

# Numerical Oscillations of Runge-Kutta Methods for Differential Equations with Piecewise Constant Arguments of Alternately Advanced and Retarded Type

Qi Wang

Faculty of Applied Mathematics, Guangdong University of Technology, Guangzhou 510006, China  
Email: bmwzqwq@126.com

FengLian Fu

Faculty of Environmental Science and Engineering, Guangdong University of Technology, Guangzhou 510006, China  
Email: fufenglian2006@163.com

**Abstract**—The purpose of this paper is to study the numerical oscillations of Runge-Kutta methods for the solution of alternately advanced and retarded differential equations with piecewise constant arguments. The conditions of oscillations for the Runge-Kutta methods are obtained. It is proven that the Runge-Kutta methods preserve the oscillations of the analytic solution. In addition, the relationship between stability and oscillations are shown. Some numerical examples are given to confirm the theoretical results.

**Index Terms**—Runge-Kutta methods, numerical solution, piecewise constant arguments, oscillation

## I. INTRODUCTION

The theory of the oscillation is one of the most interesting topics for applications. In recent years, the oscillations of difference equations [1,2], dynamic equations [3,4] and delay differential equations [5-8] have been studied and developed by many authors. Among these investigations, oscillations of solutions of differential equations with piecewise constant arguments (EPCA) have also been the subject of many recent investigations [9-12].

It is well known that studies of EPCA were motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both the differential and difference equations. These equations play an important role in numerous applications [13,14]. And the theory of EPCA was developed intensively in the last few decades, such as [15-19]. For a brief summary of the theory, the reader is referred to the book by Wiener [20].

Recently, some important results on the properties of the numerical solutions of EPCA have been obtained, such as the stability [21-24] and the oscillations [25,26]. In [25,26], the authors considered the oscillations of numerical solutions in  $\theta$ -methods and Runge-Kutta methods for the same equation, respectively. Our paper contains an attempt to enrich the gap by considering oscillations of the Runge-Kutta methods for a more

complicated equation and discussing the relationship between stability and oscillations.

In the present work, we shall consider the equation

$$\begin{cases} x'(t) = ax(t) + bx([t + 0.5]), & t \geq 0 \\ x(0) = x_0, \end{cases} \quad (1)$$

where  $a, b, x_0$  are real constants and  $[.]$  denotes the greatest integer function. Since the argument deviation of (1) is positive in  $[n, n + 0.5)$  and negative in  $[n + 0.5, n + 1)$ , (1) is said to be of alternately advanced and retarded type. We aim to investigate the oscillations of the numerical solutions in the Runge-Kutta methods for (1), and get some relationships between stability and oscillations.

The paper is organized in the following manner. In the next section, we give known definitions and results that will be needed further. Section 3 discusses the oscillations and non-oscillations of the numerical solution. We will prove that the oscillations of the analytic solution are preserved by the Runge-Kutta methods. In Section 4, we will obtain the relationship between stability and oscillations. Appropriate numerical examples are given to illustrate our results in Section 5.

## II. PRELIMINARIES

In this section, we shall introduce some definitions and theorems.

**Definition 1**[20] A solution of (1) on  $[0, \infty)$  is a function  $x(t)$  satisfies the conditions:

- I  $x(t)$  is continuous on  $[0, \infty)$  ;
- I The derivative  $x'(t)$  exists at each point  $t \in [0, \infty)$ , with the possible exception of the points  $t = n + 0.5, n = 0, 1, 2, \dots$ , where one-sided derivatives exist,
- I Eq. (1) is satisfied on  $[0, 0.5)$  and each interval  $[n - 0.5, n + 0.5)$  for  $n = 1, 2, \dots$ .

The solution of (1) is given in the following theorem.

