-i+1} g_{i-1,k+1}^{(j)} < 0$, then

$$|\varepsilon_i^{k+1}| \le (1 + \tau L)\|\mathbf{E}^k\|_\infty \le (1 + \tau L)^{k+1}\|\mathbf{E}^0\|_\infty \le e^{LT}\|\mathbf{E}^0\|_\infty,$$

i.e., $\|\mathbf{E}^k\|_\infty \le e^{LT}\|\mathbf{E}^0\|_\infty = K\|\mathbf{E}^0\|_\infty$. From Definition (3.2), this shows that the explicit Euler approximation defined by (3.6) and (3.8) is stable. ∎

Now we consider the convergence of the explicit Euler approximation. We suppose that the continuous problem (2.5)–(2.7) has a smooth solution $u(x, t) \in C_{x,t}^{1+\overline{\alpha},2}(\Omega)$.

Let $u(x_i, t_k)$ $(i = 1, 2, \ldots, m-1; k = 1, 2, \ldots, n)$ be the exact solution of (2.5)–(2.7) at mesh point (x_i, t_k). Define

$$\eta_i^k = u(x_i, t_k) - u_i^k, i = 1, 2, \ldots, m-1; k = 1, 2, \ldots, n$$

and

$$\mathbf{Y}^k = (\eta_1^k, \eta_2^k, \ldots, \eta_{m-1}^k)^T.$$

Using $\mathbf{Y}^0 = \mathbf{0}$ and $u_i^k = u(x_i, t_k) - \eta_i^k$, substitution into (3.6) leads to

$$\begin{aligned}\eta_i^{k+1} &= \eta_i^k + \mu_i^k(\eta_i^k - \eta_{i-1}^k) + r_{i,k}^{(1)} \sum_{j=0}^{i+1} g_{i+1,k}^{(j)} \eta_{i-j+1}^k \\ &\quad + r_{i,k}^{(2)} \sum_{j=0}^{m-i+1} g_{i-1,k}^{(j)} \eta_{i+j-1}^k + \tau[f(u(x_i,t_k),x_i,t_k) - f(u_i^k, x_i, t_k)] + R_i^k,\end{aligned} \tag{4.2}$$

where $i = 1, 2, \ldots, m-1, k = 0, 1, 2, \ldots, n-1$.

Similarly, we can obtain the following theorem.

THEOREM 4.2 (convergence of the explicit Euler approximation). *Suppose that the continuous problem* (2.5)–(2.7) *has a smooth solution* $u(x,t) \in C_{x,t}^{1+\overline{\alpha},2}(\Omega)$. *Let* u_i^k *be the numerical solution computed by using* (3.6) *and* (3.8). *If* $S_c < 1$, *then there is a positive constant* C *independent of* i, k, h, *and* τ *such that*

$$|u_i^k - u(x_i, t_k)| \le C(\tau + h), \quad i = 1, 2, \ldots, m-1; k = 1, 2, \ldots, n. \tag{4.3}$$

Proof. From $1 - \mu_i^k - \alpha_i^k r_{i,k}^{(1)} \ge 0 - \alpha_i^k r_{i,k}^{(2)} \ge 0$, using (4.2), we obtain

$$\begin{aligned}|\eta_i^{k+1}| &\le (1 - \mu_i^k - \alpha_i^k r_{i,k}^{(1)} - \alpha_i^k r_{i,k}^{(2)})|\eta_i^k| + \mu_i^k |\eta_{i-1}^k| \\ &\quad + r_{i,k}^{(1)} \sum_{j=0, j\ne 1}^{i+1} g_{i+1,k}^{(j)} |\eta_{i+1-j}^k| + r_{i,k}^{(2)} \sum_{j=0, j\ne 1}^{m-i+1} g_{i-1,k}^{(j)} |\eta_{i-1+j}^k| \\ &\quad + \tau L |\eta_i^k| + C_1(\tau^2 + \tau h) \\ &\le \|\mathbf{Y}^k\|_\infty + \mu_i^k (\|\mathbf{Y}^k\|_\infty - \|\mathbf{Y}^k\|_\infty) \\ &\quad + r_{i,k}^{(1)} \sum_{j=0}^{i+1} g_{i+1,k}^{(j)} \|\mathbf{Y}^k\|_\infty + r_{i,k}^{(2)} \sum_{j=0}^{m-i+1} g_{i-1,k}^{(j)} \|\mathbf{Y}^k\|_\infty \\ &\quad + \tau L \|\mathbf{Y}^k\|_\infty + C_1(\tau^2 + \tau h).\end{aligned}$$

Since $\sum_{j=0}^{i+1} g_{i+1,k+1}^{(j)} < 0$ and $\sum_{j=0}^{m-i+1} g_{i-1,k+1}^{(j)} < 0$, then

$$|\eta_i^{k+1}| \le (1 + \tau L)\|\mathbf{Y}^k\|_\infty + C_1(\tau^2 + \tau h).$$

Further,

$$\|\mathbf{Y}^{k+1}\|_\infty \le (1 + \tau L)\|\mathbf{Y}^k\|_\infty + C_1(\tau^2 + \tau h).$$

