J. Math. Soc. Japan
Vol. 58, No. 2, 2006

Numerical radius Haagerup norm and square factorization through Hilbert spaces

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(Received Apr. 7, 2004)
(Revised Mar. 22, 2005)

Abstract. We study a factorization of bounded linear maps from an operator space A to its dual space A^*. It is shown that $T : A \longrightarrow A^*$ factors through a pair of column Hilbert space $\mathscr{H}_c$ and its dual space if and only if T is a bounded linear form on $A \otimes A$ by the canonical identification equipped with a numerical radius type Haagerup norm. As a consequence, we characterize a bounded linear map from a Banach space to its dual space, which factors through a pair of Hilbert spaces.

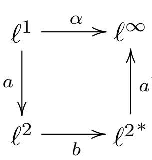
1. Introduction.

Factorization through a Hilbert space of a linear map plays one of the central roles in the Banach space theory (c.f. [**17**]). Also in the C^*-algebra and the operator space theory, many important factorization theorems have been proved related to the Grothendieck type inequality in several situations [**8**], [**5**], [**18**], [**21**].

Let α be a bounded linear map from ℓ^1 to ℓ^∞, $\{e_i\}_{i=1}^\infty$ the canonical basis of ℓ^1, and $\boldsymbol{B}(\ell^2)$ the bounded operators on ℓ^2. We regard α as the infinite matrix $[\alpha_{ij}]$ where $\alpha_{ij} = \langle e_i, \alpha(e_j)\rangle$. The Schur multiplier S_α on $\boldsymbol{B}(\ell^2)$ is defined by $S_\alpha(x) = \alpha \circ x$ for $x = [x_{ij}] \in \boldsymbol{B}(\ell^2)$ where $\alpha \circ x$ is the Schur product $[\alpha_{ij} x_{ij}]$. Let $w(\cdot)$ be the numerical radius norm on $\boldsymbol{B}(\ell^2)$. In [**12**], it was shown that

$$\|S_\alpha\|_w = \sup_{x \neq 0} \frac{w(\alpha \circ x)}{w(x)} \leq 1$$

if and only if α has the following factorization with $\|a\|^2 \|b\| \leq 1$:



where a^t is the transposed map of a.

Motivated by the above result, we will show a square factorization theorem of a bounded linear map through a pair of column Hilbert spaces $\mathscr{H}_c$ between an operator

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2000 *Mathematics Subject Classification.* Primary 46L07; Secondary 47L25, 46B28, 47L05.

Key Words and Phrases. operator space, C^*-algebra, Completely bounded map, Numerical radius Haagerup norm, Factorization through Hilbert spaces.

space and its dual space. More precisely, let us suppose that A is an operator space in $\boldsymbol{B}(\mathscr{H})$ and $A \otimes A$ is the algebraic tensor product. We define the numerical radius Haagerup norm of an element $u \in A \otimes A$ by

$$\|u\|_{wh} = \inf\left\{\frac{1}{2}\big\|[x_1, \dots, x_n, y_1^*, \dots, y_n^*]\big\|^2 \mid u = \sum_{i=1}^{n} x_i \otimes y_i\right\}.$$

Let $T : A \longrightarrow A^*$ be a bounded linear map. We show that $T : A \longrightarrow A^*$ has an extension T' which factors through a pair of column Hilbert spaces $\mathscr{H}_c$ so that

$$\begin{array}{ccc} C^*(A) & \xrightarrow{T'} & C^*(A)^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^*} \\ \mathscr{H}_c & \xrightarrow[b]{} & \mathscr{H}_c \end{array}$$

with $\inf\{\|a\|_{cb}^2\|b\|_{cb} \mid T' = a^*ba\} \leq 1$ if and only if $T \in (A \otimes_{wh} A)^*$ with $\|T\|_{wh^*} \leq 1$ by the natural identification $\langle x, T(y)\rangle = T(x \otimes y)$ for $x, y \in A$.

We also study a variant of the numerical radius Haagerup norm in order to get the factorization without using the $*$ structure.

As a consequence, the above result and/or the variant read a square factorization of a bounded linear map through a pair of Hilbert spaces from a Banach space X to its dual space X^*. The norm on $X \otimes X$ corresponding to the numerical radius Haagerup norm is as follows:

$$\|u\|_{wH} = \inf\left\{\sup\left\{\left(\sum_{i=1}^{n} |f(x_i)|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} |f(y_i)|^2\right)^{\frac{1}{2}}\right\}\right\},$$

where the supremum is taken over all $f \in X^*$ with $\|f\| \leq 1$ and the infimum is taken over all representations $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes X$.

The norm $\| \ \|_{wH}$ is equivalent to the norm $\| \ \|_H$ (see Remark 4.4) introduced by Grothendieck in [**7**]. However, $\| \ \|_{wH}$ will give us a different view to factorization problems of bounded linear operators through Hilbert spaces. Let $\pi_2(a)$ be the 2-summing norm (c.f. [**17**] or see section 4) of a linear map a from X to $\mathscr{H}$. We show that $T : X \longrightarrow X^*$ has the factorization

$$\begin{array}{ccc} X & \xrightarrow{T} & X^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^t} \\ \mathscr{H} & \xrightarrow[b]{} & \mathscr{H}^* \end{array}$$

with $\inf\{\pi_2(a)^2\|b\| \mid T = a^tba\} \leq 1$ if and only if $T \in (X \otimes_{wH} X)^*$ with $\|T\|_{wH^*} \leq 1$. Moreover we characterize a linear map $X \longrightarrow X^*$ which has a square factorization by a Lindenstrauss and Pelczynski type condition (c.f. [**14**] or see Remark 4.4).

We refer to [**6**], [**15**], [**20**] for background on operator spaces, [**17**], [**19**] for factorization through a Hilbert space, and [**16**], [**22**], [**23**], [**24**] for completely bounded maps related to the numerical radius norm.

