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NUMERICAL RANGE FOR THE MATRIX EXPONENTIAL FUNCTION*

CHRISTOS CHORIANOPOULOS† AND CHUN-HUA GUO‡

Abstract. For a given square matrix A, the numerical range for the exponential function e^{At}, $t \in \mathbb{C}$, is considered. Some geometrical and topological properties of the numerical range are presented.

Key words. Matrix exponential function, Numerical range.

AMS subject classifications. 15A16, 15A60.

1. Introduction. The numerical range of matrices (and operators in general) has been a topic of extensive research for many decades. The numerical range of a matrix, its related notions and its extensions reveal a great deal of information about the matrix. The *numerical range* $F(A)$ (also known as the *field of values*) of a matrix $A \in \mathbb{C}^{n\times n}$ is the compact and convex set

$$\begin{aligned} F(A) &= \{x^*Ax \in \mathbb{C} : \, x \in \mathbb{C}^n, \, x^*x = 1\} \\ &= \{\mu \in \mathbb{C} : \, \|A - \lambda I_n\|_2 \geq |\mu - \lambda|, \, \forall \, \lambda \in \mathbb{C}\}, \end{aligned}$$

where $\|\cdot\|_2$ denotes the spectral matrix norm. The latter definition for $F(A)$ can be found in [9]. Note also that $F(A)$ contains all eigenvalues of A.

In the last few decades, the numerical range of matrix polynomials has also been studied extensively; see [8] in particular. If $P(\mu) = A_m\mu^m + A_{m-1}\mu^{m-1} + \cdots + A_0$ is an $n \times n$ matrix polynomial, then the numerical range of $P(\mu)$ is defined as

$$W(P) = \{\mu \in \mathbb{C} : 0 \in F(P(\mu))\}.$$

More generally, one can define numerical ranges for analytic functions of square matrices [7]. Here, we will focus on the exponential function e^{At}, where $A \in \mathbb{C}^{n\times n}$ is fixed and $t \in \mathbb{C}$ is the variable. We will describe the set of all t values for which the numerical range of the matrix e^{At} contains 0, i.e., the Crawford number for the

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†Papadiamanti 26, 143 43, Athens, Greece (cchorian@gmail.com).

‡Department of Mathematics and Statistics, University of Regina, Regina, SK S4S 0A2, Canada (Chun-Hua.Guo@uregina.ca). This author was supported in part by an NSERC Discovery Grant.

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matrix e^{At} is 0 (see [2] for a discussion of the Crawford number for powers of an operator). Note that our purpose here is not to study the exponential function e^A using the numerical range of the matrix A (see [1] for some discussions along that line). Our topic here has some connection to [4], where the author studies classes of operators with 0 in the closure of their numerical range.

For $A \in \mathbb{C}^{n\times n}$, the exponential of A is defined as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

and the series always converges. We collect below some basic properties of the matrix exponential (see [5, 6]).

Let $A, B \in \mathbb{C}^{n\times n}$. If $AB = BA$ then $e^A e^B = e^{A+B}$. It follows that e^A is always invertible and $(e^A)^{-1} = e^{-A}$. If B is invertible then $e^{BAB^{-1}} = Be^A B^{-1}$. It follows that the eigenvalues of e^A are e^λ, where λ are eigenvalues of A. If A is Hermitian ($A^* = A$), then e^A is Hermitian positive definite.

The structure of this paper is as follows: In Section 2, we define the numerical range of e^{At} and show that the numerical range is a nonempty set if and only if A is not a scalar matrix. Some basic properties of the set are then presented in Section 3 when A is not a scalar matrix. The special case where the matrix A is Hermitian (or skew-Hermitian) is treated in Section 4. Some illustrative examples are given in Section 5.

2. Numerical range of e^{At}. For fixed $A \in \mathbb{C}^{n\times n}$, we let $E_A(t) = e^{At}$, $t \in \mathbb{C}$.

Definition 1. The numerical range of $E_A(t)$ is

$$W(E_A) = \{t \in \mathbb{C} : 0 \in F(e^{At})\}. \tag{2.1}$$

Thus, we also have

$$\begin{aligned} W(E_A) &= \{t \in \mathbb{C} : x^* e^{At} x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\} \\ &= \{t \in \mathbb{C} : \|e^{At} - \lambda I_n\|_2 \geq |\lambda|, \ \ \forall\, \lambda \in \mathbb{C}\}. \end{aligned}$$

It follows immediately that for any square matrix A the origin cannot be in $W(E_A)$ since $e^{0A} = I_n$. We now determine when $W(E_A)$ is nonempty.

