Numerical schemes for discontinuous value functions of Optimal Control

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Abstract - In this paper we explain that various (possibly discontinuous) value functions for optimal control problem under state-constraints can be approached by a sequence of value functions for suitable discretized systems. The key-point of this approach is the characterization of epigraphs of the value functions as suitable viability kernels. We provide new results for estimation of the convergence rate of numerical schemes and discuss conditions for the convergence of discrete optimal controls to the optimal control for the initial problem.

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1 Introduction

Let us consider the following control system

$$x'(t) = f(x(t), u(t)) \tag{1}$$

where the state space x belongs to $\boldsymbol{R}^N$ and $u(\cdot) : \boldsymbol{R} \mapsto U$ is a measurable function into a compact set U of some finite dimensional vector space. Let $\mathcal{U}(t_0, T)$ be the set of measurable controls from $[t_0, T]$ to U (or $\mathcal{U}(t_0)$ when $T = +\infty$) and $x(\cdot, t_0, x_0, u)$ the solution to (1) starting at $x_0 = x(t_0)$. We are interested in three kinds of value functions.

- **Minimal time** for target problem: Let $C \subset \boldsymbol{R}^N$ be a closed target. Then the minimal time function is defined by

$$\Theta_C(x_0) := \inf\{\tau > 0 \,|\, \exists\, u \in \mathcal{U}(0),\ x(\tau, 0, x_0, u) \in C\} \tag{2}$$

- **Bolza Problem** Let $g : \boldsymbol{R} \times \boldsymbol{R}^N \times U \mapsto \boldsymbol{R}$ and $\phi : \boldsymbol{R}^N \mapsto \boldsymbol{R}$ be two given functions. We define

$$V(t_0, x_0) := \inf_{u \in \mathcal{U}(t_0,T)} \int_{t_0}^{T} g(t, x(t, t_0, x_0, u), u(t))dt + \phi(x(T, t_0, x_0, u)) \tag{3}$$

- **Infinite Horizon Problem** Let $m : \boldsymbol{R}^N \times U \mapsto \boldsymbol{R}$ be a function and λ a positive real number

$$W(x_0) := \inf_{u \in \mathcal{U}(0)} \int_0^{+\infty} e^{-\lambda t} m(x(t, 0, x_0, u), u(t))dt \tag{4}$$

The purpose of this paper is to give a unified approach to the approximation of the above value functions and to study the rate of convergence of the proposed numerical schemes.

There is a huge literature on the approximation of value function and on the rate of convergence of the numerical schemes. Let us quote in particular [5], [6], [7], [9], [13], [15] and the references therein. These works are based on the discretization of Hamilton-Jacobi Equations. They are only concerned with continuous value functions.

In this paper, we are mainly interested in the approximation of lower semi-continuous value functions. Although these problems are often encountered in practice, there are very few works on the subject: Let us quote the articles of Bardi, Bottacin and Falcone [4] based on the discretization of some Hamilton-Jacobi Equation, of Aubin and Frankowska [2] and of the author's

[10], [11], based on viability methods. In these works, there is no result concerning the rate of convergence of the approximating schemes.

The main purpose of this paper is to give some estimations of this rate of convergence. We also provide a unified approach of the approximation of the above value functions by using viability methods. The basic idea is the following: As in [2], [3], [10], we establish that the epigraphs of the value functions are suitable viability kernels. Then we use the numerical schemes computing viability kernels (the so-called "viability kernel algorithm", see [20], [18], [11]) for computing the value functions. Although no rate of convergence for the viability kernel algorithm is known in general, we show that in the particular case of the approximation of value functions, it is possible to establish such a rate of convergence. We finally prove that this method also allows to approximate the optimal controls.

Let us briefly describe how this paper is organized. The first section is devoted to basic statements of viability theory and to the characterization of epigraphs of value functions in terms of viability kernels. The second section concerns algorithms for computing these functions and results about rate of convergence. The third section is devoted to some questions concerning the approximation of optimal control.