2. Factorization on operator spaces.

Let $\boldsymbol{B}(\mathscr{H})$ be the space of all bounded operators on a Hilbert space $\mathscr{H}$. Throughout this paper, let us suppose that A and B are operator spaces in $\boldsymbol{B}(\mathscr{H})$. We denote by $C^*(A)$ the C^*-algebra in $\boldsymbol{B}(\mathscr{H})$ generated by the operator space A. We define the numerical radius Haagerup norm of an element $u \in A \otimes B$ by

$$\|u\|_{wh} = \inf\left\{\frac{1}{2}\big\|[x_1, \dots, x_n, y_1^*, \dots, y_n^*]\big\|^2 \mid u = \sum_{i=1}^n x_i \otimes y_i\right\},$$

where $[x_1, \dots, x_n, y_1^*, \dots, y_n^*] \in M_{1,2n}(C^*(A+B))$, and denote by $A \otimes_{wh} B$ the completion of $A \otimes B$ with the norm $\| \ \|_{wh}$.

Recall that the Haagerup norm on $A \otimes B$ is

$$\|u\|_h = \inf\left\{\big\|[x_1, \dots, x_n]\big\| \, \big\|[y_1, \dots, y_n]^t\big\| \mid u = \sum_{i=1}^n x_i \otimes y_i\right\},$$

where $[x_1, \dots, x_n] \in M_{1,n}(A)$ and $[y_1, \dots, y_n]^t \in M_{n,1}(B)$.

By the identity

$$\inf_{\lambda>0} \frac{\lambda\alpha + \lambda^{-1}\beta}{2} = \sqrt{\alpha\beta} \tag{$\star$}$$

for positive real numbers $\alpha, \beta \geq 0$, the Haagerup norm can be rewritten as

$$\|u\|_h = \inf\left\{\frac{1}{2}\left(\big\|[x_1, \dots, x_n]\big\|^2 + \big\|[y_1^*, \dots, y_n^*]\big\|^2\right) \mid u = \sum_{i=1}^n x_i \otimes y_i\right\}.$$

Then it is easy to check that

$$\frac{1}{2}\|u\|_h \leq \|u\|_{wh} \leq \|u\|_h$$

and $\|u\|_{wh}$ is a norm. We use the notation $x\alpha \odot y^t$ for $\sum_{i=1}^n \sum_{j=1}^m x_i\alpha_{ij} \otimes y_j$, where $x = [x_1, \dots, x_n] \in M_{1,n}(A)$, $\alpha = [\alpha_{ij}] \in M_{n,m}(\boldsymbol{C})$ and $y^t = [y_1, \dots, y_m]^t \in M_{m,1}(B)$. We note the identity $x\alpha \odot y^t = x \odot \alpha y^t$.

First we show that the numerical radius Haagerup norm has the injectivity.

PROPOSITION 2.1. *Let $A_1 \subset A_2$ and $B_1 \subset B_2$ be operator spaces in $\boldsymbol{B}(\mathscr{H})$. Then the canonical inclusion Φ of $A_1 \otimes_{wh} B_1$ into $A_2 \otimes_{wh} B_2$ is isometric.*

Proof. The inequality $\|\Phi(u)\|_{wh} \leq \|u\|_{wh}$ is trivial. To get the reverse inequality, let $u = \sum_{i=1}^{n} x_i \otimes y_i \in A_1 \otimes B_1$. We may assume that $\{y_1, \ldots, y_k\} \subset B_2$ is linearly independent and there exists an $n \times k$ matrix of scalars $L \in M_{nk}(\boldsymbol{C})$ such that $[y_1, \ldots, y_n]^t = L[y_1, \ldots, y_k]^t$. We put $z^t = [y_1, \ldots y_k]^t$. Then we have

$$\begin{aligned} u &= x \odot y^t = x \odot Lz^t \\ &= xL(L^*L)^{-1/2} \odot (L^*L)^{1/2} z^t \end{aligned}$$

and

$$\big\| \big[xL(L^*L)^{-1/2}, ((L^*L)^{1/2}z^t)^*\big] \big\| \leq \big\| \big[x, (y^t)^*\big] \big\|.$$

So we can get a representation $u = [x'_1, \ldots, x'_k] \odot [y'_1, \ldots, y'_k]^t$ with

$$\big\| \big[x'_1, \ldots, x'_k, {y'_1}^*, \ldots, {y'_k}^*\big] \big\| \leq \big\| \big[x, (y^t)^*\big] \big\|$$

and $\{y'_1, \ldots, y'_k\}$ is linearly independent. This implies that $x'_1, \ldots, x'_k \in A_1$.

Applying the same argument for $\{x'_1, \ldots, x'_k\}$ instead of $\{y_1, \ldots, y_n\}$, we can get a representation $u = [x''_1, \ldots, x''_l] \odot [y''_1, \ldots, y''_l]^t$ with

$$\big\| \big[x''_1, \ldots, x''_l, {y''}^*_1, \ldots, {y''}^*_l\big] \big\| \leq \big\| \big[x, (y^t)^*\big] \big\|$$

and $x''_i \in A_1$ and $y''_i \in B_1$. It follows that $\|\Phi(u)\|_{wh} \geq \|u\|_{wh}$. □

We also define a norm of an element $u \in C^*(A) \otimes C^*(A)$ by

$$\|u\|_{Wh} = \inf\left\{ \big\|[x_1, \ldots, x_n]^t\big\|^2 w(\alpha) \mid u = \sum x_i^* \alpha_{ij} \otimes x_j \right\},$$

where $w(\alpha)$ is the numerical radius norm of $\alpha = [\alpha_{ij}]$ in $M_n(\boldsymbol{C})$.

$A \otimes_{Wh} A$ is defined as the closure of $A \otimes A$ in $C^*(A) \otimes_{Wh} C^*(A)$.

Theorem 2.2. *Let A be an operator space in $\boldsymbol{B}(\mathscr{H})$. Then $A \otimes_{wh} A = A \otimes_{Wh} A$.*

Proof. By Proposition 2.1 and the definition of $A \otimes_{Wh} A$, it is sufficient to show that $C^*(A) \otimes_{wh} C^*(A) = C^*(A) \otimes_{Wh} C^*(A)$.

Given $u = \sum_{i=1}^{n} x_i \otimes y_i \in C^*(A) \otimes C^*(A)$, we have

$$u = \big[x_1, \ldots, x_n, y_1^*, \ldots, y_n^*\big] \begin{bmatrix} 0_n & 1_n \\ 0_n & 0_n \end{bmatrix} \odot \big[x_1^*, \ldots, x_n^*, y_1, \ldots, y_n\big]^t.$$

Since $w\left(\left[\begin{smallmatrix} 0_n & 1_n \\ 0_n & 0_n \end{smallmatrix}\right]\right) = \frac{1}{2}$, $\|u\|_{wh} \geq \|u\|_{Wh}$.

