Numerical methods for Volterra integral equations of the first kind

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This paper contains a study of numerical methods for solving linear Volterra integral equations of the first kind. A number of convergent approximation schemes are given, but it is found that certain other 'obvious' approaches yield unstable algorithms. Means for improving the results of the convergent methods are discussed.

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1. Introduction

The numerical solution of the Volterra integral equation of the first kind

$$\int_0^x K(x, t)y(t)dt = f(x) \tag{1.1}$$

may be accomplished by converting it, by differentiation, to an equation of the second kind,

$$y(x) + \int_0^x \frac{K^{10}(x, t)}{K(x, x)} y(t)\, dt = \frac{f'(x)}{K(x, x)}, \tag{1.2}$$

where we use the notation

$$K^{ij}(x, t) = \frac{\partial^i}{\partial \xi^i} \frac{\partial^j}{\partial \eta^j} K(\xi, \eta)|_{\xi = x, \eta = t}.$$

Since there are many acceptable methods for solving equations of the second kind, the problem, in principle, is solved. However, because of their simplicity, direct methods, that is, methods based on replacing the integral in (1.1) by a numerical quadrature, are of interest and are recommended by some authors (Collatz, 1960; Mikhlin and Smolitsky, 1967). Most commonly, the trapezoidal rule is used for this, and since it produces good approximations it is often conjectured that higher order quadratures lead to similar results. As we will show, this conjecture is erroneous. Jones (1961) has investigated the trapezoidal method for convolution kernels $K(x, t) = K(x - t)$, but apart from this very little theoretical work has appeared in the literature. The present paper contains a summary of some general results obtained by the author.

2. Assumptions and notation

In the succeeding analysis we shall make the following assumptions:

(*a*) $f(0) = 0$,
(*b*) $K(x, x) \neq 0$ for all x in the range of integration,
(*c*) $K(x, t)$ and $f(x)$ are bounded and sufficiently smooth so that all derivatives used in the succeeding analysis exist.

It is a standard result that under these conditions equation (1.1) has a unique and continuous solution. Furthermore, provided $K(x, t)$ and $f(x)$ are sufficiently smooth, the solution $y(x)$ will also be sufficiently smooth. This follows immediately from differentiating (1.2).

To solve (1.1) in an interval $[0, a]$ we divide it into smaller intervals of width h, the ith point of subdivision being denoted by x_i, such that $x_i = ih$, $i = 0, 1, 2, \ldots, N$ and $Nh = a$. The approximate solution will be defined at these meshpoints and denoted by Y_i.

Definition 1. Let $Y_0(h), Y_1(h), \ldots$ denote the approximation obtained by a given method using stepsize h. Then the method is said to be convergent if and only if

$$\max_{0 \leqslant i \leqslant N} |Y_i(h) - y(x_i)| \to 0$$

as $h \to 0$, $N \to \infty$, such that $Nh = a$.

Definition 2. A method is said to be of order p if p is the largest number for which there exists a finite constant C such that

$$\max_{0 \leqslant i \leqslant N} |Y_i(h) - y(x_i)| \leqslant Ch^p$$

for all $0 < h \leqslant a$.

We shall also need the following lemma.

Lemma 1. If

$$|\xi_n| \leqslant A \sum_{i=0}^{n-1} |\xi_i| + B, \quad \text{for } n = 1, 2, \ldots$$

with $$A > 0,\ B > 0,$$

then $$|\xi_n| \leqslant (B + A|\xi_0|)(1 + A)^{n-1}. \tag{2.1}$$

If $A = hK$ and $nh = x$, then

$$|\xi_n| \leqslant (B + hK|\xi_0|)e^{Kx}. \tag{2.2}$$

The proof of this lemma follows directly by induction.

Our main aim is the investigation of the convergence properties of some common algorithms.