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$. Then $W(E_A)$ is nonempty if and only if A is not a scalar matrix.*

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Proof. If A is a scalar matrix ($A = cI_n$ for some complex c), then $W(E_A)$ is empty since $F(e^{At}) = \{e^{ct}\}$ and obviously $e^{ct} \neq 0$.

Suppose that A is not a scalar matrix. We will show $W(E_A)$ is nonempty by induction on the size n of A. Let U be unitary such that $A = U^*RU$ and R is upper triangular. We have $e^{At} = U^*e^{Rt}U$, and thus, $F(e^{At}) = F(e^{Rt})$ for each $t \in \mathbb{C}$. It follows that $W(E_A) = W(E_R)$. Therefore, we may assume that A is already in upper triangular form.

Let A be a 2×2 complex matrix $\begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}$. Then by formula (10.40) in [5]

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & a\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1} \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

when $\lambda_1 \neq \lambda_2$; if $\lambda_1 = \lambda_2 = \lambda$ then $e^{At} = \begin{bmatrix} e^{\lambda t} & e^{\lambda t}at \\ 0 & e^{\lambda t} \end{bmatrix}$.

If A has two distinct eigenvalues, then since $F(e^{At})$ is convex for all t, it suffices to show that there is always a complex t for which the line segment connecting $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ contains the origin. In other words, we want a $t \in \mathbb{C}$ such that for some $\rho \in (0, 1)$

$$\rho e^{\lambda_1 t} + (1-\rho)e^{\lambda_2 t} = 0,$$

that is, $e^{(\lambda_1-\lambda_2)t} = -\frac{1-\rho}{\rho}$, or $e^{(\lambda_1-\lambda_2)t-(2k+1)\pi i} = \frac{1-\rho}{\rho}$, where $k \in \mathbb{Z}$. Therefore, we can take

$$t = \frac{\ln\frac{1-\rho}{\rho} + (2k+1)\pi i}{\lambda_1 - \lambda_2}, \quad k \in \mathbb{Z}. \tag{2.2}$$

Since $\rho \in (0, 1)$ can be arbitrary, we know from (2.2) that $W(E_A)$ contains infinitely many parallel straight lines (these lines are horizontal when $\lambda_1 - \lambda_2$ is real).

If $\lambda_1 = \lambda_2 = \lambda$, then

$$F(e^{At}) = \{z \in \mathbb{C} : |z - e^{\lambda t}| \leq |\frac{at}{2}||e^{\lambda t}|\}$$

and $a \neq 0$ because otherwise A would be a scalar matrix. So, $0 \in F(e^{At})$ if and only if $|t| \geq \frac{2}{|a|}$. Therefore, $W(E_A) = \{t \in \mathbb{C} : |t| \geq \frac{2}{|a|}\}$.

Suppose that $W(E_A) \neq \emptyset$ for every $k \times k$ non-scalar upper triangular matrix A, where $k \geq 2$. So, there is a $t_0 \in \mathbb{C}$ such that $x_0^* e^{At_0} x_0 = 0$ for some nonzero $x_0 \in \mathbb{C}^k$. Now for any $(k+1) \times (k+1)$ non-scalar upper triangular matrix A, there are three possibilities:

$$A = \begin{bmatrix} \hat{A} & y \\ 0 & \xi \end{bmatrix}, \quad \text{or} \quad A = \begin{bmatrix} \xi & y^T \\ 0 & \hat{A} \end{bmatrix},$$

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or

$$A = \begin{bmatrix} \lambda & & & a \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{bmatrix},$$

where $y \in \mathbb{C}^k$, $\xi \in \mathbb{C}$, $a \neq 0$, and $\hat{A}$ is a $k \times k$ non-scalar upper triangular matrix.

We will treat the first case. The second case can be treated similarly. The third case can be reduced to the first case by applying a permutation similarity that interchanges row $k+1$ with row 2 and column $k+1$ with column 2.

By induction hypothesis, there is a $t_0 \in \mathbb{C}$ such that $x_0^* e^{\hat{A}t_0} x_0 = 0$ for some nonzero $x_0 \in \mathbb{C}^k$. Then

$$e^{At_0} = \sum_{j=0}^{\infty} \frac{(At_0)^j}{j!} = \begin{bmatrix} e^{\hat{A}t_0} & z_0 \\ 0 & e^{\xi t_0} \end{bmatrix},$$

where $z_0 \in \mathbb{C}^k$. Now for $w = \begin{bmatrix} x_0^T & 0 \end{bmatrix}^T$, we have

$$w^* e^{At_0} w = \begin{bmatrix} x_0^* & 0 \end{bmatrix} \begin{bmatrix} e^{\hat{A}t_0} & z_0 \\ 0 & e^{\xi t_0} \end{bmatrix} \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = x_0^* e^{\hat{A}t_0} x_0 = 0.$$

Thus, $t_0 \in W(E_A)$. $\square$

By an argument similar to the one at the end of the proof, we can show that $W(E_{\hat{A}}) \subseteq W(E_A)$ for any upper triangular matrix A and any leading principal submatrix $\hat{A}$.