Using the discretized Gronwall lemma, we obtain

$$|\eta_i^{k+1}| \le \|\mathbf{Y}^{k+1}\|_\infty \le C_1 e^{Lk\tau}(\tau + h) \le C_1 e^{LT}(\tau + h) = C(\tau + h).$$

Thus we see that for any x and t, as h and τ approach 0 in such a way that $(ih, k\tau) \to (x, t)$, u_i^k approaches $u(x,t)$. This proves that u_i^k converges to $u(x_i, t_k)$ as h and τ tend to zero. Hence, the conclusion follows. ∎

5. Stability and convergence of the implicit Euler approximation. To obtain the stability of the implicit Euler approximation (3.7), we rewrite (3.7) as

$$\begin{aligned} &u_i^{k+1} + \mu_i^{k+1}(u_i^{k+1} - u_{i-1}^{k+1}) \\ &-r_{i,k+1}^{(1)} \sum_{j=0}^{i+1} g_{i+1,k+1}^{(j)} u_{i-j+1}^{k+1} - r_{i,k+1}^{(2)} \sum_{j=0}^{m-i+1} g_{i-1,k+1}^{(j)} u_{i+j-1}^{k+1} \\ &= u_i^k + \tau f(u_i^k, x_i, t_k). \end{aligned} \tag{5.1}$$

For $i = 0, 1, 2, \ldots, m; \quad j = 0, 1, 2, \ldots, n$, we have

$$\begin{aligned} &\varepsilon_i^{k+1} + \mu_i^{k+1}(u_i^{k+1} - u_{i-1}^{k+1}) \\ &-r_{i,k+1}^{(1)} \sum_{j=0}^{i+1} g_{i+1,k+1}^{(j)} \varepsilon_{i-j+1}^{k+1} + r_{i,k+1}^{(2)} \sum_{j=0}^{m-i+1} g_{i-1,k+1}^{(j)} \varepsilon_{i+j-1}^{k+1} \\ &= \varepsilon_i^k + \tau f(\widetilde{u}_i^k, x_i, t_k) - f(u_i^k, x_i, t_k). \end{aligned} \tag{5.2}$$

Thus, the following theorem can be obtained.

THEOREM 5.1 (stability of the implicit Euler approximation). *The implicit Euler approximation defined by* (3.7) *and* (3.8) *is unconditionally stable, and we have*

$$\|\mathbf{E}^k\|_\infty \le K\|\mathbf{E}^0\|_\infty, k = 1, 2, \ldots, n,$$

where K is a positive number independent of k, h, and τ.

Proof. Assume that $|\varepsilon_{i_0}^{k+1}| = \max_{1\le i\le m-1} |\varepsilon_i^{k+1}| = \|\mathbf{E}^{k+1}\|_\infty$. Using $\sum_{j=0}^{i_0+1} g_{i_0+1,k+1}^{(j)} < 0$ and $\sum_{j=0}^{m-i_0+1} g_{i_0-1,k+1}^{(j)} < 0$, we have

$$\begin{aligned} |\varepsilon_{i_0}^{k+1}| &\le |\varepsilon_{i_0}^{k+1}| + \mu_{i_0}^{k+1}(|\varepsilon_{i_0}^{k+1}| - |\varepsilon_{i_0}^{k+1}|) \\ &\quad - r_{i_0,k+1}^{(1)} \sum_{j=0}^{i_0+1} g_{i_0+1,k+1}^{(j)} |\varepsilon_{i_0}^{k+1}| - r_{i_0,k+1}^{(2)} \sum_{j=0}^{m-i_0+1} g_{i_0-1,k+1}^{(j)} |\varepsilon_{i_0}^{k+1}| \\ &\le |\varepsilon_{i_0}^{k+1}| + \mu_{i_0}^{k+1}(|\varepsilon_{i_0}^{k+1}| - |\varepsilon_{i_0-1}^{k+1}|) \\ &\quad - r_{i_0,k+1}^{(1)} \sum_{j=0}^{i_0+1} g_{i_0+1,k+1}^{(j)} |\varepsilon_{i_0-j+1}^{k+1}| - r_{i_0,k+1}^{(2)} \sum_{j=0}^{m-i_0+1} g_{i_0-1,k+1}^{(j)} |\varepsilon_{i_0+j-1}^{k+1}| \\ &\le |\varepsilon_{i_0}^{k+1} + \mu_{i_0}^{k+1}(\varepsilon_{i_0}^{k+1} - \varepsilon_{i_0-1}^{k+1}) \\ &\quad - r_{i_0,k+1}^{(1)} \sum_{j=0}^{i_0+1} g_{i_0+1,k+1}^{(j)} \varepsilon_{i_0-j+1}^{k+1} - r_{i_0,k+1}^{(2)} \sum_{j=0}^{m-i_0+1} g_{i_0-1,k+1}^{(j)} \varepsilon_{i_0+j-1}^{k+1}|. \end{aligned}$$