**Theorem 1**[20] If

$$b \neq \frac{a}{e^{a/2} - 1},$$

then (1) has on  $[0, \infty)$  a unique solution

$$x(t) = \omega(\tau(t))\lambda^{[t+0.5]}x_0, \quad (2)$$

where

$$\omega(t) = e^{at} + (e^{at} - 1)a^{-1}b,$$

$$\tau(t) = t - [t + 0.5],$$

$$\lambda = \frac{\omega(1/2)}{\omega(-1/2)}.$$

**Definition 2** A non-trivial solution of (1) is said to be oscillatory if there exists a sequence  $\{t_k\}_{k=1}^\infty$  such that  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$  and  $x(t_k)x(t_{k-1}) \leq 0$ ; otherwise it is called non-oscillatory. We say (1) is oscillatory if all the non-trivial solutions of (1) are oscillatory; we say (1) is non-oscillatory if all the non-trivial solutions of (1) are non-oscillatory.

In [20,27], the authors establish the following oscillation results.

**Theorem 2** If any of the conditions

$$b > \frac{a}{e^{a/2} - 1}$$

and

$$b < -\frac{ae^{a/2}}{e^{a/2} - 1}$$

holds true, then (1) is oscillatory. Eq. (1) is non-oscillatory if and only if condition

$$-\frac{ae^{a/2}}{e^{a/2} - 1} < b < \frac{a}{e^{a/2} - 1}$$

is satisfied.

From Definition 2, we can see that if a solution  $x(t)$  of (1) is non-oscillatory and continuous, then it must be eventually positive or negative. That is, there exists a  $T \in \mathbb{R}$  such that  $x(t) > 0$  or  $x(t) < 0$  for  $t \geq T$ .

### III. OSCILLATION ANALYSIS

#### A. Runge-Kutta methods

Similar to [24], let  $h = 1/(2m)$ , the  $\nu$ -stage Runge-Kutta methods  $(A, B, C)$  with  $A = (a_{ij})_{\nu \times \nu}$ ,  $B = (b_1, b_2, \dots, b_\nu)^T$  and  $C = (c_1, c_2, \dots, c_\nu)^T$  applied to (1) yield the recurrence relation

$$x_{2km+l+1} = S(x)x_{2km+l} + \frac{b}{a}(S(x)-1)x_{2km}, \quad (3)$$

where  $x = ha$  and  $S(x) = 1 + xB^T(I - xA)^{-1}e$  is the stability function of the method. From [24], we know that the Runge-Kutta methods preserve the original order for (1).

We assume that  $S(x) \neq 0$  and

$$b \neq \frac{a}{S(x)^m - 1}.$$

Then it follows from (3) that

$$x_l = Q(l)x_0, \quad l \in M(0),$$

$$x_{2km} = \rho x_{2(k-1)m}, \quad k \geq 1, \quad (4)$$

$$x_{2km+l} = Q(l)x_{2km}, \quad l \in M(k), \quad (5)$$

where

$$Q(l) = S(x)^l + \frac{b}{a}(S(x)^l - 1), \quad -m \leq l \leq m,$$

$$M(k) = \{0, 1, \dots, m-1\}, \text{ for } k = 0,$$

$$M(k) = \{-m, -m+1, \dots, m-2, m-1\}, \text{ for } k \geq 1.$$

and

$$\rho = \frac{Q(m)}{Q(-m)}.$$

#### B. Numerical oscillations and non-oscillations

For any given Runge-Kutta methods, we assume that  $\delta_1 < 0 < \delta_2$  such that

$$0 < S(x) < 1, \text{ for } \delta_1 < x < 0,$$

and

$$1 < S(x) < \infty, \text{ for } 0 < x < \delta_2,$$

which implies

$$0 < \frac{S(x)-1}{x} < \infty, \text{ for } \delta_1 < x < \delta_2.$$

For the sake of simplicity, we omit the definition of oscillations of (5) (see [26]). The following theorem gives the relationship of the non-oscillations between  $\{x_n\}$  and  $\{x_{2km}\}$ .

**Theorem 3**  $\{x_n\}$  and  $\{x_{2km}\}$  are given by (5) and (4), respectively, then  $\{x_n\}$  is non-oscillatory if and only if  $\{x_{2km}\}$  is non-oscillatory.