To establish the reverse inequality, suppose that $u = \sum_{i,j=1}^{n} x_i^* \alpha_{ij} \otimes x_j \in C^*(A) \otimes C^*(A)$ with $w(\alpha) = 1$ and $\|[x_1, \ldots, x_n]^t\|^2 = 1$. It is enough to see that there exist $c_i, d_i \in$

$C^*(A)(i = 1, \dots, m)$ such that $u = \sum_{i=1}^m c_i \otimes d_i$ with $\|[c_1, \dots, c_m, d_1^*, \dots, d_m^*]\|^2 \le 2$. By the assumption $w(\alpha) = 1$ and Ando's Theorem [**1**], we can find a self-adjoint matrix $\beta \in M_n(\boldsymbol{C})$ for which $P = \left[\begin{smallmatrix} 1+\beta & \alpha \\ \alpha^* & 1-\beta \end{smallmatrix}\right]$ is positive definite in $M_{2n}(\boldsymbol{C})$.

Set $[c_1, \dots, c_{2n}] = [x_1^*, \dots, x_n^*, 0, \dots, 0]P^{\frac{1}{2}}$ and $[d_1, \dots d_{2n}]^t = P^{\frac{1}{2}}[0, \dots, 0, x_1, \dots, x_n]^t$. We note that $u = [c_1, \dots, c_{2n}] \odot [d_1, \dots, d_{2n}]^t$. Then we have

$$\begin{aligned} \big\|\big[c_1, \dots, c_{2n}, d_1^*, \dots, d_{2n}^*\big]\big\|^2 &= \big\|\big[x_1^*, \dots, x_n^*, 0, \dots, 0\big]P[x_1, \dots, x_n, 0, \dots, 0]^t \\ &\qquad + \big[0, \dots, 0, x_1^*, \dots, x_n^*\big]P[0, \dots, 0, x_1, \dots, x_n]^t\big\| \\ &= \big\|\big[x_1^*, \dots, x_n^*\big](1+\beta)[x_1, \dots, x_n]^t \\ &\qquad + \big[x_1^*, \dots, x_n^*\big](1-\beta)[x_1, \dots, x_n]^t\big\| \\ &= 2\big\|[x_1, \dots, x_n]^t\big\|^2 = 2. \end{aligned}$$

□

We recall the column (resp. row) Hilbert space $\mathscr{H}_c$(resp. $\mathscr{H}_r$) for a Hilbert space $\mathscr{H}$. For $\xi = [\xi_{ij}] \in M_n(\mathscr{H})$, we define a map $C_n(\xi)$ by

$$C_n(\xi) : \boldsymbol{C}^n \ni [\lambda_1, \dots, \lambda_n] \longmapsto \left[\sum_{j=1}^n \lambda_j \xi_{ij}\right]_i \in \mathscr{H}^n$$

and denote the column matrix norm by $\|\xi\|_c = \|C_n(\xi)\|$. This operator space structure on $\mathscr{H}$ is called the column Hilbert space and denoted by $\mathscr{H}_c$.

To consider the row Hilbert space, let $\overline{\mathscr{H}}$ be the conjugate Hilbert space for $\mathscr{H}$. We define a map $R_n(\xi)$ by

$$R_n(\xi) : \overline{\mathscr{H}}^n \ni [\overline{\eta}_1, \dots, \overline{\eta}_n] \longmapsto \left[\sum_{j=1}^n (\xi_{ij}|\eta_j)\right]_i \in \boldsymbol{C}^n$$

and the row matrix norm by $\|\xi\|_r = \|R_n(\xi)\|$. This operator space structure on $\mathscr{H}$ is called the row Hilbert space and denoted by $\mathscr{H}_r$.

Let $a : C^*(A) \longrightarrow \mathscr{H}_c$ be a completely bounded map. We define a map $d : C^*(A) \longrightarrow \overline{\mathscr{H}}$ by $d(x) = \overline{a(x^*)}$. It is not hard to check that $d : C^*(A) \longrightarrow \overline{\mathscr{H}}_r$ is completely bounded and $\|a\|_{cb} = \|d\|_{cb}$ when we introduce the row Hilbert space structure to $\overline{\mathscr{H}}$. In this paper, we define the adjoint map a^* of a by the transposed map of d, that is, $d^t : ((\overline{\mathscr{H}})_r)^* = ((\mathscr{H}^*)_r)^* = (\mathscr{H}^{**})_c = \mathscr{H}_c \longrightarrow C^*(A)^*$ (c.f. [**5**]). More precisely, we define

$$\langle a^*(\eta), x\rangle = \langle \eta, d(x)\rangle = (\eta | a(x^*)) \quad \text{for} \quad \eta \in \mathscr{H}, x \in C^*(A).$$

A linear map $T : A \longrightarrow A^*$ can be identified with the bilinear form $A \times A \ni (x, y) \longmapsto \langle x, T(y)\rangle \in \boldsymbol{C}$ and also the linear form $A \otimes A \longrightarrow \boldsymbol{C}$. We also use T to denote both of the bilinear form and the linear form, and use $\|T\|_{\beta^*}$ to denote the norm when $A \otimes A$ is equipped with a norm $\| \ \|_\beta$.

We are going to prove the main theorem. The main result below can be shown by modifying arguments for the original Haagerup norm in [**4**].