Using (4.2), we obtain

$$\begin{aligned} |\varepsilon_{i_0}^{k+1}| &\le |\varepsilon_{i_0}^k + \tau[f(\widetilde{u}_{i_0}^k, x_{i_0}, t_k) - f(u_{i_0}^k, x_{i_0}, t_k)]| \\ &\le \|\mathbf{E}^k\|_\infty + \tau L|\widetilde{u}_{i_0}^k - u_{i_0}^k| \\ &\le (1 + \tau L)\|\mathbf{E}^k\|_\infty. \end{aligned}$$

Hence,

$$\|\mathbf{E}^k\|_\infty \le (1 + \tau L)^k \|\mathbf{E}^0\|_\infty \le e^{k\tau L}\|\mathbf{E}^0\|_\infty \le e^{LT}\|\mathbf{E}^0\|_\infty, \quad k = 1, 2, \ldots, n,$$

i.e., $\|\mathbf{E}^k\|_\infty \le e^{LT}\|\mathbf{E}^0\|_\infty = K\|\mathbf{E}^0\|_\infty$. From Definition 3.2, this shows that the implicit Euler approximation defined by (3.7) and (3.8) is unconditionally stable. $\square$

Similarly, it can be verified that $\eta_i^k = u(x_i, t_k) - u_i^k$ satisfies

$$\eta_i^{k+1} = \eta_i^k + r_{i,k+1}^{(1)} \sum_{j=0}^{i+1} g_{i+1,k+1}^{(j)} \eta_{i-j+1}^{k+1} + r_{i,k+1}^{(2)} \sum_{j=0}^{m-i+1} g_{i-1,k+1}^{(j)} \eta_{i+j-1}^{k+1} + \tau[f(u(x_i,t_k),x_i,t_k) - f(u_i^k,x_i,t_k)] + R_i^{k+1}, \tag{5.3}$$

where $i = 1,2,\ldots,m-1, k = 0,1,2,\ldots,n-1$.

As a result, the following convergence theorem is obtained.

THEOREM 5.2 (convergence of the implicit Euler approximation). *Suppose that the continuous problem* (2.5)–(2.7) *has a smooth solution* $u(x,t) \in C_{x,t}^{1+\overline{\alpha},2}(\Omega)$. *Let* u_i^k *be the numerical solution computed by use of* (3.7) *and* (3.8). *Then there is a positive constant* C *independent of* i, k, h, *and* τ *such that*

$$|u_i^k - u(x_i,t_k)| \le C(\tau + h), \quad i = 1,2,\ldots,m-1; k = 1,2,\ldots,n. \tag{5.4}$$

Proof. Assume that $|\eta_{i_0}^{k+1}| = \max_{1\le i\le m-1}|\eta_i^{k+1}| = \|\mathbf{Y}^{k+1}\|_\infty$. Using $\sum_{j=0}^{i_0+1} g_{i_0+1,k+1}^{(j)} < 0$ and $\sum_{j=0}^{m-i_0+1} g_{i_0-1,k+1}^{(j)} < 0$, we have

$$\begin{aligned}
|\eta_{i_0}^{k+1}| &\le |\eta_{i_0}^{k+1}| + \mu_{i_0}^{k+1}(|\eta_{i_0}^{k+1}| - |\eta_{i_0}^{k+1}|) \\
&\quad - r_{i_0,k+1}^{(1)} \sum_{j=0}^{i_0+1} g_{i_0+1,k+1}^{(j)} |\eta_{i_0}^{k+1}| - r_{i_0,k+1}^{(2)} \sum_{j=0}^{m-i_0+1} g_{i_0-1,k+1}^{(j)} |\eta_{i_0}^{k+1}| \\
&\le |\eta_{i_0}^{k+1}| + \mu_{i_0}^{k+1}(|\eta_{i_0}^{k+1}| - |\eta_{i_0-1}^{k+1}|) \\
&\quad - r_{i_0,k+1}^{(1)} \sum_{j=0}^{i_0+1} g_{i_0+1,k+1}^{(j)} |\eta_{i_0-j+1}^{k+1}| - r_{i_0,k+1}^{(2)} \sum_{j=0}^{m-i_0+1} g_{i_0-1,k+1}^{(j)} |\eta_{i_0+j-1}^{k+1}| \\
&\le |\eta_{i_0}^{k+1} + +\mu_{i_0}^{k+1}(\eta_{i_0}^{k+1} - \eta_{i_0-1}^{k+1}) \\
&\quad - r_{i_0,k+1}^{(1)} \sum_{j=0}^{i_0+1} g_{i_0+1,k+1}^{(j)} \eta_{i_0-j+1}^{k+1} - r_{i_0,k+1}^{(2)} \sum_{j=0}^{m-i_0+1} g_{i_0-1,k+1}^{(j)} \eta_{i_0+j-1}^{k+1}|.
\end{aligned}$$