**Proof:** The necessity is obvious. In the following, we prove the sufficiency. If  $\{x_{2km}\}$  is non-oscillatory, without loss of generality, we assume that  $\{x_{2km}\}$  is an eventually negative solution of (4), that is, there exists a  $k_0 \in \mathbb{R}$  such that  $x_{2km} < 0$  for  $k > k_0$ . We will prove  $x_{2km+l} < 0$  for all  $k > k_0 + 1$  and  $l \in M(k)$ . Suppose  $b < 0$ , if  $a < 0$ , then  $0 < S(x) < 1$  and  $S(x)^m \leq S(x)^l$ , hence

$$x_{2km+l} = \left( S(x)^l + \frac{b}{a}(S(x)^l - 1) \right) x_{2km}$$

$$\begin{aligned} &\leq \left( S(x)^m + \frac{b}{a} (S(x)^m - 1) \right) x_{2km} \\ &= x_{2km+m} < 0. \end{aligned}$$

If  $a > 0$ , then  $1 < S(x) < \infty$  and  $S(x)^{-m} \leq S(x)^{-l}$ , so

$$\begin{aligned} S(x)^{-l} x_{2km+l} &= \left( 1 + \frac{b}{a} (1 - S(x)^{-l}) \right) x_{2km} \\ &\leq \left( 1 + \frac{b}{a} (1 - S(x)^{-m}) \right) x_{2km} \\ &= S(x)^{-m} x_{2km+m} < 0. \end{aligned}$$

Therefore  $x_{2km+l} < 0$ . The proof is complete.

By Theorem 3, we get the following corollary.

**Corollary 1**  $\{x_n\}$  and  $\{x_{2km}\}$  are given by (5) and (4), respectively, then  $\{x_n\}$  is oscillatory if and only if  $\{x_{2km}\}$  is oscillatory.

Set

$$\begin{aligned} \Delta_1 &= \frac{a}{e^{a/2} - 1}, & \Delta_1(m) &= \frac{a}{S(x)^m - 1}, \\ \Delta_2 &= -\frac{ae^{a/2}}{e^{a/2} - 1}, & \Delta_2(m) &= -\frac{aS(x)^m}{S(x)^m - 1}, \end{aligned}$$

then we have

**Lemma 1**

- (i)  $\Delta_1 < \Delta_1(m)$  and  $\Delta_2 > \Delta_2(m)$  if  $e^x > S(x)$  for  $a > 0$  or  $e^x < S(x)$  for  $a < 0$ ;
- (ii)  $\Delta_1 \geq \Delta_1(m)$  or  $\Delta_2 \leq \Delta_2(m)$  if  $e^x \leq S(x)$  for  $a > 0$  or  $e^x \geq S(x)$  for  $a < 0$ ;
- (iii)  $\Delta_1(m) \rightarrow \Delta_1$  as  $h \rightarrow 0$  and  $\Delta_2(m) \rightarrow \Delta_2$  as  $h \rightarrow 0$ .

Proof: (i) If  $a > 0$  and  $e^x > S(x)$ , then

$$e^{a/2} > S(x)^m,$$

which is equivalent to

$$\frac{1}{e^{a/2} - 1} < \frac{1}{S(x)^m - 1}, \tag{6}$$

so we have  $\Delta_1 < \Delta_1(m)$ .

By (6), we also have

$$1 + \frac{1}{e^{a/2} - 1} < 1 + \frac{1}{S(x)^m - 1},$$

that is

$$\frac{e^{a/2}}{e^{a/2} - 1} < \frac{S(x)^m}{S(x)^m - 1},$$

which is equivalent to

$$-\frac{ae^{a/2}}{e^{a/2} - 1} > -\frac{aS(x)^m}{S(x)^m - 1},$$

hence  $\Delta_2 > \Delta_2(m)$ . With similar process, the other cases can be proved. This completes the proof.

**Theorem 4** Eq. (4) is oscillatory if and only if  $b > \Delta_1(m)$  or  $b < \Delta_2(m)$ ; (4) is non-oscillatory if and only if  $\Delta_2(m) < b < \Delta_1(m)$ .

Proof: We can get this result from the fact that (4) is oscillatory if and only if the corresponding characteristic equation has no positive roots.

**Definition 3**[26] We say the Runge-Kutta methods preserve the oscillations of (1) if (1) oscillates then there is a  $h_0 > 0$  such that (5) oscillates for  $h < h_0$ . Similarly, we say the Runge-Kutta methods preserve the non-oscillations of (1) if (1) non-oscillates then there is a  $h_0 > 0$  such that (5) non-oscillates for  $h < h_0$ .