Theorem 2.3. *Suppose that A is an operator space in $\boldsymbol{B}(\mathscr{H})$, and that $T : A \times A \longrightarrow \boldsymbol{C}$ is bilinear. Then the following are equivalent*:

(1) $\|T\|_{wh^*} \leq 1$.
(2) *There exists a state p_0 on $C^*(A)$ such*

$$|T(x,y)| \leq p_0(xx^*)^{\frac{1}{2}} p_0(y^*y)^{\frac{1}{2}} \quad \textit{for } x, y \in A.$$

(3) *There exist a $*$-representation $\pi : C^*(A) \longrightarrow \boldsymbol{B}(\mathscr{K})$, a unit vector $\xi \in \mathscr{K}$ and a contraction $b \in \boldsymbol{B}(\mathscr{K})$ such that*

$$T(x,y) = (\pi(x) b \pi(y)\xi \mid \xi) \quad \textit{for } x, y \in A.$$

(4) *There exist an extension $T' : C^*(A) \longrightarrow C^*(A)^*$ of T and completely bounded maps $a : C^*(A) \longrightarrow \mathscr{K}_c$, $b : \mathscr{K}_c \longrightarrow \mathscr{K}_c$ such that*

$$\begin{array}{ccc} C^*(A) & \xrightarrow{\;T'\;} & C^*(A)^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^*} \\ \mathscr{K}_c & \xrightarrow[\;b\;]{} & \mathscr{K}_c \end{array}$$

$$\textit{i.e.,} \quad T' = a^* b a \quad \textit{with} \quad \|a\|_{cb}^2 \|b\|_{cb} \leq 1.$$

Proof. (1)⇒(2) By Proposition 2.1, we can extend T on $C^*(A) \otimes_{wh} C^*(A)$ and also denote it by T. We may assume $\|T\|_{wh^*} \leq 1$. By the identity ($\star$), it is sufficient to show the existence of a state $p_0 \in S(C^*(A))$ such that

$$|T(x,y)| \leq \frac{1}{2} p_0(xx^* + y^*y) \quad \text{for } x, y \in C^*(A).$$

Moreover it is enough to find $p_0 \in S(C^*(A))$ such that

$$\operatorname{Re} T(x,y) \leq \frac{1}{2} p_0(xx^* + y^*y) \quad \text{for } x, y \in C^*(A).$$

Define a real valued function $T_{\{x_1,\dots,x_n,y_1,\dots,y_n\}}(\,\cdot\,)$ on $S(C^*(A))$ by

$$T_{\{x_1,\dots,x_n,y_1,\dots,y_n\}}(p) = \sum_{i=1}^{n} \frac{1}{2} p\big(x_i x_i^* + y_i^* y_i\big) - \operatorname{Re} T(x_i, y_i),$$

for $x_i, y_i \in C^*(A)$. Set

$$\triangle = \{T_{\{x_1,\ldots,x_n,y_1,\ldots,y_n\}} \mid x_i, y_i \in C^*(A),\ n \in \boldsymbol{N}\}.$$

It is easy to see that $\triangle$ is a cone in the set of all real functions on $S(C^*(A))$. Let $\bigtriangledown$ be the open cone of all strictly negative functions on $S(C^*(A))$. For any $x_1,\ldots,x_n,y_1,\ldots,y_n \in C^*(A)$, there exists $p_1 \in S(C^*(A))$ such that $p_1(\sum x_i x_i^* + y_i^* y_i) = \|\sum x_i x_i^* + y_i^* y_i\|$. Since

$$\begin{aligned} T_{\{x_1,\ldots,x_n,y_1,\ldots,y_n\}}(p_1) &= \frac{1}{2}p_1\Big(\sum x_i x_i^* + y_i^* y_i\Big) - \operatorname{Re}\sum T(x_i, y_i) \\ &= \frac{1}{2}\Big\|\sum x_i x_i^* + y_i^* y_i\Big\| - \operatorname{Re}\sum T(x_i, y_i) \\ &\geq \frac{1}{2}\Big\|\sum x_i x_i^* + y_i^* y_i\Big\| - \Big|\sum T(x_i, y_i)\Big| \\ &\geq 0, \end{aligned}$$

we have $\triangle \cap \bigtriangledown = \varnothing$.

By the Hahn-Banach Theorem, there exists a measure μ on $S(C^*(A))$ such that $\mu(\triangle) \geq 0$ and $\mu(\bigtriangledown) < 0$. So we may assume that μ is a probability measure. Now put $p_0 = \int p d\mu(p)$. Since $T_{\{x,y\}} \in \triangle$,

$$\frac{1}{2}p_0(xx^* + y^*y) - \operatorname{Re} T(x,y) = \int T_{\{x,y\}}(p)d\mu(p) \geq 0.$$

(2)⇒(1) Since

$$\begin{aligned} \Big|\sum T(x_i, y_i)\Big| &\leq \sum p_0\big(x_i x_i^*\big)^{\frac{1}{2}} p_0\big(y_i^* y_i\big)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\sum p_0\big(x_i x_i^* + y_i^* y_i\big) \\ &\leq \frac{1}{2}\big\|\,[x_1,\ldots,x_n,y_1^*,\ldots,y_n^*]\,\big\|^2 \end{aligned}$$

for $x, y \in A$, we have that $T \in (A \otimes_{wh} A)^*$ with $\|T\|_{wh^*} \leq 1$.

(1)⇒(3) As in the proof of the implication (1)⇒(2), we can find a state $p \in S(C^*(A))$ such that $|T(x,y)| \leq p(xx^*)^{\frac{1}{2}} p(y^*y)^{\frac{1}{2}}$ for $x, y \in C^*(A)$. By the GNS construction, we let $\pi : C^*(A) \longrightarrow \boldsymbol{B}(\mathscr{K})$ be the cyclic representation with the cyclic vector ξ and $p(x) = (\pi(x)\xi \mid \xi)$ for $x \in C^*(A)$. Define a sesquilinear form on $\mathscr{K} \times \mathscr{K}$ by $\langle \pi(y)\xi, \pi(x)\xi\rangle = T(x^*, y)$. This is well-defined and bounded since

$$|\langle \pi(y)\xi, \pi(x)\xi\rangle| \leq p(x^*x)^{\frac{1}{2}} p(y^*y)^{\frac{1}{2}} = \|\pi(x)\xi\|\,\|\pi(y)\xi\|.$$

Thus there exists a contraction $b \in \boldsymbol{B}(\mathscr{K})$ such that $T(x^*, y) = (b\pi(y)\xi | \pi(x)\xi)$.

(3)⇒(4) Set $a(x) = \pi(x)\xi$ for $x \in C^*(A)$ and consider the column Hilbert structure for $\mathscr{K}$. Then it is easy to see that $a : C^*(A) \longrightarrow \mathscr{K}_c$ is a complete contraction. The composition $T' = a^*ba$ is an extension of T and $\|a\|_{cb}^2 \|b\|_{cb} \leq 1$.