Using (5.2), we obtain

$$\begin{aligned}
|\eta_{i_0}^{k+1}| &\le |\eta_{i_0}^k + R_{i_0}^{k+1} + \tau[f(u(x_{i_0},t_k,x_{i_0},t_k)) - f(u_{i_0}^k,x_{i_0},t_k)]| \\
&\le |\eta_{i_0}^k| + |R_{i_0}^{k+1}| + \tau|f(u(x_{i_0},t_k),x_{i_0},t_k) - f(u_{i_0}^k,x_{i_0},t_k)| \\
&\le \|\mathbf{Y}^k\|_\infty + C_2(\tau^2 + \tau h) + \tau L|u(x_{i_0},t_k) - u_{i_0}^k| \\
&\le \|\mathbf{Y}^k\|_\infty + C_2(\tau^2 + \tau h) + \tau L\|\mathbf{Y}^k\|_\infty.
\end{aligned}$$

Also,

$$\|\mathbf{Y}^{k+1}\|_\infty \le (1+\tau L)\|\mathbf{Y}^k\|_\infty + C_2(\tau^2 + \tau h), \quad k = 0,1,2,\ldots,n-1.$$

Using the discretized Gronwall lemma, we have

$$\|\mathbf{Y}^k\|_\infty \le e^{k\tau L} k C_2(\tau^2 + \tau h) \le e^{LT} C_2 T(\tau + h) = C(\tau + h), \quad k = 1,2,\ldots,n.$$

Thus we see that for any x and t, as h and τ approach 0 in such a way that $(ih, k\tau) \to (x,t)$, u_i^k approaches $u(x,t)$.This proves that u_i^k converges to $u(x_i,t_k)$ as h and τ tend to zero. Hence, the conclusion is obtained. $\square$

6. Other numerical methods.

6.1. Fractional method of lines. The method of lines is a well-known technique for solving parabolic-type partial differential equations. The fractional method of lines was described in Liu, Anh, and Turner [16] (see also Meerschaert and Tadjeran [22]) and has been used to solve fractional partial differential equations. Essentially, this method proceeds by leaving the derivatives along one chosen axis untouched (usually in time), while the fractional partial derivatives (in space) are discretized using a method such as that discussed in section 3. The system is thereby reduced from its partial differential equation to a system of ordinary differential equations, then integrated in time. In the fractional method of lines, time integration is accomplished by using a differential-algebraic equation integrator. Brenan, Campbell, and Petzold [2] developed the differential-algebraic system solver known as DASSL, which is based on the backward difference formulas. DASSL approximates the derivatives using the kth order backward difference, where k ranges from one to five. At every step it selects the order k and stepsize based on the behavior of the solution. In this work, we used DASSL as our solver. The fractional differential equation (2.5) can be discretized as follows.

For $i = 1, 2, \ldots, n$,

$$\frac{du_i(t)}{dt} = -\nu_i(t)\frac{u_i(t) - u_{i-1}(t)}{h} + \frac{\kappa_i(t)c_{+,i}(t)}{h^{\alpha_i(t)}}\sum_{j=0}^{i+1} g_i^{(j)}(t)u_{i-j+1}(t)$$
$$+\frac{\kappa_i(t)c_{-,i}(t)}{h^{\alpha_i(t)}}\sum_{j=0}^{m-i+1} g_i^{(j)}(t)u_{i+j-1}(t) + f(u_i(t), x_i, t),$$

where

$$u_i(t) = u(x_i, t), \quad \nu_i(t) = \nu(x_i, t), \quad \alpha_i(t) = \alpha(x_i, t),$$
$$\kappa_i(t) = \kappa(x_i, t), \quad c_{+,i}(t) = c_+(x_i, t), \quad c_{-,i}(t) = c_-(x_i, t),$$
$$g_i^{(0)}(t) = 1, \quad g_i^{(j)}(t) = -\frac{\alpha_i(t) - j + 1}{j} g_i^{(j-1)}(t) \quad , j = 0, 1, \ldots, m.$$

6.2. Extrapolation method. The implicit difference method was shown to be stable above. This method is consistent with a local truncation error which is $O(\tau + h)$. Further, if problem (2.5)–(2.7) has a sufficiently smooth solution, using Taylor formula, we can obtain

$$\frac{\partial u(x_i, t_{k+1})}{\partial t} = \frac{u(x_i, t_{k+1}) - u(x_i, t_k)}{\tau} + \frac{1}{2!}\frac{\partial^2 u(x_i, t_{k+1})}{\partial t^2}\tau - \frac{1}{3!}\frac{\partial^3 u(x_i, t_{k+1})}{\partial t^3}\tau^2 + \cdots.$$

Combining with Proposition 3.1 in [23], we can obtain the following local truncation error:

$$C_1^{(1)}h + C_1^{(2)}\tau + C_2^{(1)}h^2 + C_2^{(2)}\tau^2 + \cdots,$$

where the coefficients $C_i^{(1)}$ and $C_i^{(2)}$ do not depend on the grid size h and τ. In order to improve the low order of convergence in time and space, we choose $h = \tau$. Hence, the local truncation error can be rewritten as

$$C_1 h + C_2 h^2 + \cdots + C_n h^n + \cdots,$$

where C_i do not depend on the grid size h.