Combining Theorems 2, 3, 4 and Corollary 1, we obtain

**Theorem 5**

- (i) The Runge-Kutta methods preserve the oscillations of (1) if and only if  $\Delta_1 \geq \Delta_1(m)$  or  $\Delta_2 \leq \Delta_2(m)$ ;
- (ii) The Runge-Kutta methods preserve the non-oscillations of (1) if and only if  $\Delta_1 < \Delta_1(m)$  and  $\Delta_2 > \Delta_2(m)$ .

Before giving the conditions that the Runge-Kutta methods preserve oscillations and non-oscillations of (1), we introduce the following corollary.

**Corollary 2** [21,22] Suppose  $S(z) = \varphi(z) / \psi(z)$  (where  $\varphi(z), \psi(z)$  are polynomials) is the  $(r, s)$ -Pad é approximation to  $e^z$ . Then

- a)  $S(x) < e^x$  if and only if  $s$  is even for all  $x > 0$ ,
  - b)  $S(x) > e^x$  if and only if  $s$  is odd for all  $0 < x < \xi$ ,
  - c)  $S(x) > e^x$  if and only if  $r$  is even for all  $x < 0$ ,
  - d)  $S(x) < e^x$  if and only if  $r$  is odd for all  $\eta < x < 0$ ,
- where  $\xi$  is a real zero of  $\psi_s(z)$  and  $\eta$  is a real zero of  $\varphi_r(z)$ .

Applying Theorem 5, Lemma 1 and Corollary 2, we can obtain the following theorems.

**Theorem 6** Suppose  $S(z)$  is the  $(r, s)$ -Pad é approximation to  $e^z$ , the Runge-Kutta methods preserve oscillations of (1) if any of the following conditions is satisfied:

- (i)  $a > 0, h < h_1$  and  $s$  is odd,
- (ii)  $a < 0, h < h_2$  and  $r$  is odd,

where  $h_1 = -\delta_1 / a, h_2 = -\delta_2 / a$ .

**Theorem 7** Suppose  $S(z)$  is the  $(r, s)$ -Pad é approximation to  $e^z$ , the Runge-Kutta methods preserve non-oscillations of (1) if any of the following conditions is satisfied:

- (i)  $a > 0, h < h_1$  and  $s$  is even,

(ii)  $a < 0, h < h_2$  and  $r$  is even, where  $h_1 = -\delta_1 / a, h_2 = -\delta_2 / a$ .

Tables 1-2 visually illustrate Theorems 6 and 7 respectively, where " - " denotes no limitation to parameters. And we give the conditions that the  $A$ -stable higher order Runge-Kutta methods preserve oscillations and non-oscillations of (1) (see Tables 3-4).

TABLE 1  
PRESERVATION OF OSCILLATIONS

$a$	$h$	$r$	$s$
$a > 0$	$h < h_1$	-	odd
$a < 0$	$h < h_2$	odd	-

TABLE 2  
PRESERVATION OF NON-OSCILLATIONS

$a$	$h$	$r$	$s$
$a > 0$	$h < h_1$	-	even
$a < 0$	$h < h_2$	even	-

TABLE 3  
PRESERVATION OF OSCILLATIONS FOR HIGHER ORDER RUNGE-KUTTA METHODS

	Gauss-Legendre	Radau IA, IIA	Lobatto IIIA, IIIB	Lobatto IIIC
$(r, s)$	$(v, v)$	$(v-1, v)$	$(v-1, v-1)$	$(v-2, v)$
$a > 0$	$v$ is odd	$v$ is odd	$v$ is even	$v$ is odd
$a < 0$	$v$ is odd	$v$ is even	$v$ is even	$v$ is odd

TABLE 4  
PRESERVATION OF NON-OSCILLATIONS FOR HIGHER ORDER RUNGE-KUTTA METHODS

	Gauss-Legendre	Radau IA, IIA	Lobatto IIIA, IIIB	Lobatto IIIC
$(r, s)$	$(v, v)$	$(v-1, v)$	$(v-1, v-1)$	$(v-2, v)$
$a > 0$	$v$ is even	$v$ is even	$v$ is odd	$v$ is even
$a < 0$	$v$ is even	$v$ is odd	$v$ is odd	$v$ is even