2 Characterization of value functions

2.1 Preliminaries

As usual, we associate with (1) the differential inclusion

$$(5) \qquad x'(t) \in F(x(t)) \text{ for almost every } t \geq 0$$

where $F : \boldsymbol{R}^N \rightsquigarrow \boldsymbol{R}^N$ is the set-valued map defined by $F(x) := \bigcup_{u \in U} f(x,u)$. We say that F is Marchaud, if F upper semi-continuous with compact convex nonempty values and linear growth (these conditions are satisfied in particular if f is continuous, affine with respect to u and has a linear growth). We denote by $S_F(x_0)$ the set of absolutely continuous solutions to (5) starting at $x(0) = x_0$.

If K is a closed subset of $\boldsymbol{R}^N$, a solution to (5) satisfying $x(t) \in K$ for any $t \geq 0$ is called a *viable* solution in K. If from any point of a set D starts at least one viable solution in D, we say that D is viable under F.

The viability kernel - denoted by $Viab_F(K)$ - of a closed set K for a dynamic F is the set of initial condition $x_0 \in K$ starting from which a solution to (5) viable in K exists:

$$(6) \qquad \mathrm{Viab}_F(K) := \left\{ x_0 \in K \mid \exists x(\cdot) \in S_F(x_0) \text{ such that } x(t) \in K, \forall t \geq 0 \right\}$$

When F is Marchaud, then $Viab_F(K)$ is the largest closed viable subset of K (cf [3]).

Throughout this paper, B shall denote the closed unit ball of the space $\boldsymbol{R}^N$ and $d_K(x)$ the distance of a point x to a set K.

We shall need the following assumptions on the cost of value functions:

Assumptions H1:

- $(x,u) \mapsto f(x,u)$ is continuous, $\ell-$Lipschitz with respect to x and affine with respect to u.
- $(t,x,u) \mapsto g(t,x,u)$ is continuous, $\ell-$Lipschitz with respect to x and convex with respect to u and bounded by M:

$$\forall (t,x,u) \in \boldsymbol{R}^{N+1} \times U, \ |g(t,x,u)| \leq M$$

- $x \mapsto \phi(x)$ is a lower semi-continuous function bounded from below:

$$\forall x \in \boldsymbol{R}^N, \ \phi(x) \geq -M$$

- $(x,u) \mapsto m(x,u)$ is continuous, $\ell-$Lipschitz with respect to x, convex with respect to u and bounded by M:

$$\forall (t,x,u) \in \boldsymbol{R}^{N+1} \times U, \ |m(x,u)| \leq M$$

Remark — Under the previous assumptions, it is known that W is Hölder continuous. Moreover,

$$\forall x \in \boldsymbol{R}^N,\ |W(x)| \leq \frac{M}{\lambda}, \qquad |W(x) - W(x')| \leq C\|x - x'\|^\gamma$$

for some constant C, where $\gamma = 1$ if $\lambda > \ell$, $\gamma = \frac{\lambda}{\ell}$ if $\lambda < \ell$ and γ is any arbitrary number less than 1 if $\lambda = \ell$. $\square$

2.2 Characterization of epigraphs of value functions

For a real-valued function $q : \boldsymbol{R}^N \mapsto \boldsymbol{R}$, we denote its epigraph by $Epi(q) := \{(x, y) \,|\, q(x) \leq y\}$ and by $Epi^T(q)$ its $T-$epigraph:

$$Epi^T(q) := \{\, (x, y) \in\ \boldsymbol{R}^N \times [-T, T] \,|\, q(x) \leq y \,\}$$

The epigraphs of the value functions are suitable viability kernels as follows:

Theorem 2.1 *Let C be closed subset of $\boldsymbol{R}^N$. Suppose assumptions* **H1** *holds true. Then*
1- $Epi(\Theta_C) = Viab_\Phi(\boldsymbol{R}^N \times \boldsymbol{R}^+)$ *where*

$$\Phi(x, y) := \begin{cases} F(x) \times \{-1\} \text{ if } \ x \in X \backslash C \\ \overline{Co}((F(x) \times \{-1\}) \cup (0, 0)) \text{ otherwise} \end{cases}$$