3. Properties of $W(E_A)$. In this section, we provide several basic properties of $W(E_A)$.

PROPOSITION 1. *Suppose $A \in \mathbb{C}^{n \times n}$ is not a scalar matrix. Then $W(E_A)$ is closed but it is never bounded. Moreover, $W(E_A)$ always contains infinitely many parallel straight lines.*

Proof. To prove that it is closed suppose that there is a sequence $\{t_n\} \subseteq W(E_A)$ with $t_n \to \hat{t}$. Therefore, there is a sequence of unit vectors $\{x_n\}$ such that $x_n^* e^{At_n} x_n = 0$ for all $n \in \mathbb{N}$. We need to prove that $\hat{t} \in W(E_A)$. Since x_n is bounded, there is a subsequence x_{n_k} converging to a unit vector $\hat{x}$. Therefore, $0 = \lim_{n_k \to \infty} x_{n_k}^* e^{At_{n_k}} x_{n_k} = \hat{x}^* e^{A\hat{t}} \hat{x}$, so $\hat{t} \in W(E_A)$.

In the proof of Theorem 1, we have shown that $W(E_A)$ contains infinitely many parallel straight lines for all 2×2 non-scalar matrices A. The proof by induction there

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also reveals that, for $n \geq 2$, $W(E_A)$ always contains infinitely many parallel straight lines for all $n \times n$ non-scalar matrices A. □

So, if A is not a scalar matrix a $t \in W(E_A)$ can be as large in magnitude as we like. A question that naturally arises is: how small in magnitude can t be?

Proposition 2. *Suppose $A \in \mathbb{C}^{n\times n}$ is not a scalar matrix and $t \in W(E_A)$. Then*

$$|t| \geq \frac{\ln 2}{\inf\limits_{k\in\mathbb{C}} \|A - kI_n\|_2}. \tag{3.1}$$

Proof. Since $t \in W(E_A)$, $\|e^{At} - \lambda I_n\|_2 \geq |\lambda|, \quad \forall\ \lambda \in \mathbb{C}$. For $\lambda = 1$ we have $\|e^{At} - I_n\|_2 \geq 1$. Since $0 \in F(e^{At})$, $0 \in e^{-kt}F(e^{At}) = F(e^{At-ktI_n})$, for all $k \in \mathbb{C}$. Therefore

$$\|e^{At-ktI_n} - I_n\|_2 \geq 1. \tag{3.2}$$

It is known (Corollary 6.2.32 in [6]) that

$$\|e^{X+Y} - e^X\|_2 \leq (e^{\|Y\|_2} - 1)e^{\|X\|_2},$$

where X, Y are square matrices of the same size. For $Y = At - ktI_n$ and $X = 0$ the inequality becomes

$$\|e^{At-ktI_n} - I_n\|_2 \leq e^{\|At-ktI_n\|_2} - 1. \tag{3.3}$$

Combining inequalities (3.2) and (3.3) we have

$$1 \leq e^{\|At-ktI_n\|_2} - 1,$$

or

$$\ln 2 \leq |t|\|A - kI_n\|_2.$$

Inequality (3.1) follows readily. □

The term $\inf\limits_{k\in\mathbb{C}} \|A - kI_n\|_2$ in (3.1) gives the distance from the matrix A to the set of all scalar matrices. The infimum is achieved for a particular scalar matrix since $\inf\limits_{k\in\mathbb{C}} \|A - kI_n\|_2 = \inf\limits_{|k|\leq 2\|A\|_2} \|A - kI_n\|_2$. In estimating this distance, we may reduce A to upper triangular form by the Schur triangularization. If A is Hermitian, then we readily find that the distance is $\frac{1}{2}(\lambda_n - \lambda_1)$, where λ_n and λ_1 are the largest and the smallest eigenvalues of A, respectively.

The constant $\ln 2$ in the lower bound in (3.1) is unlikely to be sharp. But it will be seen later that the constant (valid for all $n \times n$ non-scalar matrices) is at most $\pi/2$.

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The next result is about how $W(E_A)$ will change if a shift or a scalar multiplication is applied to A.