IV. RELATIONSHIP BETWEEN STABILITY AND OSCILLATIONS

**Theorem 8**[20] The solution of (1) is asymptotically stable (the solution  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ ) for any given  $x_0$ , if and only if

$$-\frac{a(e^a + 1)}{(e^{a/2} - 1)^2} < b < -a, \text{ for } a > 0,$$

$$b < -a \text{ or } b > -\frac{a(e^a + 1)}{(e^{a/2} - 1)^2}, \text{ for } a < 0, \quad (7)$$

$$b < 0, \text{ for } a = 0.$$

**Theorem 9**[24] The numerical solution of (1) is asymptotically stable ( $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ ) if and only if

$$-\frac{a(S(x)^{2m} + 1)}{(S(x)^m - 1)^2} < b < -a, \text{ for } a > 0,$$

$$b < -a \text{ or } b > -\frac{a(S(x)^{2m} + 1)}{(S(x)^m - 1)^2}, \text{ for } a < 0, \quad (8)$$

$$b < 0, \text{ for } a = 0.$$

Let

$$\Delta_3 = -\frac{a(e^a + 1)}{(e^{a/2} - 1)^2},$$

and

$$\Delta_3(m) = -\frac{a(S(x)^{2m} + 1)}{(S(x)^m - 1)^2},$$

by Theorems 2, 4, 8 and 9 we get the following theorems.

**Theorem 10** For  $a > 0$ , the analytic solution of (1) is

- a) oscillatory and unstable if  $b \in (-\infty, \Delta_3)$  or  $b \in (\Delta_1, +\infty)$ ;
- b) oscillatory and asymptotically stable if  $b \in (\Delta_3, \Delta_2)$ ;
- c) non-oscillatory and asymptotically stable if  $b \in (\Delta_2, -a)$ ;
- d) non-oscillatory and unstable if  $b \in (-a, \Delta_1)$ .

for  $a < 0$ , the analytic solution of (1) is

- a) oscillatory and asymptotically stable if  $b \in (-\infty, \Delta_2)$  or  $b \in (\Delta_3, +\infty)$ ;
- b) non-oscillatory and asymptotically stable if  $b \in (\Delta_2, -a)$ ;
- c) non-oscillatory and unstable if  $b \in (-a, \Delta_1)$ ;
- d) oscillatory and unstable if  $b \in (\Delta_1, \Delta_3)$ .

**Theorem 10** For  $a > 0$ , the numerical solution of (1) is

- a) oscillatory and unstable if  $b \in (-\infty, \Delta_3(m))$  or  $b \in (\Delta_1(m), +\infty)$ ;
- b) oscillatory and asymptotically stable if  $b \in (\Delta_3(m), \Delta_2(m))$ ;
- c) non-oscillatory and asymptotically stable if  $b \in (\Delta_2(m), -a)$ ;
- d) non-oscillatory and unstable if  $b \in (-a, \Delta_1(m))$ .

for  $a < 0$ , the numerical solution of Eq. (1) is

- a) oscillatory and asymptotically stable if  $b \in (-\infty, \Delta_2(m))$  or  $b \in (\Delta_3(m), +\infty)$ ;
- b) non-oscillatory and asymptotically stable if  $b \in (\Delta_2(m), -a)$ ;
- c) non-oscillatory and unstable if  $b \in (-a, \Delta_1(m))$ ;

d) oscillatory and unstable if  $b \in (\Delta_1(m), \Delta_3(m))$ .