3. The rectangular method

The simplest approximation is obtained by using the rectangular rule, where in each interval $x_i \leqslant t \leqslant x_{i+1}$ the kernel $K(x, t)$ is replaced by $K(x, x_i)$. Then

$$h \sum_{i=0}^{n-1} K(x_n, x_i) Y_i = f(x_n), \tag{3.1}$$

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or

$$Y_{n-1} = \frac{f(x_n)}{hK(x_n, x_{n-1})} - \sum_{i=0}^{n-2} \frac{K(x_n, x_i)}{K(x_n, x_{n-1})} Y_i,$$

$$\text{for } n = 1, 2, \ldots \quad (3.2)$$

Theorem 1. The approximation method (3.2) is convergent with order at least one.
Proof: Let $\epsilon_r = y(x_r) - Y_r$. Then, from (1.1) and (3.1),

$$h \sum_{i=0}^{n-1} K(x_n, x_i)\epsilon_i = h \sum_{i=0}^{n-1} K(x_n, x_i) y(x_i) - \int_0^{x_n} K(x_n, t) y(t) dt.$$

If we subtract this equation for n from the same equation for $n + 1$ we have

$$K(x_{n+1}, x_n)\epsilon_n + \sum_{i=0}^{n-1} \{K(x_{n+1}, x_i) - K(x_n, x_i)\}\epsilon_i$$
$$= K(x_{n+1}, x_n) y(x_n) - \frac{1}{h}\int_{x_n}^{x_{n+1}} K(x_{n+1}, t) y(t) dt$$
$$+ \sum_{i=0}^{n-1} \{K(x_{n+1}, x_i) - K(x_n, x_i)\} y(x_i)$$
$$- \frac{1}{h}\int_0^{x_n} \{K(x_{n+1}, t) - K(x_n, t)\} y(t) dt,$$
$$\text{for } n = 1, 2, \ldots \quad (3.3)$$

Provided everything is sufficiently smooth the right-hand side is clearly $O(h)$. Since $K(x, x) \neq 0$, for sufficiently small h, $K(x_{n+1}, x_n) \neq 0$. Thus, there exist constants M_1, M_2, M_3 such that

$$|\epsilon_n| \leqslant hM_1 \sum_{i=0}^{n-1} |\epsilon_i| + hM_2,$$

with $|\epsilon_0| \leqslant hM_3$. Then, according to (2.2),

$$|\epsilon_n| \leqslant (hM_2 + h^2 M_1 M_3) e^{M_1 x_n},$$

which proves the theorem.

4. The midpoint method

Here we replace $K(x, t)$ by its value at the centre point $t = x_{i+1/2} = x_i + h/2$. Then

$$h \sum_{i=0}^{n-1} K(x_n, x_{i+1/2}) Y_{i+1/2} = f(x_n), \quad (4.1)$$

$$Y_{n-1/2} = \frac{f(x_n)}{hK(x_n, x_{n-1/2})} - \sum_{i=0}^{n-2} \frac{K(x_n, x_{i+1/2})}{K(x_n, x_{n-1/2})} Y_{i+1/2}. \quad (4.2)$$

Theorem 2. The approximation (4.2) is convergent with order at least two.
Proof: Again, from (1.1) and (4.1),

$$h \sum_{i=0}^{n-1} K(x_n, x_{i+1/2})\epsilon_{i+1/2} = h \sum_{i=0}^{n-1} K(x_n, x_{i+1/2}) y(x_{i+1/2}) - \int_0^{x_n} K(x_n, t) y(t) dt$$

$$K(x_{n+1}, x_{n+1/2})\epsilon_{n+1/2} + \sum_{i=0}^{n-1} \{K(x_{n+1}, x_{i+1/2}) - K(x_n, x_{i+1/2})\}\epsilon_{i+1/2}$$
$$= K(x_{n+1}, x_{n+1/2}) y(x_{n+1/2}) - \frac{1}{h}\int_{x_n}^{x_{n+1}} K(x_{n+1}, t) y(t) dt$$
$$+ \sum_{i=0}^{n-1} \{K(x_{n+1}, x_{i+1/2}) - K(x_n, x_{i+1/2})\} y(x_{i+1/2})$$
$$- \frac{1}{h}\int_0^{x_n} \{K(x_{n+1}, t) - K(x_n, t)\} y(t) dt,$$
$$\text{for } n = 1, 2, \ldots$$

Since the midpoint method is a second order quadrature method the right-hand side is $O(h^2)$, and again we can find constants N_1, N_2, N_3, such that

$$|\epsilon_{n+1/2}| \leqslant hN_1 \sum_{i=0}^{n-1} |\epsilon_{i+1/2}| + h^2 N_2, \qquad n = 1, 2, \ldots$$

with $|\epsilon_{1/2}| \leqslant h^2 N_3$.