(4)⇒(1) Since $T'(x,y) = (ba(y)|a(x^*))$ for $x, y \in C^*(A)$, we have

$$\begin{aligned}
\Big|\sum_{i,j=1}^{n} T'\big(x_i^*\alpha_{ij}, x_j\big)\Big| &= \Big|\sum_{i,j=1}^{n} (b\alpha_{ij}a(x_j) \mid a(x_i))\Big| \\
&= \left|\left(\begin{bmatrix} b & & 0 \\ & \ddots & \\ 0 & & b \end{bmatrix}\begin{bmatrix} \alpha_{ij} \end{bmatrix}\begin{bmatrix} a(x_1) \\ \vdots \\ a(x_n) \end{bmatrix} \;\middle|\; \begin{bmatrix} a(x_1) \\ \vdots \\ a(x_n) \end{bmatrix}\right)\right| \\
&\le w\left(\begin{bmatrix} b & & 0 \\ & \ddots & \\ 0 & & b \end{bmatrix}\begin{bmatrix} \alpha_{ij} \end{bmatrix}\right)\left\|\begin{bmatrix} a(x_1) \\ \vdots \\ a(x_n) \end{bmatrix}\right\|^2 \\
&\le \|b\|_{cb} w(\alpha) \|a\|_{cb}^2 \left\|\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right\|^2
\end{aligned}$$

for $\sum_{i,,j=1}^{n} x_i^*\alpha_{ij} \otimes x_j \in C^*(A) \otimes C^*(A)$. At the last inequality, we use $w(cd) \le \|c\|w(d)$ for double commuting operators c, d as well as the fact that $\boldsymbol{B}(\mathscr{K}, \mathscr{K})$ is completely isometric onto $CB(\mathscr{K}_c, \mathscr{K}_c)$. Hence we obtain that $\|T\|_{wh^*} \le \|T'\|_{wh^*} \le 1$. □

Remark 2.4. (i) If we replace the linear map $\langle T(x), y\rangle = T(x,y)$ with $\langle x, T(y)\rangle = T(x,y)$ in Theorem 2.3, then we have a factorization of T through a pair of the row Hilbert spaces $\mathscr{H}_r$. More precisely, the following condition (4)$'$ is equivalent to the conditions in Theorem 2.3.

(4)$'$ There exist an extension $T' : C^*(A) \longrightarrow C^*(A)^*$ of T and completely bounded maps $a : C^*(A) \longrightarrow \mathscr{K}_r$, $b : \mathscr{K}_r \longrightarrow \mathscr{K}_r$ such that

$$\begin{array}{ccc}
C^*(A) & \xrightarrow{\;T'\;} & C^*(A)^* \\
{\scriptstyle a}\big\downarrow & & \big\uparrow{\scriptstyle a^*} \\
\mathscr{K}_r & \xrightarrow[\;b\;]{} & \mathscr{K}_r
\end{array}$$

$$\text{i.e.,}\quad T' = a^*ba \quad \text{with} \quad \|a\|_{cb}^2\|b\|_{cb} \le 1.$$

(ii) Let ℓ_n^2 be an n-dimensional Hilbert space with the canonical basis $\{e_1, \dots, e_n\}$. Given $\alpha : \ell_n^2 \longrightarrow \ell_n^2$ with $\alpha(e_j) = \sum_i \alpha_{ij}e_i$, we set the map $\dot\alpha : \ell_n^2 \longrightarrow {\ell_n^2}^*$ by $\dot\alpha(e_j) = \sum_i \alpha_{ij}\bar e_i$ where $\{\bar e_i\}$ is the dual basis. For notational convenience, we shall also denote $\dot\alpha$ by α. For $\sum_{i=1}^n x_i \otimes e_i \in C^*(A) \otimes \ell_n^2$, we define a norm by $\|\sum_{i=1}^n x_i \otimes e_i\| = \|[x_1, \dots, x_n]^t\|$. Let $T : C^*(A) \longrightarrow C^*(A)^*$ be a bounded linear map. Consider $T \otimes \alpha : C^*(A) \otimes \ell_n^2 \longrightarrow C^*(A)^* \otimes {\ell_n^2}^*$ with a numerical radius type norm $w(\cdot)$ given by

$$w(T \otimes \alpha) = \sup\Big\{\Big|\Big\langle \sum x_i^* \otimes e_i, T \otimes \alpha\Big(\sum x_i \otimes e_i\Big)\Big\rangle\Big| \;\Big|\; \Big\|\sum x_i \otimes e_i\Big\| \le 1\Big\}.$$

Then we have

$$\sup\left\{\frac{w(T\otimes\alpha)}{w(\alpha)}\,\middle|\,\alpha:\ell_n^2\longrightarrow\ell_n^2,\ n\in\boldsymbol{N}\right\}=\|T\|_{wh^*},$$

since $T(\sum x_i^*\alpha_{ij}\otimes x_j)=\langle\sum x_i^*\otimes e_i, T\otimes\alpha(\sum x_i\otimes e_i)\rangle$.

(iii) Let $u=\sum x_i\otimes y_i\in C^*(A)\otimes C^*(A)$. It is straightforward from Theorem 2.3 that

$$\|u\|_{wh}=\sup w\Big(\sum\varphi(x_i)b\varphi(y_i)\Big)$$

where the supremum is taken over all $*$- preserving complete contractions φ and contractions b.

3. A variant of the numerical radius Haagerup norm.

In this section, we study a factorizaion of $T:A\longrightarrow A^*$ through a column Hilbert space $\mathscr{K}_c$ and its dual operator space $\mathscr{K}_c{}^*$. Arguments required in this section are very similar to those in section 2, and instead of repeating them we will just emphasize differences.

We define a variant of the numerical radius Haagerup norm of an element $u\in A\otimes B$ by

$$\|u\|_{wh'}=\inf\left\{\frac{1}{2}\left\|[x_1,\dots,x_n,y_1,\dots,y_n]^t\right\|^2\mid u=\sum_{i=1}^n x_i\otimes y_i\right\},$$

where $[x_1,\dots,x_n,y_1,\dots,y_n]^t\in M_{2n,1}(A+B)$, and denote by $A\otimes_{wh'}B$ the completion of $A\otimes B$ with the norm $\|\ \|_{wh'}$.

