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(continued after index)

Dale R. Durran

Numerical Methods for Wave Equations in Geophysical Fluid Dynamics

With 93 Illustrations

Springer

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To every hand that's touched the Wall

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM).*

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

This book is designed to serve as a textbook for graduate students or advanced undergraduates studying numerical methods for the solution of partial differential equations governing wave-like flows. Although the majority of the schemes presented in this text were introduced in either the applied-mathematics or atmospheric-science literature, the focus is not on the nuts-and-bolts details of various atmospheric models but on fundamental numerical methods that have applications in a wide range of scientific and engineering disciplines. The prototype problems considered include tracer transport, shallow-water flow and the evolution of internal waves in a continuously stratified fluid.

A significant fraction of the literature on numerical methods for these problems falls into one of two categories, those books and papers that emphasize theorems and proofs, and those that emphasize numerical experimentation. Given the uncertainty associated with the messy compromises actually required to construct numerical approximations to real-world fluid-dynamics problems, it is difficult to emphasize theorems and proofs without limiting the analysis to classical numerical schemes whose practical application may be rather limited. On the other hand, if one relies primarily on numerical experimentation it is much harder to arrive at conclusions that extend beyond a specific set of test cases. In an attempt to establish a clear link between theory and practice, I have tried to follow a middle course between the theorem-and-proof formalism and the reliance on numerical experimentation. There are no formal proofs in this book, but the mathematical properties of each method are derived in a style familiar to physical scientists. At the same time, numerical examples are included that illustrate these theoretically derived properties and facilitate the intercomparison of various methods.

A general course on numerical methods for geophysical fluid dynamics might draw on portions of the material presented in Chapters 2 through 6. Chapter 2 describes the largely classical theory of finite-difference approximations to the one-way wave equation (or alternatively the constant-wind-speed advection equation). The extension of these results to systems of equations, several space dimensions, dissipative flows and nonlinear problems is discussed in Chapter 3. Chapter 4 introduces series-expansion methods with emphasis on the Fourier and spherical-harmonic spectral methods and the finite-element method. Finite-volume methods are discussed in Chapter 5 with particular attention devoted to methods for simulating the transport of scalar fields containing poorly resolved spatial gradients. Semi-Lagrangian schemes are analyzed in Chapter 6. Both theoretical and applied problems are provided at the end of each chapter. Those problems that require numerical computation are marked by an asterisk.

In addition to the core material in Chapters 2 through 6, the introduction in Chapter 1 discusses the relation between the equations governing wave-like geophysical flows and other types of partial differential equations. Chapter 1 concludes with a short overview of the strategies for numerical approximation that are considered in detail throughout the remainder of the book. Chapter 7 examines schemes for the approximation of slow moving waves in fluids that support physically insignificant fast waves. The emphasis in Chapter 7 is on atmospheric applications in which the slow wave is either an internal gravity wave and the fast waves are sound waves, or the slow wave is a Rossby wave and the fast waves are both gravity waves and sound waves. Chapter 8 examines the formulation of wave-permeable boundary conditions for limited-area models with emphasis on the shallow-water equations in one and two dimensions and on internally stratified flow.

Many numerical methods for the simulation of internally stratified flow require the repeated solution of elliptic equations for pressure or some closely related variable. Due to the limitations of my own expertise and to the availability of other excellent references I have not discussed the solution of elliptic partial differential equations in any detail. A thumbnail sketch of some solution strategies is provided in Section 7.1.3; the reader is referred to Chapter 5 of Ferziger and Perić (1997) for an excellent overview of methods for the solution of elliptic equations arising in computational fluid dynamics.

I have attempted to provide sufficient references to allow the reader to further explore the theory and applications of many of the methods discussed in the text, but the reference list is far from encyclopedic and certainly does not include every worthy paper in the atmospheric science or applied mathematics literature. References to the relevant literature in other disciplines and in foreign language journals is rather less complete.[1]

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[1] Those not familiar with the atmospheric science literature may be surprised by the number of references to *Monthly Weather Review*, which despite its title, has become the primary American journal for the publication of papers on numerical methods in atmospheric science.

This book would not have been written without the generous assistance of several colleagues. Christopher Bretherton, in particular, provided many perceptive answers to my endless questions. J. Ray Bates, Byron Boville, Michael Cullen, Marcus Grote, Robert Higdon, Randall LeVeque, Christoph Schär, William Skamarock, Piotr Smolarkiewicz, and David Williamson all provided very useful comments on individual chapters. Many students used earlier versions of this manuscript in my courses in the Atmospheric Sciences Department at the University of Washington, and their feedback helped improve the clarity of the manuscript. Two students to whom I am particularly indebted are Craig Epifanio and Donald Slinn. I am also grateful to James Holton for encouraging me to undertake this project.