2- $Epi(V) = Viab_\Psi([0, T] \times \boldsymbol{R}^{N+1})$ *where*

$$\Psi(t, x, \varepsilon) := \begin{cases} \{(1, f(t, x, u), \theta) \mid -M \leq \theta \leq -g(t, x, u),\ u \in U\} \\ \qquad\qquad if\ t \neq T \text{ or } (t = T \ \&\ \varepsilon < \phi(x)) \\ \overline{Co}\,\{\{0\}\ \cup\ \{(1, f(t, x, u), \theta) \mid -M \leq \theta \leq -g(t, x, u)u \in U,\ \}\} \\ \qquad\qquad \text{otherwise} \end{cases}$$

3- $Epi^T(W) = Viab_\Lambda\left(\boldsymbol{R}^N \times [-T, T]\right)$ *where $2M/\lambda$ and*

$$\Lambda(x, \varepsilon) := \{(f(x, u), \theta) \mid u \in U,\ \lambda\varepsilon - M \leq \theta \leq \lambda\varepsilon - m(x, u)\}\ .$$

Furthermore the functions Θ_C and V are lower semi-continuous while W is Hölder continuous, and there exist optimal controls.

Proof — The lower semi-continuity of the value functions and the existence of optimal control can be classically obtained using **H1**. The part 1 of the Theorem can be found in [10], the part 3 is very similar to an approach developed in [2]. So we only prove the part 2 which is one of the new results of the present paper.

First note that Ψ is a Marchaud map.

Let $(t_0, x_0) \in [0, T] \times \boldsymbol{R}^N$, with $t_0 < T$, $\theta_0 \geq 0$, $x(\cdot)$ be an optimal solution and $u(\cdot)$ be an associated control. Let $\varepsilon(\cdot)$ be a solution to the differential equation:

$$\begin{cases} \varepsilon'(t) = -g(t, x(t), u(t)) \\ \varepsilon(t_0) = \theta_0 + V(t_0, x_0) = \theta_0 + \int_{t_0}^T g(\sigma, x(\sigma), u(\sigma))d\sigma + \phi(x(T)) \end{cases}$$

Then $\varepsilon(t) = \theta_0 + \int_t^T g(\sigma, x(\sigma), u(\sigma))d\sigma + \phi(x(T))$, so that $\varepsilon(T) = \theta_0 + \phi(x(T))$. Let us now define $(t_1(\cdot), x_1(\cdot), \varepsilon_1(\cdot))$ by

$$\begin{cases} t_1(s) := t_0 + s & \text{if } s \leq T - t_0 & \text{and} & t_1(s) := T \text{ otherwise} \\ x_1(s) := x(t_0 + s) & \text{if } s \leq T - t_0 & \text{and} & x_1(s) := x(T) \text{ otherwise} \\ \varepsilon_1(t_0 + s) := \varepsilon(s) & \text{if } s \leq T - t_0 & \text{and} & \varepsilon_1(s) := \theta_0 + \phi(x(T)) \text{ otherwise} \end{cases}$$

Then $(t_1(\cdot), x_1(\cdot), \varepsilon_1(\cdot))$ is a solution to the differential inclusion for Ψ, starting from $(t_0, x_0, \theta_0 + V(t_0, x_0))$, which remains in $[0, T] \times \boldsymbol{R}^{N+1}$. Thus $(t_0, x_0, \theta_0 + V(t_0, x_0))$ belongs to the viability kernel of $[0, T] \times \boldsymbol{R}^{N+1}$ for Ψ. So $Epi(V)$ is contained in the viability kernel $Viab_\Psi([0, T] \times \boldsymbol{R}^{N+1})$.