V. NUMERICAL EXPERIMENTS

In order to give a numerical illustration to the main theorems in the paper, we consider the following four equations:

$$\begin{cases} x'(t) = x(t) - 10x([t + 0.5]), & t \geq 0 \\ x(0) = 1, \end{cases} \quad (9)$$

$$\begin{cases} x'(t) = -2x(t) - 5x([t + 0.5]), & t \geq 0 \\ x(0) = 1, \end{cases} \quad (10)$$

$$\begin{cases} x'(t) = 3x(t) - 2x([t + 0.5]), & t \geq 0 \\ x(0) = 1, \end{cases} \quad (11)$$

$$\begin{cases} x'(t) = -x(t) - x([t + 0.5]), & t \geq 0 \\ x(0) = 1, \end{cases} \quad (12)$$

According to Theorem 2, the analytic solutions of (9) and (10) are oscillatory, the analytic solutions of (11) and (12) are non-oscillatory. In Figs. 1-4, we draw the figures of the analytic solutions and the numerical solutions using 1-Gauss-Legendre method, 2-Radau IA method and 2-Lobatto IIC method. From these figures, we can see that the numerical solutions of (9) and (10) are oscillatory, the numerical solutions of (11) and (12) are non-oscillatory, which are in agreement with Theorems 6 and 7.

In Fig. 1, let  $m = 50$ , so  $\Delta_1 \mathbf{B} 1.5415$ ,  $\Delta_2 \mathbf{B} -2.5415$ ,  $\Delta_3 \mathbf{B} -8.8354$ ,  $\Delta_1(m) \mathbf{B} 1.5415$ ,  $\Delta_2(m) \mathbf{B} -2.5415$  and  $\Delta_3(m) \mathbf{B} -8.8353$ , Obviously,  $b = -10 \in (-\infty, \Delta_3)$  and  $b = -10 \in (-\infty, \Delta_3(m))$ . Therefore, the analytic solutions and the numerical solutions of (9) are both oscillatory and unstable according to Theorems 10 and 11, which are in agreement with Fig. 1. For (10)-(12), we can verify the consistency in the same way (see Figs. 2-4).

All these numerical examples confirm our theoretical findings.

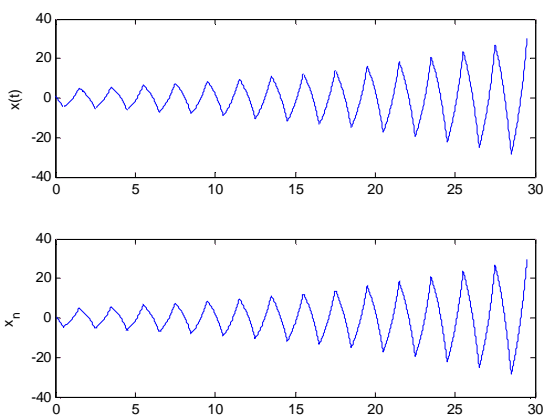


Figure 1. The analytic solution and the numerical solution for (9) by 1-Gauss-Legendre method with  $m=50$

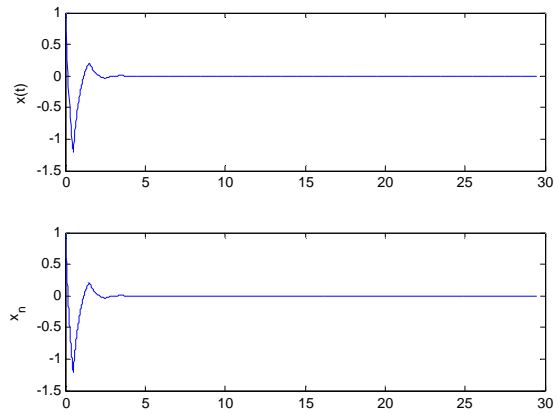


Figure 2. The analytic solution and the numerical solution for (10) by 2-Radau IA method with  $m=50$

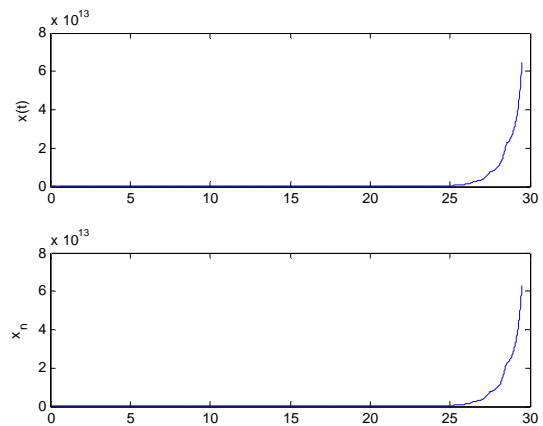


Figure 3. The analytic solution and the numerical solution for (11) by 2-Lobatto IIC method with  $m=50$