Thus

$$|\epsilon_{n+1/2}| \leqslant (h^2 N_2 + h^3 N_1 N_3) e^{N_1 x_n},$$

and the theorem is proven.

5. The trapezoidal method

Using the trapezoidal quadrature scheme we get

$$\tfrac{1}{2}K(x_n, x_0) Y_0 + \sum_{i=1}^{n-1} K(x_n, x_i) Y_i + \tfrac{1}{2}K(x_n, x_n) Y_n = \frac{f(x_n)}{h}, \quad (5.1)$$

$$Y_n = \frac{2f(x_n)}{hK(x_n, x_n)} - \frac{K(x_n, x_0)}{K(x_n, x_n)} Y_0 - 2\sum_{i=1}^{n-1} \frac{K(x_n, x_i)}{K(x_n, x_n)} Y_i, \quad \text{for } n = 1, 2, \ldots . \quad (5.2)$$

The first value Y_0 can be obtained exactly since

$$y(0) = \frac{f'(0)}{K(0, 0)}.$$

Theorem 3. The approximation (5.2) is convergent with order at least two.
Proof: The proof of this theorem can be carried out in a fashion similar to the previous two cases, but it is more tedious and hence will be omitted here. The proof for convolution kernels was given by Jones (1961) and a detailed proof for the general case may be found in Linz (1967).

6. Higher order approximations

In general, one might attempt to use a qth order quadrature formula with weights w_{ri}. The corresponding error equation is

$$\epsilon_r = -\sum_{i=0}^{r-1} \frac{w_{ri} K(x_r, x_i)}{w_{rr} K(x_r, x_r)} \epsilon_i + O(h^{q-1}). \quad (6.1)$$

It is not at all obvious that $\epsilon_r \to 0$ as $h \to 0$; indeed it is frequently untrue. For example, take the third order Gregory formula

$$w_{r0} = w_{rr} = 3/8,$$
$$w_{r1} = w_{r,\, r-1} = 7/6,$$
$$w_{r2} = w_{r,\, r-2} = 23/24,$$
$$w_{ri} = 1 \quad \text{otherwise.}$$

For simplicity, take $K(x, t) = 1$. Then (6.1) yields

$$\epsilon_r - \epsilon_{r-1} = -\tfrac{28}{9}\epsilon_{r-1} + \tfrac{5}{9}\epsilon_{r-2} - \tfrac{1}{9}\epsilon_{r-3} + O(h^q).$$

The homogeneous part of this equation has solution

$$\epsilon_r^H = c_1 z_1^r + c_2 z_2^r + c_3 z_3^r,$$

where z_1, z_2, z_3 are the solutions of

$$x^3 + \tfrac{19}{9}x^2 - \tfrac{5}{9}x + \tfrac{1}{9} = 0.$$

This equation has a root near $x = -2 \cdot 5$, thus

$$|\epsilon_r| \to \infty.$$

This indicates that the method diverges, and numerical results bear out this prediction. Similar results hold for Simpson's rule and higher order Gregory formulae. In fact, we have not found any convergent methods based on higher order quadrature methods. While it is possible that such methods do exist, it is clear that their convergence properties have to be examined carefully. This problem does not seem to have been recognised previously, and one finds the use of higher order methods suggested in some text-books. Hopkins and Hamming (1957) encountered some difficulties with equations of the first kind, but did not give any details.