Conversely, let $(t_0, x_0, \varepsilon_0) \in Viab_\Psi([0,T]\times \boldsymbol{R}^{N+1})$. There is at least one solution $(t(\cdot), x(\cdot), \varepsilon(\cdot))$ to the differential inclusion for Ψ, starting from $(t_0, x_0, \varepsilon_0)$, which remains in $[0,T]\times \boldsymbol{R}^{N+1}$. From the Measurable Selection Theorem [1] and the very definition of Ψ, there is some measurable $u(\cdot)$ and $\theta(\cdot)$ such that $(t(\cdot), x(\cdot), \varepsilon(\cdot))$ is, on $[t_0, T[$, a solution to

$$\begin{cases} t(s) = t_0 + s \\ x'(s) = f(s, x(s), u(s)) \\ \varepsilon(s) = \varepsilon_0 + \int_{t_0}^{t} \theta(\tau)d\tau \leq \varepsilon_0 - \int_{t_0}^{t} g(\tau, x(\tau), u(\tau))d\tau \end{cases}$$

At time T, one has necessarily $\varepsilon(T) \geq \phi(x(T))$ because, otherwise, the solution would leave $[0,T] \times \boldsymbol{R}^{N+1}$ from the very definition of the dynamic Ψ. Thus

$$\varepsilon_0 - \int_{t_0}^{T} g(\tau, x(\tau), u(\tau))d\tau \geq \phi(x(T))$$

So $\varepsilon_0 \geq V(t_0, x_0)$ and $(t_0, x_0, \varepsilon_0)$ belongs to the epigraph of V. The proof is complete.

Q.E.D.

3 Approximation of value functions

To study the numerical approximation we need the following

Assumption H2 : The function f is bounded:

$$\exists M > 0,\ \forall x \in \boldsymbol{R}^N,\ \forall u \in U,\ \|f(x,u)\| \leq M\ .$$

3.1 Viability Algorithms

In this section we summarize the ideas of viability algorithms ([3], [11]) and prove a new result which is the key point for the speed of convergence of approximation of value functions. Throughout this section the state space is denoted by X (it could be $\boldsymbol{R}^N, \boldsymbol{R}^{N+1}\ldots$ later on).

3.1.1 Approximation of the viability kernel

The basic idea is to approach the continuous dynamic by a discrete one. Let us define some notions for discrete systems.

Let $G : X \rightsquigarrow X$ be a Marchaud map and consider the discrete inclusion system of the form

(7) $$\forall n \geq 0\,,\ x_{n+1} \in G(x_n)\ .$$

To any initial position x_0, we associate a sequence $\overrightarrow{x} := (x_n)_{n\in\boldsymbol{N}}$ solution to (7). We denote by $\overrightarrow{S}_G(x_0)$ the set of solutions to (7) starting from x_0. A solution $\overrightarrow{x}$ to (7) satisfying $x_n \in K$ for any $n \in \boldsymbol{N}$ is called a *discrete viable* solution in K.

The discrete viability kernel of a closed set K for a dynamic G is the set of initial values $x_0 \in K$ of a sequence $(x_n)_n$ solution to (7) satisfying $x_n \in K$ for any $n \in \boldsymbol{N}$. If G is upper semi-continuous with compact values, this set is closed and is denoted by

(8) $$\overrightarrow{\mathrm{Viab}}_G(K) := \left\{ x_0 \in K \mid \begin{array}{l} \exists\ \overrightarrow{x}\ \text{ solution to (7) with} \\ x(0) = x_0 \text{ and } x_n \in K\ \forall n \in \boldsymbol{N} \end{array} \right\}$$

In the point of view of approximation, it is proved in [10], Proposition 2.14, that the set $\overrightarrow{\mathrm{Viab}}_G(K)$ is the limit of the following nonincreasing sequence of closed sets K^n:

(9) $$K^0 := K\ \&\ K^{n+1} := \{x \in K^n \mid G(x) \cap K^n \neq \emptyset\}\ .$$

The point is to approximate the viability kernel of K for F by discrete viability kernels of K for some suitable dynamics G_h.