PROPOSITION 3. *Suppose $A \in \mathbb{C}^{n\times n}$ is not a scalar matrix and let $\alpha \in \mathbb{C}$. Then $W(E_{A+\alpha I}) = W(E_A)$ and for $\alpha \neq 0$, $W(E_{\alpha A}) = \frac{1}{\alpha}W(E_A)$.*

Proof.

$$\begin{aligned} W(E_{A+\alpha I}) &= \{t \in \mathbb{C} : x^* e^{t(A+\alpha I)}x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\} \\ &= \{t \in \mathbb{C} : x^* e^{tA}e^{t\alpha I}x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\} \\ &= \{t \in \mathbb{C} : x^* e^{tA}x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\} \\ &= W(E_A), \end{aligned}$$

and

$$\begin{aligned} W(E_{\alpha A}) &= \{t \in \mathbb{C} : x^* e^{t\alpha A}x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\} \\ &= \{\frac{w}{\alpha} \in \mathbb{C} : x^* e^{wA}x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\} \\ &= \frac{1}{\alpha}W(E_A). \quad \square \end{aligned}$$

We now give a condition under which $W(E_A)$ contains some points on the real axis.

PROPOSITION 4. *Let λ and μ be any two eigenvalues of A such that $\lambda - \mu = a + bi$ with $a, b \in \mathbb{R}$ and $b \neq 0$. Then $W(E_A) \supseteq \{(2k + 1)\pi/b : k \in \mathbb{Z}\}$.*

Proof. From the proof by induction for Theorem 1, we only need to prove the result here when A is an upper triangular matrix with eigenvalues λ and μ. In this case, we know from (2.2) that $W(E_A)$ contains all points of the form

$$t = \frac{\ln \frac{1-\rho}{\rho} + (2k + 1)\pi i}{a + bi}, \quad \rho \in (0, 1),\ k \in \mathbb{Z}.$$

We get $t = (2k + 1)\pi/b$ by taking $\rho \in (0, 1)$ with $\ln \frac{1-\rho}{\rho} = (2k + 1)\pi\frac{a}{b}$. $\square$

It is well known that the solution to the differential equation

$$\frac{dx}{dt} = Ax$$

is given by $x(t) = e^{At}x(0)$. The above proposition says that for any matrix A with a nonzero imaginary part for the difference of any two eigenvalues, $W(E_A)$ contains some $t > 0$. It follows that $\hat{x}^* e^{At}\hat{x} = 0$ for some nonzero $\hat{x} \in \mathbb{C}^n$. Therefore, for $x(0) = \hat{x}$, we have $x(0)^*x(t) = \hat{x}^* e^{At}\hat{x} = 0$, i.e., $x(t)$ and $x(0)$ are orthogonal.

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We now show that there are no isolated points in $W(E_A)$.

PROPOSITION 5. *Suppose* $A \in \mathbb{C}^{n\times n}$ *is not a scalar matrix. Then* $W(E_A)$ *does not have any isolated points.*

Proof. Let t_0 be a point in $W(E_A)$. We separate two cases. First $\mathrm{Int}(F(e^{t_0 A})) = \emptyset$ and then $\mathrm{Int}(F(e^{t_0 A})) \neq \emptyset$.

Let $t_0 \in W(E_A)$ and suppose that $\mathrm{Int}(F(e^{t_0 A})) = \emptyset$. Then $F(e^{t_0 A})$ is a line segment passing through the origin. By Theorem 1.6.3 of [6], the endpoints of $F(e^{t_0 A})$ are eigenvalues of $e^{t_0 A}$, say $e^{t_0\lambda_1}$ and $e^{t_0\lambda_2}$, where λ_1 and λ_2 are eigenvalues of A. So the origin cannot be an endpoint of the line segment, and there is an $r \in (0,1)$ such that

$$0 = re^{t_0\lambda_1} + (1-r)e^{t_0\lambda_2}, \quad \frac{-r}{1-r} = e^{t_0(\lambda_2-\lambda_1)} < 0. \tag{3.4}$$

We will show that $0 \in F(e^{(t_0+\varepsilon e^{i\theta})A})$ for a suitable $\theta \in [0, 2\pi]$ and all $\varepsilon > 0$. We just need to show the existence of $s \in (0,1)$ such that $se^{(t_0+\varepsilon e^{i\theta})\lambda_1} + (1-s)e^{(t_0+\varepsilon e^{i\theta})\lambda_2} = 0$, i.e., $-s/(1-s) = e^{(t_0+\varepsilon e^{i\theta})\lambda_2}/e^{(t_0+\varepsilon e^{i\theta})\lambda_1} = e^{t_0(\lambda_2-\lambda_1)}e^{\varepsilon e^{i\theta}(\lambda_2-\lambda_1)}$. For this, we need to show that $e^{t_0(\lambda_2-\lambda_1)}e^{\varepsilon e^{i\theta}(\lambda_2-\lambda_1)}$ is a negative real number. Since $e^{t_0(\lambda_2-\lambda_1)}$ is a negative real number by (3.4), we just need to choose θ such that $e^{\varepsilon e^{i\theta}(\lambda_2-\lambda_1)}$ is a positive real number, so we choose θ such that $e^{i\theta}(\lambda_2-\lambda_1) = |\lambda_2-\lambda_1|$.