Therefore, according to Lax's equivalence theorem [27], it converges at this rate. We then apply the implicit Euler approximation on a (coarse) grid $\Delta t = \tau, \Delta x = h$, and then on a finer grid $\Delta t = \tau/2, \Delta x = h/2$. The extrapolated solution is then computed from $u(x_i, t) \approx 2u_{2i}^{2k}(h/2, \tau/2) - u_i^k(h, \tau)$, $i = 1, 2, \ldots, m-1$, where $t = t_k, x = x_i$ on the coarse grid, while $t = t_{2k}, x = x_{2i}$ on the fine grid. $u_i^k(h, \tau)$ and $u_{2i}^{2k}(h/2, \tau/2)$ are the numerical solutions on the coarse grid and the fine grid, respectively. Thus, the extrapolation method may be used to obtain a solution with convergence order $O(\tau^2 + h^2)$ (see [9, 23]).

6.3. Matrix transfer technique (MTT). In order to introduce the MTT, we consider an approximation of the Riesz fractional derivative. Suppose that $x \in [a, b]$, and $x_i = a + ih, i = 0, 1, \ldots, m$, and $h = \frac{b-a}{m}$. For convenience, let $y_i = y(x_i), i = 0, 1, \ldots, m$, so that

$$\frac{d^2 y(x_i)}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i+1}}{h^2} + O(h^2).$$

If $y_0 = y_m = 0$, then $\Delta \mathbf{y} \approx -h^{-2} A \mathbf{y}$, i.e., $-\Delta \mathbf{y} \approx h^{-2} A \mathbf{y}$, where

$$\Delta \mathbf{y} = \left(\frac{d^2 y(x_1)}{dx^2}, \frac{d^2 y(x_2)}{dx^2}, \ldots, \frac{d^2 y(x_{m-1})}{dx^2} \right)^T, \quad \mathbf{y} = (y_1, y_2, \ldots, y_{m-1})^T,$$

and

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}_{(m-1)\times(m-1)}.$$

We know that the eigenvalues of the $(m-1) \times (m-1)$ real symmetric matrix A are $\mu_i = 4 \sin^2(\frac{i\pi}{2m})$, $\quad i = 1, 2, \ldots, m-1$, and the eigenvector corresponding to μ_i is

$$\mathbf{v}_i = (v_i^{(1)}, v_i^{(2)}, \ldots, v_i^{(m-1)})^T,$$

where $v_i^{(j)} = \sin \frac{ij\pi}{m}$, $\quad j = 1, 2, \ldots, m-1$. It can be seen that all the eigenvalues of the matrix A are positive and different. Further, the following propositions can be obtained.

PROPOSITION 6.1. *The matrix A is symmetric and positive definite.*

PROPOSITION 6.2. *Suppose that $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{m-1}$ are the eigenvectors corresponding to the eigenvalues $\mu_1, \mu_2, \ldots, \mu_{m-1}$ of the matrix A, then $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{m-1}$ form an orthonormal set.*

PROPOSITION 6.3. *The matrix $\mathbf{A}$ is a symmetric positive definite matrix. There exists an orthogonal matrix $\mathbf{P}$ such that*

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \mathbf{\Lambda},$$

where $\mathbf{\Lambda}$ is a diagonal matrix with diagonal entries being the eigenvalues of $\mathbf{A}$.

In fact, setting $\mathbf{P}_i = \mathbf{v}_i / \|\mathbf{v}_i\|_2 = \sqrt{2/m}\mathbf{v}_i$, $\quad i = 1, 2, \ldots, m-1$, then we can take

$$\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_{m-1}).$$

Hence, we have

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{\Lambda} = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_{m-1} \end{bmatrix}$$

or

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T.$$

Because $\mathbf{P}$ is an orthogonal matrix and $\mathbf{A}$ is a positive definite matrix, from Propositions 6.1–6.3, we can obtain the following proposition.

PROPOSITION 6.4 (see [21]). *Suppose that the matrix* $\mathbf{A}$ *is positive definite, and*

$$\mathbf{A} = \mathbf{P}\text{diag}(\mu_1, \mu_2, \ldots, \mu_{m-1})\mathbf{P}^T,$$

where $\mathbf{P}$ *is an orthogonal matrix. Then for an arbitrary real* α*,*

$$\mathbf{A}^\alpha = \mathbf{P}\text{diag}(\mu_1^\alpha, \mu_2^\alpha, \ldots, \mu_{m-1}^\alpha)\mathbf{P}^T$$

may be uniquely determined by $\mathbf{A}$ *and* α*.*

We consequently adopt the following approximation [8]:

$$-(-\Delta)^{\frac{\alpha}{2}}\mathbf{y} \approx -\frac{1}{h^\alpha}A^{\frac{\alpha}{2}}\mathbf{y}, \tag{6.1}$$

where

$$\mathbf{A}^{\frac{\alpha}{2}} = \mathbf{P}\mathbf{\Lambda}^{\frac{\alpha}{2}}\mathbf{P}^T = \sum_{j=1}^{m-1}\mu_j^{\frac{\alpha}{2}}\mathbf{P}_j\mathbf{P}_j^T = \sum_{j=1}^{m-1}\frac{2}{m}\mu_j^{\frac{\alpha}{2}}\mathbf{v_j}\mathbf{v_j}^T.$$