Let F_h be an approximation of F satisfying the following properties (When f satisfies **H1** and **H2**, one can take $F_h(x) := F(x) + \ell M h B$):

$$
(10) \qquad \begin{array}{lll} i) & \forall h > 0, & F_h \text{ is Marchaud} \\ ii) & \forall h > 0, & \mathrm{Graph}(F_h) \subset \mathrm{Graph}(F) + o(h)B \\ iii) & \forall h > 0, & \forall x \in X,\ \bigcup_{\|y-x\| \le Mh} F(y) \subset F_h(x) \end{array}
$$

Then the suitable time discretization of the initial dynamical system (5) is

$$
(11) \qquad x^{n+1} \in G_h(x^n)
$$

where $G_h(x) := x + hF_h(x)$. This discretization allows us to approach the continuous viability kernel.

Theorem 3.1 *[11] Let F be a Marchaud set-valued map and K be a closed set. Consider any approximation $F_h(x)$ of F satisfying (10) and set $G_h(x) := x + hF_h(x)$. Then*

$$
(12) \qquad Viab_F(K) \subset \overrightarrow{\mathrm{Viab}}_{G_h}(K)
$$

and

$$
(13) \qquad \mathrm{Lim}_{h \longrightarrow 0} \overrightarrow{\mathrm{Viab}}_{G_h}(K) = \mathrm{Viab}_F(K)
$$

where Lim *denotes the Painlevé-Kuratowski limit (See [1]).*

3.1.2 Stability properties for viability kernels

The next result explains that the rate of convergence of $\overrightarrow{\mathrm{Viab}}_{G_h}(K)$ to $Viab_F(K)$ is related with some stability property of the viability kernel with respect to the dynamics.

For simplicity, we assume that F_h is also bounded by M (actually, from assumption (10-ii), it is bounded by $M + o(h)$), i.e.,

$$
(14) \qquad \forall x \in X,\ \forall h \in (0,1],\ \forall v \in F_h(x),\ \|v\| \le M\ .
$$

Proposition 3.2 *Let F be a Marchaud map, F_h satisfying (10) and (14), $G_h(x) := x + hF_h(x)$ and let $\tilde{F}_h : X \rightsquigarrow X$ satisfy the following properties:*

$$
(15) \qquad \left\{ \begin{array}{ll} i) & \tilde{F}_h \text{ is a Marchaud set-valued map} \\ ii) & \forall x \in X, \quad \displaystyle\bigcup_{\|y-x\| \le Mh} F_h(y) \subset \tilde{F}_h(x) \end{array} \right.
$$

Then,

$$
Viab_F(K) \subset \overrightarrow{\mathrm{Viab}}_{G_h}(K) \subset Viab_{\tilde{F}_h}(K + MhB)\ .
$$

To estimate the distance between $Viab_F(K)$ and $\overrightarrow{\mathrm{Viab}}_{G_h}(K)$, it is sufficient to estimate the distance between $Viab_F(K)$ and the perturbed viability kernel $Viab_{\tilde{F}_h}(K + MhB)$.

Proof of Proposition 3.2 : Let x_0 belong to $\overrightarrow{\mathrm{Viab}}_{G_h}(K)$ and (x_n) be a solution to

$$
x_{n+1} \in G_h(x_n), \qquad x_n \in K \qquad \forall n \in \boldsymbol{N}\ .
$$

Let $x(\cdot)$ be the interpolation of (x_n):

$$
x(t) = x_n + \left(\frac{t - nh}{h} \right) (x_{n+1} - x_n) \qquad \text{if } t \in [nh, (n+1)h)\ .
$$

Obviously, $x'(t) = \frac{x_{n+1} - x_n}{h} \in F_h(x_n)$ for almost every $t \ge 0$. From the boundness assumption (14), $\|x(t) - x_n\| \le Mh$. Thus $x'(t)$ belongs to $\tilde{F}_h(x(t))$ for almost every $t \ge 0$. So $x(\cdot)$ is a solution to the differential inclusion associated with $\tilde{F}_h$. Moreover, since $x(t)$ belongs to $K + MhB$ for every t, $x(\cdot)$ is a solution of the differential inclusion for $\tilde{F}_h$ starting from x_0 which remains in $K + MhB$. Thus x_0 belongs to $Viab_{\tilde{F}_h}(K + MhB)$.