Now let $t_0 \in W(E_A)$ and suppose that $\mathrm{Int}(F(e^{At_0})) \neq \emptyset$. Since $t_0 \in W(E_A)$, we have $0 \in F(e^{At_0})$, so $0 = x_0^* e^{At_0} x_0$ for some $x_0 \in \mathbb{C}^n$ with $\|x_0\|_2 = 1$. Since $\mathrm{Int}(F(e^{At_0}) \neq \emptyset$ and $F(e^{At_0})$ is a convex set, we can take $w_1, w_2 \in F(e^{At_0})$ such that $w_1 = re^{i\theta_1}, w_2 = re^{i\theta_2}$, where $r > 0$ and the arguments θ_1 and θ_2 may be negative and satisfy $0 < \theta_2 - \theta_1 < \frac{\pi}{2}$. We also have

$$w_1 = x_1^* e^{At_0} x_1, \quad w_2 = x_2^* e^{At_0} x_2,$$

for suitable $x_1, x_2 \in \mathbb{C}^n$ with $\|x_1\|_2 = \|x_2\|_2 = 1$.

For $\epsilon > 0$, $\delta \in [0, 2\pi)$, and $j = 0, 1, 2$, let

$$w_j(\epsilon, \delta) = x_j^* e^{A(t_0+\epsilon e^{i\delta})} x_j.$$

We have

$$w_0(\epsilon,\delta) = x_0^* e^{At_0} e^{A\epsilon e^{i\delta}} x_0 = x_0^* e^{At_0} \sum_{k=0}^{\infty} \frac{1}{k!}(A\epsilon e^{i\delta})^k x_0 = \sum_{k=0}^{\infty} \frac{1}{k!}\epsilon^k e^{ik\delta} x_0^* e^{At_0} A^k x_0.$$

If $x_0^* e^{At_0} A^k x_0 = 0$ for all $k \geq 0$, then we would have $x_0^* e^{A(t_0+\epsilon e^{i\delta})} x_0 = 0$ for all $\epsilon > 0$ and all $\delta \in [0, 2\pi)$, and in particular $x_0^* e^{A0} x_0 = x_0^* x_0 = 0$, which is impossible.

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ELA
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We now let $k \geq 1$ be the smallest integer such that $x_0^* e^{At_0} A^k x_0 \neq 0$. Then

$$w_0(\epsilon, \delta) = \frac{1}{k!} \epsilon^k e^{ik\delta} x_0^* e^{At_0} A^k x_0 + o(\epsilon^k).$$

For ϵ sufficiently small, we have $|w_1(\epsilon, \delta)| \geq \frac{r}{2}$ and $|w_2(\epsilon, \delta)| \geq \frac{r}{2}$, and

$$|\arg w_1(\epsilon, \delta) - \theta_1| \leq \frac{1}{4}(\theta_2 - \theta_1), \quad |\arg w_2(\epsilon, \delta) - \theta_2| \leq \frac{1}{4}(\theta_2 - \theta_1).$$

Choose δ such that

$$e^{ik\delta} x_0^* e^{At_0} A^k x_0 = -|x_0^* e^{At_0} A^k x_0| e^{i(\theta_1 + \theta_2)/2}.$$

Then for $\epsilon > 0$ sufficiently small, 0 is inside the triangle with vertices $w_j(\epsilon, \delta)$. So $0 \in F(e^{A(t_0 + \epsilon e^{i\delta})})$. Thus, $t_0 + \epsilon e^{i\delta} \in W(E_A)$ for the chosen δ and all $\epsilon > 0$ sufficiently small. So t_0 is not an isolated point. $\square$

From the proof we know that for any $t \in W(E_A)$, $W(E_A)$ also contains a line segment ℓ_t with t as one of the two endpoints. This means that $W(E_A)$ cannot contain a circle disjoint from the rest of $W(E_A)$. It also follows that $W(E_A) = \bigcup_{t \in W(E_A)} \ell_t$. Many of the line segments ℓ_t will merge into one line segment, but $W(E_A)$ is still the union of *infinitely many* line segments by Proposition 1.

In the next example, we show that a connected component of $W(E_A)$ is not convex in general.