7. Error estimation formulae

We can get a better idea of the behaviour of the numerical solution by obtaining actual error estimates. Since the manipulations required to derive these estimates are somewhat tedious and not very instructive we shall sketch the derivation only for the simplest case, the rectangular method. We will give the final results for the other two methods; the derivations can be found in Linz (1967).

Rectangular method

From (3.3), replacing differences by derivatives, and remembering that $\epsilon_i = O(h)$, we get

$$K(x_n, x_n)\epsilon_n + h\sum_{i=0}^{n-1} K^{10}(x_n, x_i)\epsilon_i$$
$$= -\frac{h}{2}\frac{\partial}{\partial \xi}[K(x_n, \xi)y(\xi)]_{\xi = x_n}$$
$$-\frac{h^2}{2}\sum_{i=0}^{n-1}\frac{\partial}{\partial \xi}[K^{10}(x_n, \xi)y(\xi)]_{\xi = x_i} + O(h^2).$$

Introducing the scaled error $e_n = \dfrac{1}{h}\epsilon_n$, this becomes

$$e_n + h\sum_{i=0}^{n-1}\frac{K^{10}(x_n, x_i)}{K(x_n, x_n)} e_i$$
$$= -\frac{1}{2K(x_n, x_n)}\{K^{01}(x_n, x_n)y(x_n) + K(x_n, x_n)y'(x_n)$$
$$+ h\sum_{i=0}^{n-1}\frac{\partial}{\partial \xi}[K^{10}(x_n, \xi)y(\xi)]_{\xi = x_i}\} + O(h). \quad (7.1)$$

Let $e(x)$ be the solution of

$$e(x) = -\int_0^x \frac{K^{10}(x, t)}{K(x, x)} e(t)dt$$
$$-\frac{1}{2K(x, x)}\{K^{01}(x, x)y(x) + K(x, x)y'(x)$$
$$+ \int_0^x \frac{\partial}{\partial \xi}[K^{10}(x, \xi)y(\xi)]d\xi\}, \quad (7.2)$$

then the rectangular approximation to (7.2) differs from (7.1) only by terms of order h.

Thus

$$\epsilon_n = he(x_n) + O(h^2).$$

This also shows that the rectangular method is exactly of order one.

Midpoint method

If $e(x)$ is the solution of

$$e(x) = -\int_0^x \frac{K^{10}(x, t)}{K(x, x)} e(t)dt - \frac{1}{24K(x, x)}\{K^{02}(x,x)y(x)$$
$$+ 2K^{01}(x, x)y'(x) + K(x, x)y''(x)$$
$$+ K^{11}(x, x)y(x) + K^{10}(x, x)y'(x)$$
$$- K^{11}(x, 0)y(0) - K^{10}(x, 0)y'(0)\}, \quad (7.3)$$

then

$$\epsilon_{n+1/2} = h^2 e(x_{n+1/2}) + O(h^3). \quad (7.4)$$

Trapezoidal method

In this case we find that the results can no longer be represented in the above form. Indeed, it is obvious from numerical results that the approximation shows some small oscillations about the true solution. This phenomenon was pointed out by Jones, who also suggested a method for smoothing the results.

If we introduce the 'averaged' error

$$\eta_r = \tfrac{1}{2}(\epsilon_{r+1} + \epsilon_r),$$

then it can be shown that

$$\eta_r = h^2\eta(x_r) + O(h^3),$$

where $\eta(x)$ is the solution of

$$\eta(x) = -\int_0^x \frac{K^{10}(x, t)}{K(x, x)}\eta(t)dt + \frac{1}{12K(x, x)}\{K^{02}(x,x)y(x)$$
$$+ 2K^{01}(x, x)y'(x) + K(x, x)y''(x)$$
$$+ K^{11}(x, x)y(x) + K^{10}(x, x)y'(x)$$
$$- K^{11}(x, 0)y(0) - K^{10}(x, 0)y'(0)\}. \quad (7.6)$$

This result provides a rigorous justification for Jones' smoothing method which is as follows. Given the initial solution $Y_0, Y_1, \ldots$ one forms

$$\bar{Y}_k = \tfrac{1}{2}(Y_{k-1} + Y_{k+1})$$

and

$$\tilde{Y}_k = \tfrac{1}{2}(\bar{Y}_k + Y_k).$$

It is easy to show that if $\tilde{\epsilon}_k$ denotes the error in $\tilde{Y}_k$, then

$$\tilde{\epsilon}_k = \eta_k + \frac{h^2}{4}y''(x_k) + O(h^3),$$

that is, the ${}_k\tilde{Y}$ are smooth to order h^2.