Hence, $-(-\Delta)^{\frac{\alpha}{2}}\mathbf{y} \approx -\frac{1}{h^\alpha}\sum_{j=1}^{m-1}\frac{2}{m}\mu_j^{\frac{\alpha}{2}}(\mathbf{v}_j^T\mathbf{y})\mathbf{v}_j$, i.e.,

$$-(-\Delta)^{\frac{\alpha}{2}}y_i \approx -\frac{1}{h^\alpha}\sum_{j=1}^{m-1}\frac{2}{m}\mu_j^{\frac{\alpha}{2}}\sin\frac{ij\pi}{m}\sum_{l=1}^{m-1}\sin\frac{lj\pi}{m}y_l\ .$$

Letting $c_{i,l} = \sum_{j=1}^{m-1}\mu_j^{\frac{\alpha}{2}}\sin\frac{jl\pi}{m}\sin\frac{ij\pi}{m}$, we get

$$-(-\Delta)^{\frac{\alpha}{2}}y_i \approx -\frac{2}{m\cdot h^\alpha}\sum_{l=1}^{m-1}c_{i,l}y_l.$$

We consider the following equation:

$$\frac{\partial u}{\partial t} = -\nu(x,t)\frac{\partial u}{\partial x} + \kappa(x,t)(-(-\Delta)^{\frac{\alpha(x,t)}{2}}u) + f(u,x,t), \tag{6.2}$$

and propose a self-discretized difference scheme for it as

$$\begin{aligned}\frac{du_i(t)}{dt} &= -\nu_i(t)\frac{u_i(t)-u_{i-1}(t)}{h}\\ &\quad -\frac{2\kappa_i(t)}{m\cdot h^{\alpha_i(t)}}\sum_{l=1}^{m-1}c_{i,l}(t)u_l(t) + f(u_i(t),x_i,t), \quad i=1,2,\ldots,m-1,\end{aligned} \tag{6.3}$$

where

$$\nu_i(t) = \nu(x_i, t), \quad \kappa_i(t) = \kappa(x_i, t), \quad \alpha_i(t) = \alpha(x_i, t),$$

$$c_{i,l}(t) = \sum_{j=1}^{m-1} \mu_j^{\frac{\alpha_i(t)}{2}} \sin\frac{jl\pi}{m} \sin\frac{ij\pi}{m}, \quad l = 1, 2, \ldots, m-1 .$$

We can now apply the fractional method of lines to solve (6.3).

7. Numerical examples. In order to demonstrate the effectiveness of our theoretical analysis, four numerical examples are now presented. In Examples 1 and 2, we take $\alpha(x,t) = 1.5 + 0.4\sin(0.5\pi xt)$.

Example 1. Consider the following variable-order fractional advection-diffusion equation:

$$\frac{\partial u}{\partial t} = -\nu(x,t)\frac{\partial u}{\partial x} + c_+(x,t)_a D_x^{\alpha(x,t)} u + c_-(x,t)_x D_b^{\alpha(x,t)} u + f(u,x,t), \quad (x,t) \in \Omega, \tag{7.1}$$

$$u(x,0) = x^2(1-x)^2, 0 \le x \le 1; \quad u(0,t) = u(1,t) = 0, 0 \le t \le 1, \tag{7.2}$$

where

$$\begin{aligned} &\nu(x,t) = 6x^3(1-x)^3 e^t, \\ &c_+(x,t) = 0.5\Gamma(5-\alpha(x,t))x^{2+\alpha}(1-x)^4 e^t, \\ &c_-(x,t) = 0.5\Gamma(5-\alpha(x,t))x^4(1-x)^{2+\alpha} e^t, \\ &f(u,x,t) = u + u^2(-24x^2 + 2\alpha(x,t)(4-\alpha(x,t)). \end{aligned}$$

The exact solution of the above problem is $u(x,t) = e^t x^2(1-x)^2$.

A comparison of numerical solutions using explicit Euler approximation (EEA), implicit Euler approximation (IEA), fractional method of lines (MOL), and the exact solution at time $t = 2.0$ is shown in Figure 7.1. Here we take $h = 0.01$ as the space

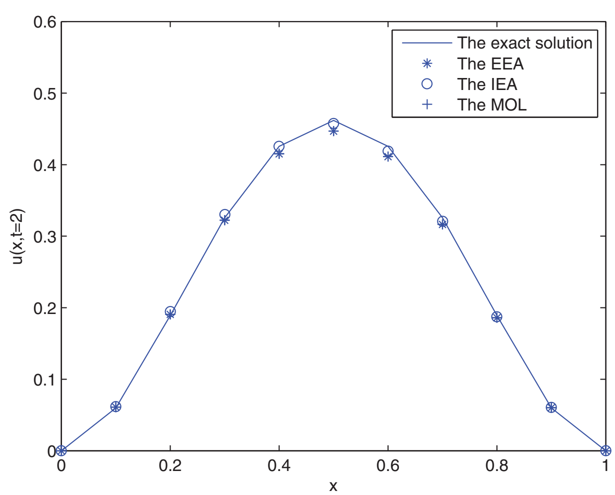

FIG. 7.1. *A comparison of numerical solutions using explicit Euler approximation, implicit Euler approximation, fractional method of lines, and the exact solution at time $t = 2.0$.*

TABLE 7.1
Maximum errors for the IEA and extrapolation method (EM), and the effect of grid size reduction at time $t = 2.0$.