Example 1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then from the proof of Theorem 1 we already know that $W(E_A) = \{t \in \mathbb{C} : |t| \geq 2\}$, which is connected but not convex.

We now examine when a point t_0 is an interior point of $W(E_A)$.

Proposition 6. *Suppose $A \in \mathbb{C}^{n \times n}$ is not a scalar matrix. If the origin is an interior point of $F(e^{At_0})$, then t_0 is an interior point of $W(E_A)$.*

Proof. We use $D(c, r)$ to denote the open disk with center c and radius r. Suppose $0 \in \mathrm{Int}(F(e^{At_0}))$. Then there exists $\epsilon > 0$ such that $D(0, 2\epsilon) \subseteq F(e^{At_0})$. In particular, for $j = 0, 1, 2$, $w_j = \epsilon e^{2j\pi/3} = x_j^* e^{At_0} x_j$ for some $x_j \in \mathbb{C}^n$ with $\|x_j\|_2 = 1$. Note that 0 is inside the triangle with vertices $w_j = x_j^* e^{At_0} x_j$ and that the functions $f_j(t) = x_j^* e^{At} x_j$ are continuous at t_0. Then there exists $\delta > 0$ such that for all $t \in D(t_0, \delta)$, 0 is inside the triangle with vertices $\widetilde{w}_j = x_j^* e^{At} x_j$, so $0 \in F(e^{At})$. Thus, $D(t_0, \delta) \subseteq W(E_A)$. $\square$

Corollary 1. *Suppose $A \in \mathbb{C}^{n \times n}$ is not a scalar matrix. If t_0 is a boundary point of $W(E_A)$, then the origin is a boundary point of $F(e^{At_0})$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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We end this section by one more observation about $W(E_A)$.

PROPOSITION 7. *Suppose $A \in \mathbb{C}^{n\times n}$ is not a scalar matrix and $t_0 \in W(E_A)$. Then*

(a) $-t_0 \in W(E_A)$,
(b) $\overline{t_0} \in W(E_{A^*})$.

Proof. For (a), it suffices to show that for every invertible matrix B, $0 \in F(B)$ implies that $0 \in F(B^{-1})$. But if $0 \in F(B)$ then $0 \in F(B^*)$, therefore $x_0^* B^* x_0 = 0$ for some unit $x_0 \in \mathbb{C}^n$. Then,

$$0 = x_0^* B^* x_0 = x_0^* B^* B^{-1} B x_0 = (Bx_0)^* B^{-1}(Bx_0),$$

so, $0 \in F(B^{-1})$ since $Bx_0 \neq 0$.

For (b), we have for some unit $x_0 \in \mathbb{C}^n$

$$0 = x_0^* e^{t_0 A} x_0 = x_0^* e^{\overline{t_0} A^*} e^{-\overline{t_0} A^*} e^{t_0 A} x_0 = (e^{t_0 A} x_0)^* e^{-\overline{t_0} A^*} (e^{t_0 A} x_0).$$

Therefore, $-\overline{t_0} \in W(E_{A^*})$ and the conclusion follows from (a). $\square$

Part (a) means that $W(E_A)$ is symmetric with respect to the origin.

4. The cases where A is Hermitian or skew-Hermitian. We have seen earlier that $W(E_A)$ contains infinitely many parallel straight lines for any non-scalar matrix A. When A is Hermitian or skew-Hermitian, we can show that $W(E_A)$ *consists of* infinitely many parallel straight lines. In what follows, we only provide proofs for the case where A is Hermitian. If A is skew-Hermitian, then iA is Hermitian and $W(E_A) = iW(E_{iA})$ by Proposition 3, so corresponding results for the skew-Hermitan case will follow readily.

PROPOSITION 8. *Suppose $A \in \mathbb{C}^{n\times n}$ is not a scalar matrix.*

(a) *If A is Hermitian then $W(E_A) \cap \mathbb{R} = \emptyset$ and $t \in W(E_A)$ if and only if $t + \alpha \in W(E_A)$ for all $\alpha \in \mathbb{R}$.*
(b) *If A is skew-Hermitian then $W(E_A) \cap i\mathbb{R} = \emptyset$ and $t \in W(E_A)$ if and only if $t + i\alpha \in W(E_A)$ for all $\alpha \in \mathbb{R}$.*

Proof. Let A be Hermitian. Then for $r \in \mathbb{R}$ the matrix exponential e^{rA} is positive definite. Therefore, $r \notin W(E_A)$. For any $t \in W(E_A)$ there is a nonzero vector x_t such that

$$0 = x_t^* e^{At} x_t = x_t^* e^{\frac{-\alpha}{2}A} e^{A(t+\alpha)} e^{\frac{-\alpha}{2}A} x_t = w_t^* e^{A(t+\alpha)} w_t,$$

where $w_t = e^{\frac{-\alpha}{2}A} x_t$ is nonzero. Thus, $t + \alpha \in W(E_A)$. $\square$