It is my pleasure to acknowledge the many years of support for my numerical modeling efforts provided by the Mesoscale Dynamic Meteorology Program of the National Science Foundation. Additional support for my atmospheric simulation studies has been provided by the Coastal Meteorology ARI of the Office of Naval Research. Part of this book was completed while I was on sabbatical at the Laboratoire d'Aérologie of the Université Paul Sabatier in Toulouse, France, and I thank Daniel Guedalia and Evelyne Richard for helping make that year productive and scientifically stimulating.

As errors in the text are identified, they will be posted on the web at http://www. atmos.washington.edu/methods.for.waves, which can be accessed directly or via Springer's home page at http://www.springer-ny.com. I would be most grateful to be advised of any typographical or other errors by electronic mail at dale.durran@atmos.washington.edu.

Seattle, Washington DALE R. DURRAN

Cover art: The three curves plot solutions to the linearized Rossby–adjustment problem. The governing equations and physical parameters for this problem are identical to those given in Problem 12 of Chapter 3, except that the spatial domain is -400 km $\leq x \leq 400$ km with open lateral boundaries, and the initial condition for the free-surface displacement is $h(x, t = 0) = \arctan(x/20 \text{ km})$. The curves shown are plots of $u(x, t = 943 \text{ s})$, $u(x, t = 1222 \text{ s})$, and $u(x, t = 1501 \text{ s})$ on an artisically cropped portion of the sub-domain $x > 0$.

Contents

Series Preface **vii**

Preface **ix**

1 Introduction **1**

1.1 Partial Differential Equations—Some Basics 2

1.1.1 First-Order Hyperbolic Equations 4

1.1.2 Linear Second-Order Equations in Two Independent Variables 7

1.2 Wave Equations in Geophysical Fluid Dynamics 11

1.2.1 Hyperbolic Equations 12

1.2.2 Filtered Equations 20

1.3 Strategies for Numerical Approximation 26

1.3.1 Approximating Calculus with Algebra 26

1.3.2 Marching Schemes 29

Problems 33

2 Basic Finite-Difference Methods **35**

2.1 Accuracy and Consistency 36

2.2 Stability and Convergence 39

2.2.1 The Energy Method 41

2.2.2 Von Neumann's Method 43

2.2.3 The Courant–Fredrichs–Lewy Condition 45

2.3 Time-Differencing 47

2.3.1 The Oscillation Equation: Phase-Speed and Amplitude Error 48

2.3.2 Single-Stage Two-Level Schemes 50
2.3.3 Multistage Methods 53
2.3.4 Three-Level Schemes 56
2.3.5 Controlling the Leapfrog Computational Mode 60
2.3.6 Higher-Order Schemes 65
2.4 Space-Differencing 72
2.4.1 Differential–Difference Equations and Wave Dispersion 73
2.4.2 Dissipation, Dispersion, and the Modified Equation . . . 80
2.4.3 Artificial Dissipation 82
2.4.4 Compact Differencing 86
2.5 Combined Time- and Space-Differencing 89
2.5.1 The Discrete-Dispersion Relation 91
2.5.2 The Modified Equation 94
2.5.3 The Lax–Wendroff Method 95
2.6 Summary Discussion of Elementary Methods 99
Problems 101

3 Beyond the One-Way Wave Equation **107**
3.1 Systems of Equations 107
3.1.1 Stability 108
3.1.2 Staggered Meshes 113
3.2 Three or More Independent Variables 117
3.2.1 Scalar Advection in Two Dimensions 117
3.2.2 Systems of Equations in Several Dimensions 126
3.3 Splitting into Fractional Steps 129
3.3.1 Split Explicit Schemes 130
3.3.2 Split Implicit Schemes 132
3.3.3 Stability of Split Schemes 134
3.4 Diffusion, Sources, and Sinks 136
3.4.1 Pure Diffusion 136
3.4.2 Advection and Diffusion 138
3.4.3 Advection with Sources and Sinks 144
3.5 Linear Equations with Variable Coefficients 147
3.5.1 Aliasing Error 148
3.5.2 Conservation and Stability 154
3.6 Nonlinear Instability 159
3.6.1 Burgers's Equation 159
3.6.2 The Barotropic Vorticity Equation 163
Problems 167

4 Series-Expansion Methods **173**
4.1 Strategies for Minimizing the Residual 173
4.2 The Spectral Method 176
4.2.1 Comparison with Finite-Difference Methods 177
4.2.2 Improving Efficiency Using the Transform Method . . . 184
4.2.3 Conservation and the Galerkin Approximation 189
4.3 The Pseudospectral Method 191
4.4 Spherical Harmonics 195
4.4.1 Truncating the Expansion 197
4.4.2 Elimination of the Pole Problem 200
4.4.3 Gaussian Quadrature and the Transform Method 202
4.4.4 Nonlinear Shallow-Water Equations 207
4.5 The Finite-Element Method 212
4.5.1 Galerkin Approximation with Chapeau Functions 214
4.5.2 Petrov–Galerkin and Taylor–Galerkin Methods 216
4.5.3 Quadratic Expansion Functions 219
4.5.4 Hermite-Cubic Expansion Functions 226
4.5.5 Two-Dimensional Expansion Functions 231
Problems 234