$h = \tau$	Max error-IEA	Error-IEA rate	Max error-EM	Error-EM rate
$\frac{1}{20}$	$3.595E-2$	-	$7.844E-3$	-
$\frac{1}{30}$	$2.147E-2$	$1.675 \approx 30/20$	$3.242E-3$	$2.419 \approx (30/20)^2$
$\frac{1}{40}$	$1.529E-2$	$1.404 \approx 40/30$	$1.784E-3$	$1.817 \approx (40/30)^2$
$\frac{1}{50}$	$1.200E-2$	$1.274 \approx 50/40$	$1.134E-3$	$1.573 \approx (50/40)^2$
$\frac{1}{60}$	$9.844E-3$	$1.219 \approx 60/50$	$7.853E-4$	$1.445 \approx (60/50)^2$

stepsize for all methods and $\tau = 0.01$ as the time stepsize for the IEA. To ensure the convergence of the EEA, we take $\tau = 0.0002$ as its time stepsize. It is apparent that the three numerical solutions are in good agreement with the exact solution.

Table 7.1 shows the numerical errors at time $t = 2.0$ between the exact solution and the implicit and extrapolation numerical solutions. The second column shows the absolute value of the maximum error in the implicit numerical solution at time $t = 2.0$. The third column shows the ratio of the error reduction as the grid is refined. Note that the behavior of this error is (almost) linear when the IEA is used. Column 4 shows the absolute value of the maximum error when the IEA solution is extrapolated. Column 5 shows the ratio of these extrapolated solution errors to examine the convergence. It is seen that the rate of convergence is of order $O(\tau^2 + h^2)$. From Table 7.1, both methods are in excellent agreement with the exact solution, and the convergence order of the EM is improved significantly.

Example 2. Consider the following variable-order fractional advection-diffusion equation with the Riesz fractional derivative:

$$\begin{gathered}\frac{\partial u}{\partial t} = -\nu(x,t)\frac{\partial u}{\partial x} - \kappa(x,t)(-\Delta)^{\frac{\alpha(x,t)}{2}}u + f(u,x,t), \quad (x,t) \in \Omega, \\ u(x,0) = x^2, \quad 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, \quad 0 \le t \le 2\ .\end{gathered} \tag{7.3}$$

We choose

$$\nu(x,t) = x^3(1-x)^3e^t, \quad \kappa(x,t) = -2\cos(0.5\pi\alpha(x,t))\Gamma(5-\alpha(x,t))(x-x^2)^{2+\alpha(x,t)},$$

and

$$f(u,x,t) = [1 - g_1(x,t) - g_2(x,t)]u + 2(1-2x)u^2,$$

where

$$\begin{aligned}g_1(x,t) &= 2x^2(1-x)^{2+\alpha(x,t)}[12x^2 + (4-\alpha(x,t))(-6x+3-\alpha(x,t))], \\ g_2(x,t) &= 2(1-x)^2x^{2+\alpha(x,t)}[12(1-x)^2 + (4-\alpha(x,t))(6x-3-\alpha(x,t))].\end{aligned}$$

The above problem has the exact solution $u(x,t) = e^tx^2(1-x)^2$.

A comparison of the exact solution and numerical solutions using IEA, fractional MOL, and MTT for Example 2 at time $t = 2.0$ is shown in Figure 7.2. It is apparent that all the numerical solutions are in excellent agreement with the exact solution.

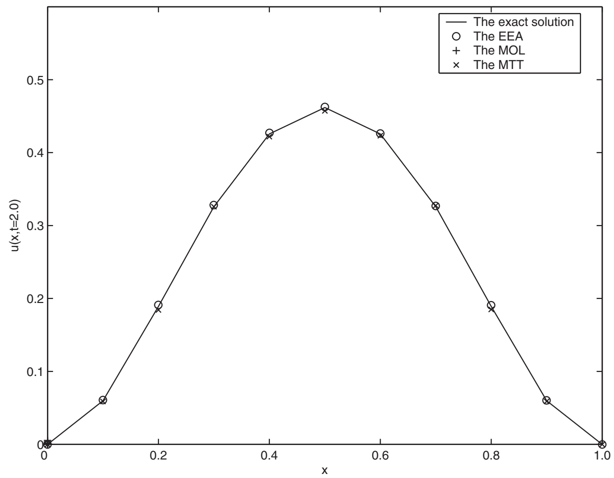

FIG. 7.2. *A comparison among the IEA, the fractional MOL, the MTT, and the exact solution.*