PROPOSITION 9. *Suppose $A \in \mathbb{C}^{n\times n}$ is not a scalar matrix.*

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Volume 31, pp. 633-645, October 2016

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(a) *If A is Hermitian then $W(E_A)$ is symmetric with respect to the real axis.*
(b) *If A is skew-Hermitian then $W(E_A)$ is symmetric with respect to the imaginary axis.*

Proof. The conclusion in (a) follows directly from Proposition 8 (a) and Proposition 7 (a). ◻

Propositions 8 and 9 reveal what $W(E_A)$ looks like for Hermitian or skew-Hermitian A. In particular, if A is Hermitian, $W(E_A)$ consists of disjoint complex stripes parallel to the real axis of the form

$$\{t \in \mathbb{C} : \gamma \leq \mathrm{Im}(t) \leq \delta,\ \gamma\delta > 0\}.$$

Observe that γ and δ should have the same sign since $W(E_A)$ has no intersection with real axis. Moreover, for each stripe in the upper plane, $W(E_A)$ contains its symmetric stripe in the lower plane.

However, it is possible to have $\gamma = \delta$. In that case, the corresponding stripes are reduced to straight lines parallel to the real axis. Indeed we have the following result.

Proposition 10. *Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian and has only two distinct eigenvalues λ and μ. Then*

$$W(E_A) = \{t \in \mathbb{C} : \mathrm{Im}(t) = (2k+1)\pi/(\lambda - \mu),\ k \in \mathbb{Z}\}.$$

Proof. Let U be unitary such that $A = U^*DU$ and D is diagonal with diagonal entries equal to λ or μ. Then $F(e^{At}) = F(e^{Dt})$ is the line segment connecting $e^{\lambda t}$ and $e^{\mu t}$, by Theorem 1.6.8 of [6]. Thus, $t = a + bi \in W(E_A)$ if and only if $0 = se^{\lambda t} + (1-s)e^{\mu t} = (se^{(\lambda-\mu)t} + (1-s))e^{\mu t}$ for some $s \in (0,1)$, if and only if $e^{(\lambda-\mu)t} = e^{(\lambda-\mu)a}e^{(\lambda-\mu)bi}$ is a negative number, if and only if $e^{(\lambda-\mu)bi} = -1$, if and only if $(\lambda-\mu)b = (2k+1)\pi$ for some $k \in \mathbb{Z}$. ◻

We also have examples showing that for an Hermitian matrix A having three distinct eigenvalues, $W(E_A)$ can have stripes with zero width and stripes with nonzero width. In Example 2 of next section, some stripes with nonzero width are displayed.

For Hermitian matrices, the lower bound in (3.1) can be improved.

Proposition 11. *Let A be an $n \times n$ Hermitian and non-scalar matrix. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A (so $\lambda_1 < \lambda_n$ since A is not scalar). Then for all $t \in W(E_A)$,*

$$|t| \geq \frac{\pi}{\lambda_n - \lambda_1},$$

and the lower bound is sharp.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Proof. By Proposition 8, $\min_{t\in W(E_A)} |t|$ is achieved at $t_a = ia \in W(E_A)$ for some $a > 0$, and ia is a boundary point of $W(E_A)$. It follows from Proposition 6 that 0 is a boundary point of $F(e^{iaA})$. Since A is Hermitian, e^{iaA} is normal. By Theorem 1.6.8 of [6] and the fact that 0 is a boundary point of $F(e^{iaA})$, we know that $0 = se^{ia\lambda_p}+(1-s)e^{ia\lambda_q}$ for two distinct eigenvalues λ_p and λ_q of A, and some $s \in (0,1)$. Thus, $e^{ia(\lambda_p-\lambda_q)}$ is a negative real number, so $a(\lambda_p - \lambda_q) = (2k+1)\pi$ for some $k \in \mathbb{Z}$. Now, $a = \frac{|2k+1|}{|\lambda_p-\lambda_q|}\pi \geq \frac{\pi}{\lambda_n-\lambda_1}$. Therefore, $|t| \geq \frac{\pi}{\lambda_n-\lambda_1}$ for all $t \in W(E_A)$. The lower bound is sharp since $t_0 = \frac{\pi i}{\lambda_n-\lambda_1} \in W(E_A)$. In fact, we have $e^{t_0(\lambda_n-\lambda_1)} = -1$ and thus $\frac{1}{2}e^{t_0\lambda_1} + \frac{1}{2}e^{t_0\lambda_n} = 0$, so $0 \in F(e^{At_0})$. $\square$

Note that we have $\inf\limits_{k\in\mathbb{C}} \|A-kI_n\|_2 = \frac{1}{2}(\lambda_n-\lambda_1)$ in Proposition 11. So the constant $\ln 2$ in (3.1) has been improved to $\pi/2$ when A is Hermitian.