5 Finite Volume Methods **241**
5.1 Conservation Laws and Weak Solutions 243
5.1.1 The Riemann Problem 244
5.1.2 Entropy-Consistent Solutions 246
5.2 Finite-Volume Methods and Convergence 249
5.2.1 Monotone Schemes 251
5.2.2 TVD Methods 252
5.3 Discontinuities in Geophysical Fluid Dynamics 254
5.4 Flux-Corrected Transport 257
5.4.1 Flux Correction: The Original Proposal 259
5.4.2 The Zalesak Corrector 260
5.4.3 Iterative Flux Correction 263
5.5 Flux-Limiter Methods 263
5.5.1 Ensuring That the Scheme Is TVD 264
5.5.2 Possible Flux Limiters 267
5.5.3 Flow Velocities of Arbitrary Sign 271
5.6 Approximation with Local Polynomials 272
5.6.1 Godunov's Method 272
5.6.2 Piecewise-Linear Functions 274
5.7 Two Spatial Dimensions 277
5.7.1 FCT in Two Dimensions 277
5.7.2 Flux-Limiter Methods for Uniform 2-D Flow 279
5.7.3 Nonuniform Nondivergent Flow 282
5.7.4 A Numerical Example 284
5.7.5 When Is a Flux Limiter Necessary? 291

5.8 Schemes for Positive Definite Advection 292
5.8.1 An FCT Approach 293
5.8.2 Antidiffusion via Upstream Differencing 294
5.9 Curvilinear Coordinates 296
Problems 297

6 Semi-Lagrangian Methods 303
6.1 The Scalar Advection Equation 305
6.1.1 Constant Velocity 305
6.1.2 Variable Velocity 310
6.2 Forcing in the Lagrangian Frame 313
6.3 Systems of Equations 318
6.3.1 Comparison with the Method of Characteristics 318
6.3.2 Semi-implicit Semi-Lagrangian Schemes 320
6.4 Alternative Trajectories 324
6.4.1 A Noninterpolating Leapfrog Scheme 325
6.4.2 Interpolation via Parametrized Advection 327
6.5 Eulerian or Semi-Lagrangian? 330
Problems 331

7 Physically Insignificant Fast Waves 335
7.1 The Projection Method 336
7.1.1 Forward-in-Time Implementation 337
7.1.2 Leapfrog Implementation 339
7.1.3 Solving the Poisson Equation for Pressure 340
7.2 The Semi-implicit Method 342
7.2.1 Large Time Steps and Poor Accuracy 343
7.2.2 A Prototype Problem 345
7.2.3 Semi-implicit Solution of the Shallow-Water Equations . 347
7.2.4 Semi-implicit Solution of the Euler Equations 350
7.2.5 Numerical Implementation 356
7.3 Fractional-Step Methods 359
7.3.1 Complete Operator Splitting 359
7.3.2 Partially Split Operators 365
7.4 Summary of Schemes for Nonhydrostatic Models 371
7.5 The Hydrostatic Approximation 372
7.6 Primitive Equation Models 374
7.6.1 Pressure and σ Coordinates 375
7.6.2 Spectral Representation of the Horizontal Structure . . . 379
7.6.3 Vertical Differencing 381
7.6.4 Energy Conservation 383
7.6.5 Semi-implicit Time-Differencing 387
Problems 389

8 Nonreflecting Boundary Conditions **395**
8.1 One-Dimensional Flow 397
8.1.1 Well-Posed Initial–Boundary Value Problems. 397
8.1.2 The Radiation Condition 400
8.1.3 Time-Dependent Boundary Data 401
8.1.4 Reflections at an Artificial Boundary—
The Continuous Case 402
8.1.5 Reflections at an Artificial Boundary—
The Discretized Case 403
8.1.6 Stability in the Presence of Boundaries 409
8.2 Two-Dimensional Shallow-Water Flow 412
8.2.1 One-Way Wave Equations 414
8.2.2 Numerical Implementation 419
8.3 Two-Dimensional Stratified Flow 419
8.3.1 Lateral Boundary Conditions 420
8.3.2 Upper Boundary Conditions 424
8.3.3 Numerical Implementation of the Radiation Upper
Boundary Condition 429
8.4 Wave-Absorbing Layers 431
8.5 Summary 436
Problems 437

Appendix Numerical Miscellany **439**
A.1 Finite-Difference Operator Notation 439
A.2 Tridiagonal Solvers 440
A.2.1 Code for a Tridiagonal Solver 440
A.2.2 Code for a Periodic Tridiagonal Solver 441

Bibliography **443**

Index **457**