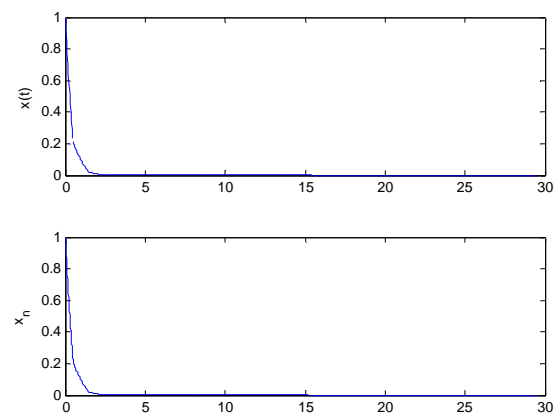


Figure 4. The analytic solution and the numerical solution for (12) by 2-Lobatto IIC method with  $m=50$

VI. CONCLUSIONS

In this paper, numerical oscillations of Runge-Kutta methods for one important class of EPCA of alternately advanced and retarded type are discussed. The preservation of oscillations of the Runge-Kutta method

are obtained and the relationships between the stability and the oscillations are considered for the analytic solution and the numerical solution, respectively. Four numerical examples have shown the correctness of the main result. We will consider the Euler-Maruyama method in the future work.

#### ACKNOWLEDGMENT

The authors wish to thank Professor Mingzhu Liu for his helpful comments and constructive suggestions. This work was supported by the National Natural Science Foundation of China (No. 51008084), the Natural Science Foundation of Guangdong Province (No. 9451009001002753) and the Youth Foundation of Guangdong University of Technology (No. 092027).

