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NUMERICAL RANGES OF TENSOR PRODUCTS OF OPERATORS

TEISHIRÔ SAITÔ

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1. In a recent paper [1], A. Brown and C. Pearcy proved that if A and B are bounded linear operators on a Hilbert space, then $\sigma(A\otimes B) = \sigma(A)\cdot\sigma(B)$, where $\sigma(\cdot)$ denotes the spectrum of an operator. In connection with this theorem, we shall discuss an analogous relation among the numerical ranges of A, B and $A\otimes B$.

2. In the sequel, spaces under consideration will be complex Hilbert spaces, and all operators will be assumed to be bounded and linear. The spectrum of any operator T is denoted, as above, by $\sigma(T)$ and the numerical range of T is denoted by $W(T)$. For an arbitrary set S in the complex plane, $\mathrm{co}(S)$ means the convex hull of S and we write $\overline{S}$ for the closure of S. It is known that $\mathrm{co}(\sigma(T))$ is closed. Motivated by the above theorem by A. Brown and C. Pearcy, the following questions are naturally raised.

(I) Does the following relation (*) hold for any operators A and B?

$$(*)\qquad \overline{W(A\otimes B)} = \overline{\mathrm{co}}(W(A)\cdot W(B)).$$

(II) If the relation (*) is not always true, when does the relation (*) hold?

3. In this section we shall prove the following two propositions.

PROPOSITION 1. *Let A and B be operators on Hilbert space H. Then the condition* $\overline{W(A\otimes B)} = \mathrm{co}(\sigma(A\otimes B))$ *implies the relation* (*).

PROPOSITION 2. *There exist operators A and B on some Hilbert space such as* $\overline{W(A\otimes B)} \supsetneqq \overline{\mathrm{co}}(W(A)\cdot W(B))$.

To prove Proposition 1, we shall show the following lemma.

LEMMA 1. *For arbitrary operators A and B on a Hilbert space H*

$$\overline{W(A \otimes B)} \supseteqq \overline{\mathrm{co}}(W(A) \cdot W(B)).$$

PROOF. If $\lambda \in W(A)$ and $\mu \in W(B)$, there exist unit vectors $x, y \in H$ such as $\lambda = (Ax, x)$ and $\mu = (By, y)$. Thus we have

$$((A \otimes B)\, x \otimes y, x \otimes y) = (Ax, x)(By, y) = \lambda\mu$$

and so $W(A \otimes B) \supseteqq W(A) \cdot W(B)$. Since $\overline{W(A \otimes B)}$ is convex and closed, the conclusion of the lemma is clear.

PROOF OF PROPOSITION 1. By the theorem of A. Brown and C. Pearcy

$$\mathrm{co}(\sigma(A \otimes B)) = \mathrm{co}(\sigma(\mathrm{A}) \cdot \sigma(B)) \subseteqq \mathrm{co}(\mathrm{co}(\sigma(A)) \cdot \mathrm{co}(\sigma(B))).$$

On the other hand, we have

$$\mathrm{co}(\sigma(A)) \cdot \mathrm{co}(\sigma(B))) \subseteqq \overline{\mathrm{co}}(\overline{W(A)} \cdot \overline{W(B)}) = \overline{\mathrm{co}}(W(A) \cdot W(B))$$

since $\mathrm{co}(\sigma(T)) \subseteqq \overline{W(T)}$ for any operator T. Hence, by Lemma 1 and the hypothesis of the proposition, the relation (*) holds.

As a consequence of Proposition 1, we obtain the following result.

COROLLARY. *If both A and B are hyponormal operators*(*), *the relation* (*) *holds.*

PROOF. Since $A^*A - AA^* \geqq 0$ and $B^*B - BB^* \geqq 0$, we have

$$(A \otimes B)^*(A \otimes B) - (A \otimes B)(A \otimes B)^*$$
$$= (A^*A - AA^*) \otimes B^*B + AA^* \otimes (B^*B - BB^*) \geqq 0.$$

Thus $A \otimes B$ is also a hyponormal operator and so $A \otimes B$ satisfies the condition $\overline{W(A \otimes B)} = \mathrm{co}(\sigma(A \otimes B))$ by [2: Corollary 1]. Hence the assertion follows from Proposition 1.

To prove Proposition 2, we shall construct an example. Let A be an operator on a two-dimensional Hilbert space H with the matrix representation $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and let $B = A^*$. Then it is easy to show the following lemma and we omit the proof.

(*) An operator T on a Hilbert space is called to be hyponormal if $T^*T \geqq TT^*$.

LEMMA 2. $W(A) = W(B) = \{\lambda : |\lambda| \leqq 1/2\}$.

From Lemma 2 we obtain

LEMMA 3. $\overline{\mathrm{co}}(W(A)\cdot W(B)) \subseteqq \{\lambda : |\lambda| \leqq 1/4\}$.

PROOF OF PROPOSITION 2. By Lemma 3 it is sufficient to show that $1/2 \in W(A\otimes B)$. Since the matrix corresponding to B is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the matrix representation of $A\otimes B$ is $\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$. For a unit vector $\Phi = \begin{pmatrix} x \\ y \end{pmatrix} \in H \otimes H$, $((A\otimes B)\Phi, \Phi) = (Ax, y)$. Furthermore, for $x = \begin{pmatrix} 0 \\ 1/\sqrt{2} \end{pmatrix}$ and $y = \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}$ $\in H$, we have $(Ax, y) = 1/2$. Hence $1/2 \in W(A\otimes B)$ and the proof is completed.

ADDENDUM. Under the conditions $\overline{W(A)} = \mathrm{co}(\sigma(A))$ and $\overline{W(B)} = \mathrm{co}(\sigma(B))$, the relation (*) is equivalent to the relation $\overline{W(A\otimes B)} = \mathrm{co}(\sigma(A\otimes B)$ which is not always insured by the conditions $\overline{W(A)} = \mathrm{co}(\sigma(A))$ and $\overline{W(B)} = \mathrm{co}(\sigma(B))$. In fact, let X be the operator $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ as in the proof of Proposition 2 and Y an operator such as $\sigma(Y) = \overline{W(Y)} = \{\lambda ; |\lambda| \leqq 1/2\}$, then we have $\overline{W(S\otimes T)} \supsetneqq \mathrm{co}(\sigma(S\otimes T))$ where $S = X \oplus Y$ and $T = S^*$, while $\overline{W(S)} = \mathrm{co}(\sigma(S)) = \{\lambda : |\lambda| \leqq 1/2\}$ and $\overline{W(T)} = \mathrm{co}(\sigma(T)) = \{\lambda : |\lambda| \leqq 1/2\}$.

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THE COLLGE OF GENERAL EDUCATION
TÔHOKU UNIVERSITY,
SENDAI, JAPAN.