Q.E.D.

3.2 Numerical Schemes and rate of convergence

The main result of this paper is the following

Theorem 3.3 *Let assumptions* **H1, H2** *hold true. Then*
Part I *Let T_h^n the sequence of function defined by*

$$\begin{cases} T_h^0(x) := 0 \text{ if } x \in K, \; T_h^0(x) := +\infty \text{ else}, \\ T_h^{n+1}(x) := h + \min_{v \in F(x),\, b \in B} T_h^n(x + hv + M\ell h^2 b) \text{ if } x \in K \backslash (C + MhB) \\ T_h^{n+1}(x) := T_h^n(x) \text{ otherwise}, \end{cases}$$

Then the sequence T_h^n is nondecreasing with respect to n and, if we set

$$T_h^\infty(x) := \lim_n T_h^n(x) \,,$$

we have, for any $T \geq 0$, for any $x \in \boldsymbol{R}^N$ with $\Theta_C(x) \leq T$,

$$\inf_{\|y - x_0\| \leq rh} \Theta_C(y) - rh \leq T_h^\infty(x) \leq \Theta_C(x_0)$$

for some positive constant $r = r(T, M, \ell)$.
Part II *Let us set $\phi_h(x) = \inf_{\|y-x\| \leq Mh} \phi(y) - Mh$. We define by induction the the numerical scheme V_h^n:*

$$\begin{cases} V_h^1(t,x) := -M(T+1) \text{ if } t \leq T - h, \; V_h^1(t,x) := \phi_h(y) \text{ otherwise} \\ \\ V_h^{n+1}(t,x) := \inf_{u \in U,\, b \in B} \left\{ \begin{array}{r} V_h^n(t+h, x + hf(t,x,u) + M\ell h^2 b) \\ +hg(t,x,u) - M\ell h^2 \end{array} \right\} \\ \qquad \text{if } t < T - h \\ V_h^{n+1}(t,x) := \min[\phi_h(x) \,, \inf_{u \in U,\, b \in B} \left\{ \begin{array}{r} V_h^n(t+h, x + hf(t,x,u) + M\ell h^2 b) \\ +hg(t,x,u) - M\ell h^2 \end{array} \right\}] \\ \qquad \text{otherwise} \end{cases}$$

Then, the sequence V_h^n is nondecreasing with respect to n and, if we set

$$V_h^\infty(t,x) := \lim_{n \to +\infty} V_h^n(t,x) \,,$$

we have

$$\forall (t,x) \in [0,T] \times \boldsymbol{R}^N, \; \inf_{\|y-x\| \leq rh} \{V(y)\} - rh \leq V_h^\infty(x) \leq V(x)$$

for some constant $r = r(T, M, \ell)$.
Part III *Let $BUC(\boldsymbol{R}^N)$ be the set of bounded uniformly continuous functions on $\boldsymbol{R}^N$ and T_h be the contractive operator on $BUC(\boldsymbol{R}^N)$ defined, for any $Z \in BUC(\boldsymbol{R}^N)$ by*

$$T_h(Z)(x) = \min_{u \in U \;\&\; \|b\| \leq 1} \left\{ \frac{Z(x + hf(x,u) + ch^2 b) + hm(x,u) - ch^2}{1 + \lambda h} \right\}$$

where $c := M(\ell + \lambda)$. Let $W_h \in BUC(\boldsymbol{R}^N)$ be the unique fixed point of T_h. Then

$$\forall x \in \boldsymbol{R}^N, \; W(x) - rh^\gamma \leq W_h(x) \leq W(x)$$

for some positive constant $r = r(M, \ell, \lambda)$, where $\gamma = 1$ if $\lambda > \ell$, $\gamma = \frac{\lambda}{\ell}$ if $\lambda < \ell$ and γ is any arbitrary number less than 1 if $\lambda = \ell$.