#### REFERENCES

- [1] H. Khatibzadeh, "An oscillation criterion for a delay difference equation," *Comput. Math. Appl.*, vol. 57, pp. 37-41, 2009.
- [2] Q.L. Li, Z.G. Zhang, F. Guo, Z.Y. Liu, and H.Y. Liang, "Oscillatory criteria for Third-Order difference equation with impulses," *J. Comput. Appl. Math.*, vol. 225, pp. 80-86, 2009.
- [3] B.G. Jia, L. Erbe, and A. Peterson, "New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales," *Comput. Math. Appl.* vol.56, pp. 2744-2756, 2008.
- [4] B.G. Jia, L. Erbe, and A. Peterson, "Oscillation of sublinear Emden-Fowler dynamic equations on time scales," *J. Difference Equations Appl.* vol. 16, pp. 217-226, 2010.
- [5] J.R. Graef, J.H. Shen, and I.P. Stavroulakis, "Oscillation of impulsive neutral delay differential equations," *J. Math. Anal. Appl.* vol. 268, pp. 310-333, 2002.
- [6] I. Kubiacyk, S.H. Saker, "Oscillation and stability in nonlinear delay differential equations of population dynamics," *Math. Comput. Model.*, vol. 35, pp. 295-301, 2002.
- [7] L.P. Gimenes, M. Federson, "Oscillation by impulses for a second-order delay differential equation," *Comput. Math. Appl.*, vol. 52, pp. 819-828, 2006.
- [8] A.Z. Weng, J.T. Sun, "Oscillation of second order delay differential equations," *Appl. Math. Comput.*, vol. 198, pp. 930-935, 2008.
- [9] J.H. Shen, I.P. Stavroulakis, "Oscillatory and nonoscillatory delay equations with piecewise constant argument," *J. Math. Anal. Appl.*, vol. 248, pp. 385-401, 2000.
- [10] Z.G. Luo, J.H. Shen, "New results on oscillation for delay differential equations with piecewise constant argument," *Comput. Math. Appl.*, vol. 45, pp. 1841-1848, 2003.
- [11] Y.B. Wang, J.R. Yan, "Oscillation of a differential equation with fractional delay and piecewise constant arguments," *Comput. Math. Appl.*, vol. 52, pp. 1099-1106, 2006.
- [12] X.L. Fu, X.D. Li, "Oscillation of higher order impulsive differential equations of mixed type with constant argument at fixed time," *Math. Comput. Model.*, vol. 48, pp. 776-786, 2008.
- [13] S. Busenberg, K. Cooke, *Vertically Transmitted Diseases, Models and Dynamics*, in: *Biomathematics*, Berlin: Springer, 1993.
- [14] T. Kupper, R. Yuang, "On quasi-periodic solutions of differential equations with piecewise constant argument," *J. Math. Anal. Appl.*, vol. 267, pp. 173-193, 2002.
- [15] G. Papaschinopoulos, "Linearization near the integral manifold for a system of differential equations with piecewise constant argument," *J. Math. Anal. Appl.*, vol. 215, pp. 317-333, 1997.
- [16] A. Alonso, J.L. Hong, "Ergodic type solutions of differential equations with piecewise constant arguments," *Int. J. Math. Math. Sci.*, vol. 28, pp. 609-619, 2001.
- [17] Y. Muroya, "Persistence, contractivity and global stability in logistic equations with piecewise constant delays," *J. Math. Anal. Appl.*, vol. 270, pp. 602-635, 2002.
- [18] R. Yuan, "On the spectrum of almost periodic solution of second order scalar functional differential equations with piecewise constant argument," *J. Math. Anal. Appl.*, vol. 303, pp. 103-118, 2005.
- [19] F. Gurcan, F. Bozkurt, "Global stability in a population model with piecewise constant arguments," *J. Math. Anal. Appl.*, vol. 360, pp. 334-342, 2009.
- [20] J. Wiener, *Generalized Solutions of Functional Differential Equations*. Singapore: World Scientific, 1993.
- [21] M.Z. Liu, M.H. Song, and Z.W. Yang, "Stability of Runge-Kutta methods in the numerical solution of equation  $u'(t) = au(t) + a_0u([t])$ ," *J. Comput. Appl. Math.*, vol. 166, pp. 361-370, 2004.
- [22] Z.W. Yang, M.Z. Liu, and M.H. Song, "Stability of Runge-Kutta methods in the numerical solution of equation  $u'(t) = au(t) + a_0u([t]) + a_1u([t+1])$ ," *Appl. Math. Comput.*, vol. 162, pp. 37-50, 2005.
- [23] M.Z. Liu, S.F. Ma, and Z.W. Yang, "Stability analysis of Runge-Kutta methods for unbounded retarded differential equations with piecewise continuous arguments," *Appl. Math. Comput.*, vol. 191, pp. 57-66, 2007.
- [24] W.J. Lv, Z.W. Yang, and M.Z. Liu, "Stability of Runge-Kutta methods for the alternately advanced and retarded differential equations with piecewise continuous arguments," *Comput. Math. Appl.*, vol. 54, pp. 326-335, 2007.
- [25] M.Z. Liu, J.F. Gao, and Z.W. Yang, "Oscillation analysis of numerical solution in the  $\theta$ -methods for equation  $x'(t) + ax(t) + a_1x([t-1]) = 0$ ," *Appl. Math. Comput.*, vol. 186, pp. 566-578, 2007.
- [26] M.Z. Liu, J.F. Gao, and Z.W. Yang, "Preservation of oscillations of the Runge-Kutta method for equation  $x'(t) + ax(t) + a_1x([t-1]) = 0$ ," *Comput. Math. Appl.*, vol. 58, pp. 1113-1125, 2009.
- [27] A.R. Aftabzadeh, J. Wiener, "Oscillatory and periodic solutions of an equation alternately of retarded and advanced type," *Appl. Anal.*, vol. 23, pp. 219-231, 1986.

**Qi Wang** was born in Yichun, Heilongjiang, China in 1978. He is a math teacher of Guangdong University of Technology. He received his B.S. degree in applied mathematics from Heilongjiang University in 2002 and the M.S. degree in pure mathematics from Harbin Institute of Technology in 2004. His research interests include numerical analysis of delay differential equation, stochastic differential equation and impulsive differential equation.

**Fenglian Fu**, a teacher of Guangdong University of Technology. She received M.S. and Ph.D. degree in environmental engineering from Sun yat-sen University in 2004 and 2007, respectively. Her research interest is theories and technology of pollution control.