We remark that $\|\ \|_{wh}$ and $\|\ \|_{wh'}$ are inequivalent, since $\|\ \|_{'h}$ (resp. $\|\ \|_h$) in [**10**] is equivalent to $\|\ \|_{wh'}$ (resp. $\|\ \|_{wh}$) while $\|\ \|_h$ and $\|\ \|_{'h}$ are inequivalent [**10**], [**13**].

Proposition 3.1. *Let $A_1\subset A_2$ and $B_1\subset B_2$ be operator spaces in $\boldsymbol{B}(\mathscr{H})$. Then the canonical inclusion Φ of $A_1\otimes_{wh'}B_1$ into $A_2\otimes_{wh'}B_2$ is isometric.*

Proof. The proof is almost the same as that given in Proposition 2.1. □

In the next theorem, we use the transposed map $a^t:(\mathscr{K}_c)^*\longrightarrow C^*(A)^*$ of $a:C^*(A)^*\longrightarrow\mathscr{K}_c$ instead of $a^*:\mathscr{K}_c\longrightarrow C^*(A)^*$. We note that $(\mathscr{K}_c)^*=(\overline{\mathscr{K}})_r$ and a, a^t are related by

$$\langle a^t(\bar\eta),x\rangle=\langle\bar\eta,a(x)\rangle=(\bar\eta|\overline{a(x)})_{\overline{\mathscr{K}}}\quad\text{for }\bar\eta\in\overline{\mathscr{K}},x\in C^*(A).$$

It seems that the fourth condition in the next theorem is simpler than the fourth one in Theorem 2.3, since we do not use $*$-structure.

Theorem 3.2. *Suppose that A is an operator space in $\boldsymbol{B}(\mathscr{H})$, and that $T:A\times A\longrightarrow\boldsymbol{C}$ is bilinear. Then the following are quivalent*:

(1) $\|T\|_{wh'^*} \leq 1$.

(2) *There exists a state* p_0 *on* $C^*(A)$ *such that*

$$|T(x,y)| \leq p_0(x^*x)^{\frac{1}{2}} p_0(y^*y)^{\frac{1}{2}} \quad \textit{for } x, y \in A.$$

(3) *There exist a* $*$*-representation* $\pi : C^*(A) \longrightarrow \boldsymbol{B}(\mathscr{K})$, *a unit vector* $\xi \in \mathscr{K}$ *and a contraction* $b : \mathscr{K} \longrightarrow \overline{\mathscr{K}}$ *such that*

$$T(x,y) = (b\pi(y)\xi \mid \overline{\pi(x)\xi})_{\overline{\mathscr{K}}} \quad \textit{for } x, y \in A.$$

(4) *There exist a completely bounded map* $a : A \longrightarrow \mathscr{K}_c$ *and a bounded map* $b : \mathscr{K}_c \longrightarrow (\mathscr{K}_c)^*$ *such that*

$$\begin{array}{ccc} A & \xrightarrow{\;T\;} & A^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^t} \\ \mathscr{K}_c & \xrightarrow[\;b\;]{} & (\mathscr{K}_c)^* \end{array}$$

$$\textit{i.e.,} \quad T = a^t b a \quad \textit{with} \quad \|a\|_{cb}^2 \|b\| \leq 1.$$

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) We can prove these implications by the similar way as in the proof of Theorem 2.3.

(3)$\Rightarrow$(4) We note that we use the norm $\| \ \|$ for b instead of the completely bounded norm $\| \ \|_{cb}$.

(4)$\Rightarrow$(1) For $x_i, y_i \in A$, we have

$$\left|\sum_{i=1}^n T(x_i, y_i)\right| = \left|\sum_{i=1}^n (ba(y_i) \mid \overline{a(x_i)})_{\overline{\mathscr{K}}}\right|$$

$$= \left|\left(\begin{bmatrix} 0 & & b & \\ & \ddots & & \ddots \\ & & 0 & & b \\ 0 & & 0 & & \\ & \ddots & & \ddots & \\ & & 0 & & 0 \end{bmatrix} \begin{bmatrix} \overline{a(x_1)} \\ \vdots \\ \overline{a(x_n)} \\ a(y_1) \\ \vdots \\ a(y_n) \end{bmatrix} \middle| \begin{bmatrix} \overline{a(x_1)} \\ \vdots \\ \overline{a(x_n)} \\ a(y_1) \\ \vdots \\ a(y_n) \end{bmatrix}\right)\right| \leq w\left(\begin{bmatrix} 0 & & b & \\ & \ddots & & \ddots \\ & & 0 & & b \\ 0 & & 0 & & \\ & \ddots & & \ddots & \\ & & 0 & & 0 \end{bmatrix}\right) \left\|\begin{bmatrix} \overline{a(x_1)} \\ \vdots \\ \overline{a(x_n)} \\ a(y_1) \\ \vdots \\ a(y_n) \end{bmatrix}\right\|^2$$

$$= \frac{1}{2}\|b\|\|a\|_{cb}^2 \left\|[x_1, \ldots, x_n, y_1, \ldots, y_n]^t\right\|^2 \leq \frac{1}{2}\left\|[x_1, \ldots, x_n, y_1, \ldots, y_n]^t\right\|^2. \qquad \square$$

4. Factorization on Banach spaces.

Let X be a Banach space. Recall that the minimal quantization $\mathrm{Min}(X)$ of X. Let Ω_X be the unit ball of X^*, that is, $\Omega_X = \{f \in X^* \mid \|f\| \leq 1\}$. For $[x_{ij}] \in M_n(X)$, $\|[x_{ij}]\|_{\min}$ is defined by

$$\|[x_{ij}]\|_{\min} = \sup\{\|[f(x_{ij})]\| \mid f \in \Omega_X\}.$$

Then $\mathrm{Min}(X)$ can be regarded as a subspace in the C^*-algebra $C(\Omega_X)$ of all continuous functions on the compact Hausdorff space Ω_X. Here we define a norm of an element $u \in X \otimes X$ by

$$\|u\|_{wH} = \inf\left\{ \sup\left\{ \left(\sum_{i=1}^n |f(x_i)|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n |f(y_i)|^2\right)^{\frac{1}{2}} \right\}\right\},$$

where the supremum is taken over all $f \in X^*$ with $\|f\| \le 1$ and the infimum is taken over all representations $u = \sum_{i=1}^n x_i \otimes y_i$.