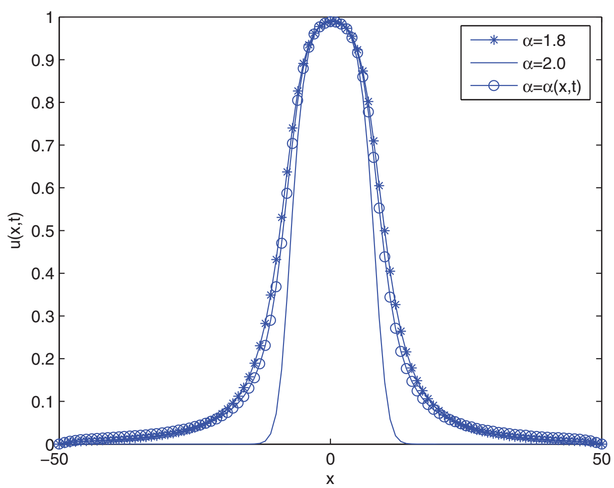

FIG. 7.3. *Comparison of numerical solutions of the nonlinear advection-diffusion equation, with* $\alpha = 2.0$*,* $\alpha = 1.8$*, and* $\alpha = \alpha(x,t) = 1.7 + 0.5e^{-\frac{x^2}{1000} - \frac{t}{50} - 1}$ *at* $t = 32$*.*

Example 3. Consider the following nonlinear variable-order fractional advection-diffusion equation with the Riesz fractional derivative:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -0.1\dfrac{\partial u}{\partial x} - 0.1(-\Delta)^{\frac{\alpha(x,t)}{2}} u + \dfrac{1}{4}u(1-u), (x,t) \in \Omega, \\ u(-50,t) = 0, \quad u(50,t) = 0, 0 \le t \le 32, \end{cases} \tag{7.4}$$

where $\Omega = (-50, 50) \times (0, 32]$, $\alpha(x,t) = 1.7 + 0.5e^{-\frac{x^2}{1000} - \frac{t}{50} - 1}$, and the initial function $u(x,0) = u_0(x)$ which takes the constant value $u = 0.8$ around the origin and rapidly decays to 0 away from the origin. The exact solution is not available.

Figure 7.3 shows the numerical solutions with $\alpha = 2$, $\alpha = 1.8$, and $\alpha = \alpha(x,t)$ (which is close to 2) at $t = 32$, respectively. Figures 7.4, 7.5, and 7.6 show the

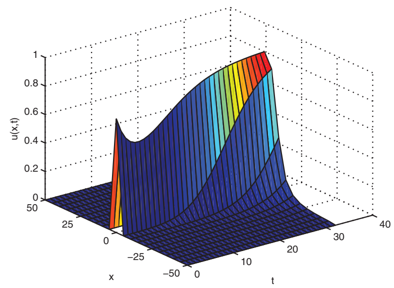

FIG. 7.4. *The approximation solution of* (7.4) *when* $\alpha(x,t) = 1.8$.

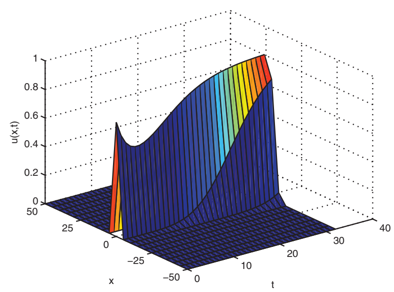

FIG. 7.5. *The approximation solution of* (7.4) *when* $\alpha(x,t) = 2.0$.

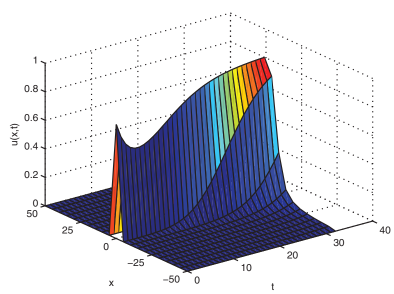

FIG. 7.6. *The approximation solution of* (7.4) *when* $\alpha = \alpha(x,t) = 1.7 + 0.5e^{-\frac{x^2}{1000} - \frac{t}{50} - 1}$.

solution behavior of (7.4) when $\alpha(x,t) = 1.8$, $\alpha(x,t) = 2.0$, and $\alpha = \alpha(x,t) = 1.7 + 0.5e^{-\frac{x^2}{1000} - \frac{t}{50} - 1}$, respectively.

From Figures 7.3, 7.4, 7.5, and 7.6, it can be seen that the solution with $\alpha = 2$ produces a rapidly decaying solution away from the origin. However, numerical solutions with $\alpha = 1.8$ and $\alpha = \alpha(x,t)$ have heavy tails, which are similar to the results reported in del Castillo-Negrete, Carreras, and Lynch [3] for the one-sided fractional reaction-diffusion equation.

8. Conclusion. In this paper, EEAs and IEAs for the variable-order fractional advection-diffusion equation with a nonlinear source term were described and demonstrated. We discussed the stability and convergence of both approximation schemes. The fractional MOL, the EM, and the MTT with the Riesz fractional derivative were also presented. The three methods provide a computationally efficient tool for simulating the behavior of solutions of the equation. In particular, the EM approximates the exact solution very well, and its convergence order is improved significantly. This type of fractional differential equation is able to describe heavy-tailed motions more accurately. These methods and techniques can be applied to solve variable-order fractional (in space and in time) partial differential equations.

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