5. Some illustrative examples. In this section, we give three matrices and plot the numerical range of the corresponding matrix exponential function for each case. The plots are obtained using an inverse numerical range Matlab file based on the algorithm described in [3].

EXAMPLE 2. Consider the Hermitian matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

with eigenvalues $\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 4$. We plot $W(E_A)$ within $[-4,4] \times [-4,4]$ in Figure 5.1. We can see 6 stripes of $W(E_A)$, symmetric about the real axis. The distance between the origin and $W(E_A)$ is seen to be around 0.52, consistent with the exact distance of $\pi/6$ obtained in Proposition 11.

EXAMPLE 3. Consider the unitary matrix

$$A = \begin{bmatrix} -\sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

We plot $W(E_A)$ within $[-7,7] \times [-7,7]$ in Figure 5.2. The figure shows that $W(E_A)$ contains some points on the real axis. It also suggests that $W(E_A)$ is symmetric about the origin and contains infinitely many parallel straight lines. All these are consistent with our theoretical results.

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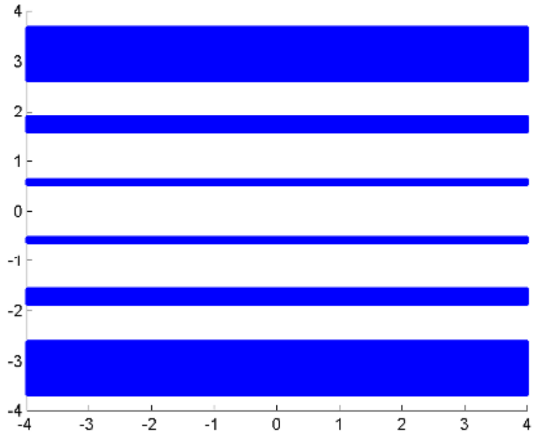

FIG. 5.1.

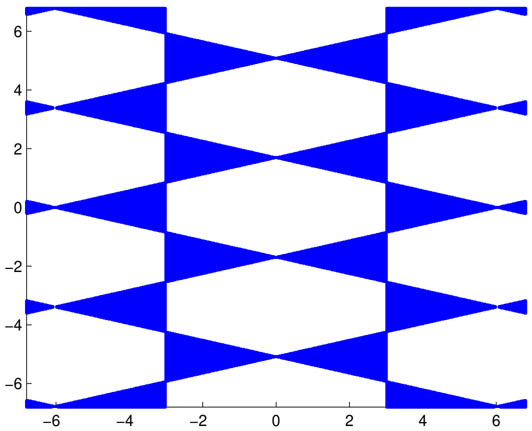

FIG. 5.2.

EXAMPLE 4. Consider the randomly generated complex matrix

$$A = \begin{bmatrix} 0.8147 + 0.9649i & 0.9134 + 0.9572i & 0.2785 + 0.1419i \\ 0.9058 + 0.1576i & 0.6324 + 0.4854i & 0.5469 + 0.4218i \\ 0.1270 + 0.9706i & 0.0975 + 0.8003i & 0.9575 + 0.9157i \end{bmatrix}.$$

We plot $W(E_A)$ within $[-2\|A\|_2, 2\|A\|_2] \times [-2\|A\|_2, 2\|A\|_2]$ in Figure 5.3. The figure shows that $W(E_A)$ contains some points on the real axis. It also suggests that $W(E_A)$

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ELA
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is symmetric about the origin and contains infinitely many parallel straight lines. All these are consistent with our theoretical results.

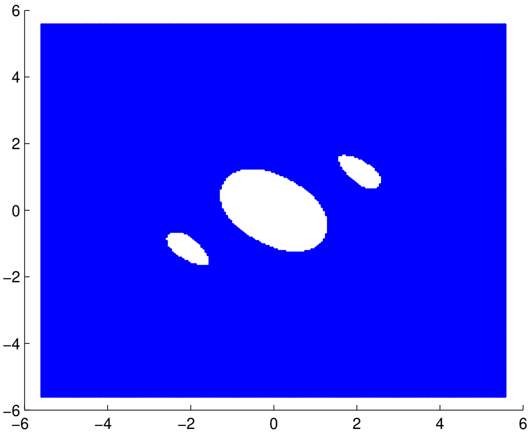

FIG. 5.3.

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