Proposition 4.1. *Let X be a Banach space. Then*

$$\mathrm{Min}(X) \otimes_{wh} \mathrm{Min}(X) = \mathrm{Min}(X) \otimes_{wh'} \mathrm{Min}(X) = X \otimes_{wH} X.$$

Proof. Let $u = \sum_{i=1}^n x_i \otimes y_i \in \mathrm{Min}(X)$. Then, using the identity $(\star)$, we have

$$\begin{aligned}
\|u\|_{wh} &= \inf\left\{ \frac{1}{2}\big\|\big[x_1, \dots, x_n, y_1^*, \dots, y_n^*\big]\big\|^2 \mid u = \sum_{i=1}^n x_i \otimes y_i \right\} \\
&= \inf\left\{ \sup\left\{ \frac{1}{2}\big\|\big[f(x_1), \dots, f(x_n), \overline{f(y_1)}, \dots, \overline{f(y_n)}\big]\big\|^2 \mid f \in \Omega_X \right\} \mid u = \sum_{i=1}^n x_i \otimes y_i \right\} \\
&= \inf\left\{ \sup\left\{ \frac{1}{2}\left(\sum_{i=1}^n |f(x_i)|^2 + |f(y_i)|^2 \right) \mid f \in \Omega_X \right\} \mid u = \sum_{i=1}^n x_i \otimes y_i \right\} \\
&= \inf\left\{ \sup\left\{ \left(\sum_{i=1}^n |f(x_i)|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n |f(y_i)|^2\right)^{\frac{1}{2}} \mid f \in \Omega_X \right\} \mid u = \sum_{i=1}^n x_i \otimes y_i \right\} \\
&= \|u\|_{wH}.
\end{aligned}$$

The equality $\|u\|_{wh'} = \|u\|_{wH}$ is obtained by the same way as above. □

Let $T : X \longrightarrow X^*$ be a bounded linear map. As in Remark 2.4(ii), we consider the map $T \otimes \alpha : X \otimes \ell_n^2 \longrightarrow X^* \otimes {\ell_n^2}^*$ and define a norm for $\sum x_i \otimes e_i \in X \otimes \ell_n^2$ by

$$\left\| \sum x_i \otimes e_i \right\| = \sup\left\{ \left(\sum |f(x_i)|^2 \right)^{\frac{1}{2}} \mid f \in \Omega_X \right\}.$$

We note that, given $x \in X$, x^* is regarded as $\langle x^*, f\rangle = \overline{f(x)}$ for $f \in X^*$ in the definition of $w(T \otimes \alpha)$, that is,

$$w(T \otimes \alpha) = \sup\left\{ \left| \left\langle \sum x_i^* \otimes e_i, T \otimes \alpha\left(\sum x_i \otimes e_i \right) \right\rangle \right| \,\Big|\, \left\| \sum x_i \otimes e_i \right\| \le 1 \right\}.$$

Let $a : X \longrightarrow Y$ be a linear map between Banach spaces. a is called a 2-summing operator if there is a constant C which satisfies the inequality

$$\Big(\sum \|a(x_i)\|^2\Big)^{\frac{1}{2}} \le C \sup\Big\{\Big(\sum |f(x_i)|^2\Big)^{\frac{1}{2}} \mid f \in \Omega_X\Big\}$$

for any finite subset $\{x_i\} \subset X$. The smallest such constant C is defined as $\pi_2(a)$, the 2-summing norm of a. The following might be well known.

Proposition 4.2. *Let X be a Banach space. If a is a linear map from X to $\mathscr{H}$, then the following are equivalent*:

(1) $\|a : \mathrm{Min(X)} \longrightarrow \mathscr{H}_{\mathrm{c}}\|_{\mathrm{cb}} \le 1$.
(2) $\|a : \mathrm{Min(X)} \longrightarrow \mathscr{H}_{\mathrm{r}}\|_{\mathrm{cb}} \le 1$.
(3) $\pi_2(a : X \longrightarrow \mathscr{H}) \le 1$.

Proof. (1)⇒(3) For any $x_1, \dots, x_n \in X$, we have

$$\begin{aligned}\sum_{i=1}^n \|a(x_i)\|^2 &= \big\|[a(x_1), \cdots, a(x_n)]^t\big\|^2 \\ &\le \|a\|_{cb}^2 \big\|[x_1, \cdots, x_n]^t\big\|_{\mathrm{Min}}^2 \\ &\le \Big\|\sum_{i=1}^n x_i^* x_i\Big\|_{\mathrm{Min}} \\ &= \sup\Big\{\sum_{i=1}^n |f(x_i)|^2 \mid f \in \Omega_X\Big\}.\end{aligned}$$

(3)⇒(1) For any $[x_{ij}] \in M_n(\mathrm{Min}(X))$, we have

$$\begin{aligned}\|[a(x_{ij})]\|_{M_n(\mathscr{H}_c)}^2 &= \sup\Big\{\sum_i \Big\|\sum_j \lambda_j a(x_{ij})\Big\|^2 \mid \sum |\lambda_j|^2 = 1\Big\} \\ &\le \sup\Big\{\pi_2(a)^2 \sup\Big\{\sum_i \Big|f\Big(\sum_j \lambda_j x_{ij}\Big)\Big|^2 \mid f \in \Omega_X\Big\} \mid \sum |\lambda_j|^2 = 1\Big\} \\ &\le \sup\{\|[f(x_{ij})]\|^2 \mid f \in \Omega_X\} \\ &\le \|[x_{ij}]\|_{M_n(\mathrm{Min}(X))}^2.\end{aligned}$$

(2)⇔(3) It follows by the same way as above. □

Corollary 4.3. *Suppose that X is a Banach space, and that $T : X \longrightarrow X^*$ is a bounded linear map. Then the following are equivalent*:

(1) $w(T \otimes \alpha) \le w(\alpha)$ *for all* $\alpha : \ell_n^2 \longrightarrow \ell_n^2$ *and* $n \in \boldsymbol{N}$.
(2) $\|T\|_{wH^*} \le 1$.