8. Improving the accuracy of the approximations

The error estimates of the previous section provide a justification for the use of extrapolation methods for improving the accuracy. For instance, using Richardson's extrapolation with the mid-point method, we compute solutions using step-size h and $3h$ (to make meshpoints coincide). Then

$$Y_n(h) = y(x_n) + h^2 e(x_n) + O(h^3),$$
$$Y_n(3h) = y(x_n) + 9h^2 e(x_n) + O(h^3),$$

and an improved solution Y_n^c is given by

$$Y_n^c = \tfrac{1}{8}(9Y_n(h) - Y_n(3h)),$$

where $$Y_n^c = y(x_n) + O(h^3).$$

Similar results can be obtained for the trapezoidal method, but only after the results have been smoothed. Alternatively, the method of deferred corrections may be used with similar results.

9. Numerical examples

In this section we give some numerical examples to illustrate the above points. In all cases the numerical results show the predicted behaviour.

Example 1. (from Jones, 1961)

$$\int_0^x \cos(x-t)y(t)dt = \sin x.$$

Exact solution: $y(x) = 1$.

Table 1 contains the results by the midpoint method. The error, somewhat surprisingly, stays constant, but this is predicted by the error equation (7.3). From (7.3) we find

$$\epsilon_k = \frac{h^2}{24} + O(h^3).$$

Table 1

Example 1 by the midpoint method

x	$h = 0\cdot1$	x	$h = 0\cdot05$
0·45	0·99958	0·475	0·99990
0·95	0·99958	0·975	0·99990
1·45	0·99958	1·475	0·99990
1·95	0·99958	1·975	0·99990

Table 2 contains the results by the trapezoidal method. The oscillations and the effect of the smoothing method are apparent.

Example 2.

$$\int_0^x \cos(x-t)y(t)dt = 1 - \cos x.$$

Exact solution: $y(x) = x$.

The results, using the midpoint method with Richardson's extrapolation, are given in **Table 3**.

Table 2

Example 1 by the trapezoidal method with $h = 0\cdot1$

x	INITIAL APPROX.	SMOOTHED RESULTS
0·1	1·00166	1·00083
0·2	1·00000	1·00083
0·3	1·00166	1·00083
0·4	1·00000	1·00083
0·5	1·00166	1·00083
0·6	1·00000	1·00083

Table 3

Example 2 by the midpoint method with Richardson's extrapolation

x	$h = 0\cdot3$	$h = 0\cdot1$	EXTRAP.
0·15	0·15057	0·15006	0·14999
0·45	0·45171	0·45019	0·45000
0·75	0·75285	0·75031	0·74999
1·05	1·05398	1·05044	1·05000
1·35	1·35512	1·35056	1·34999
1·65	1·65626	1·65069	1·64999
1·95	1·95740	1·95081	1·94999

10. Conclusions

We have shown that approximations to Volterra integral equations of the first kind can be obtained by using certain simple numerical quadrature rules, but that many of the higher order quadrature methods lead to unstable algorithms. From a computational point of view we have found the midpoint method, combined with Richardson's extrapolation, quite satisfactory. It is generally reasonably accurate and also provides some estimate of the error. The trapezoidal rule, while frequently proposed in the literature, is somewhat less satisfactory because of the oscillations in the results. Even after smoothing it tends to be less accurate than the midpoint method. This may be seen by comparing equations (7.3) and (7.6): the nonhomogeneous term in (7.6) is exactly twice as large as the corresponding term in (7.3).

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