(3) *T factors through a Hilbert space $\mathscr{K}$ and its dual space $\mathscr{K}^*$ by a 2-summing operator $a : X \longrightarrow \mathscr{K}$ and a bounded operator $b : \mathscr{K} \longrightarrow \mathscr{K}^*$ as follows*:

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & X^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^t} \\ \mathscr{K} & \xrightarrow[\ b\ ]{} & \mathscr{K}^* \end{array}$$

$$\textit{i.e.,}\quad T = a^t ba \quad \textit{with} \quad \pi_2(a)^2\|b\| \le 1.$$

(4) *T has an extension $T' : C(\Omega_X) \longrightarrow C(\Omega_X)^*$ which factors through a pair of Hilbert spaces $\mathscr{K}$ by a 2-summing operator $a : C(\Omega_X) \longrightarrow \mathscr{K}$ and a bounded operator $b : \mathscr{K} \longrightarrow \mathscr{K}$ as follows*:

$$\begin{array}{ccc} C(\Omega_X) & \xrightarrow{\ T'\ } & C(\Omega_X)^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^*} \\ \mathscr{K} & \xrightarrow[\ b\ ]{} & \mathscr{K} \end{array}$$

$$\textit{i.e.,}\quad T' = a^* ba \quad \textit{with} \quad \pi_2(a)^2\|b\| \le 1.$$

Proof. (1)⇒(2) Suppose that

$$\left|\left\langle \sum_{i=1}^m z_i^* \otimes e_i, T \otimes \alpha\left(\sum_{i=1}^m z_i \otimes e_i\right)\right\rangle\right| \le 1$$

for any $\sum_{i=1}^m z_i \otimes e_i \in X \otimes \ell_m^2$ with $\|\sum_{i=1}^m z_i \otimes e_i\| \le 1$ and $\alpha \in M_n(\boldsymbol{C})$ with $w(\alpha) \le 1$. It is easy to see that $|\sum_{i,j=1}^m \langle z_i^*, T(z_j)\rangle \alpha_{ij}| \le 1$, equivalently $|\sum_{i,j=1}^m \langle z_i, T(z_j)\rangle \overline{\alpha_{ij}}| \le 1$.

Given $\|\sum_{i=1}^n x_i \otimes y_i\|_{wH} < 1$, we may assume that

$$\frac{1}{2}\left\|[x_1, \ldots, x_n, y_1, \ldots, y_n]^t\right\|^2 \le 1.$$

Set

$$z_i = \begin{cases} \dfrac{1}{\sqrt{2}} x_i & i = 1, \ldots, n \\ \dfrac{1}{\sqrt{2}} y_{i-n} & i = n+1, \ldots, 2n \end{cases} \qquad \text{and } \alpha = \begin{bmatrix} 0_n & 2 \cdot 1_n \\ 0_n & 0_n \end{bmatrix}.$$

It turns out $\|\sum_{i=1}^{2n} z_i \otimes e_i\| \le 1$ and $w(\alpha) = 1$. Then we have $|T(\sum_{i=1}^n x_i \otimes y_i)| = |\sum_{i,j=1}^{2n} \langle z_i, T(z_j)\rangle \alpha_{ij}| \le 1$. Hence $\|T\|_{wH^*} \le 1$.

(2)⇒(1) Suppose that $\|T\|_{wH^*} \le 1$. Then T has an extension $T' \in (C(\Omega_X) \otimes_{wh} C(\Omega_X))^*$ with $\|T'\|_{wh^*} \le 1$. Given $\varepsilon > 0$ and $\alpha \in M_n(\boldsymbol{C})$, there exist $x_1, \ldots, x_n \in C(\Omega_X)$ such that $\|\sum_{i=1}^n x_i \otimes e_i\| \le 1$ (equivalently $\|[x_1, \ldots, x_n]^t\| \le 1$) and $w(T' \otimes \alpha) -$

$\varepsilon < |\sum_{i,j=1}^{n}\langle x_i^*, T'(x_j)\rangle\alpha_{ij}|$. Hence we have

$$w(T\otimes\alpha) \le w(T'\otimes\alpha) < \left|T'\left(\sum_{i,j=1}^{n} x_i^*\alpha_{ij}\otimes x_j\right)\right| + \varepsilon$$
$$\le \left\|[x_1,\ldots,x_n]^t\right\|^2 w(\alpha) + \varepsilon \le w(\alpha) + \varepsilon.$$

(2)⇔(3) It is straightforward from Theorem 3.2 and Propositions 4.1, 4.2.

(2)⇔(4) It is straightforward from Theorem 2.3 and Propositions 4.1, 4.2. □

Remark 4.4. Here we compare the above corollary with the classical factorization theorems through a Hilbert space. Let X and Y be Banach spaces. Grothendieck introduced the norm $\|\ \|_H$ on $X\otimes Y$ in [**7**] by

$$\|u\|_H = \inf\left\{\sup\left\{\left(\sum_{i=1}^{n}|f(x_i)|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}|g(y_i)|^2\right)^{\frac{1}{2}}\right\}\right\}$$

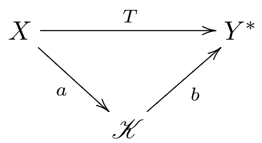
where the supremum is taken over all $f\in X^*, g\in Y^*$ with $\|f\|,\|g\|\le 1$ and the infimum is taken over all representations $u=\sum_{i=1}^{n}x_i\otimes y_i\in X\otimes Y$. In [**14**], Lindenstrauss and Pelczynski characterized the factorization by using $T\otimes\alpha : X\otimes\ell_n^2 \longrightarrow Y\otimes\ell_n^2$ for $T: X\longrightarrow Y$, however the norm on $X\otimes\ell^2$ is slightly different from the one in this paper. Their theorems with a modification are summarized for a bounded linear map $T: X\longrightarrow Y^*$ as follows:

The following are equivalent:

(1) $\|T\otimes\alpha\|\le\|\alpha\|$ for all $\alpha:\ell_n^2\longrightarrow\ell_n^2$ and $n\in\boldsymbol{N}$.

(2) $\|T\|_{H^*}\le 1$.

(3) T factors through a Hilbert space $\mathscr{K}$ by a 2-summing operator $a: X\longrightarrow\mathscr{K}$ and $b:\mathscr{K}\longrightarrow Y^*$ whose transposed b^t is 2-summing as follows:



i.e., $T = ba$ with $\pi_2(a)\pi_2(b^t)\le 1$.

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