Proof of Theorem 3.3 Part I — Following [11], let us define the discretization Φ_h of the map Φ defined in Theorem 2.1:

$$\Phi_h(x,y) := \begin{cases} \{F(x) + M\ell hB\} \times \{-1\} & \text{if } x \notin C + MhB \\ \overline{Co}\,\{\{(0,0)\} \cup \{F(x) + M\ell hB\} \times \{-1\}\} & \text{otherwise} \end{cases} \tag{16}$$

The set-valued map Φ_h satisfies (10). It is proved in [10] that the sequence T_h^n is nondecreasing, that it converges to some lower semi-continuous function T_h^∞ and that the epigraph of T_h^∞ is the discrete viability kernel of $\mathcal{K} := \boldsymbol{R}^N \times [0, +\infty)$ for G_h:

$$\overrightarrow{\mathrm{Viab}}_{G_h}(\mathcal{K}) = Epi(T_h^\infty) ,$$

where we have set as usual $G_h(x) = x + h\Phi_h(x)$. In order to apply Proposition 3.2, let us introduce

$$\tilde{\Phi}_h(x,y) := \begin{cases} \{F(x) + 2M\ell hB\} \times \{-1\} & \text{if } x \notin C + 2MhB \\ \overline{Co}\,\{\{(0,0)\} \cup \{F(x) + 2M\ell hB\} \times \{-1\}\} & \text{otherwise} \end{cases}$$

(i.e., $\tilde{\Phi}_h = \Phi_{2h}$). Let us set $\mathcal{K}_h := \boldsymbol{R}^N \times [-Mh, +\infty)$. Since $\tilde{\Phi}_h$ satisfies (15), Proposition 3.2 states that $\overrightarrow{\mathrm{Viab}}_{G_h}(\mathcal{K}) \subset Viab_{\tilde{\Phi}_h}(\mathcal{K}_h)$. Let us now estimate the distance between $Viab_{\tilde{\Phi}_h}(\mathcal{K}_h)$ and $Viab_\Phi(\mathcal{K})$.

We already know that $Viab_\Phi(\mathcal{K}) \subset Viab_{\tilde{\Phi}_h}(\mathcal{K}_h)$. Let now $(\tilde{x}_0, \varepsilon_0)$ belong to $Viab_{\tilde{\Phi}_h}(\mathcal{K}_h)$ with $\varepsilon_0 \leq T$. Let $(\tilde{x}(\cdot), \varepsilon(\cdot))$ be a solution to

$$(\tilde{x}'(t), \varepsilon'(t)) \in \tilde{\Phi}_h(y(t), \varepsilon(t)) \ \&\ \tilde{x}(0) = \tilde{x}_0,\ \varepsilon(0) = \varepsilon_0$$

which remains in $\mathcal{K}_h$ forever. As long as $\tilde{x}(t)$ does not belong to $C + 2MhB$, one has $\varepsilon'(t) = -1$. Since $\varepsilon(t) \in [-Mh, +\infty)$ for any t, there is a first time $\tau \leq \varepsilon_0 + 2Mh$ such that $\tilde{x}(\tau)$ belongs to $C + 2MhB$.

Let now z_0 belong to the projection of $\tilde{x}(\tau)$ onto C. Then $\|z_0 - \tilde{x}(\tau)\| \leq 2Mh$. Recall that $\tilde{x}(\cdot)$ is a solution to the differential inclusion for $x \rightsquigarrow F(x) + 2M\ell hB$ on $[0, \tau]$. ¿From Filippov Theorem, there is a solution $z(\cdot)$ to the differential inclusion for $-F$ starting from z_0 and satisfying

$$\forall t \in [0, \tau],\ \|z(t) - \tilde{x}(\tau - t)\| \leq 2Mh(2e^{\ell t} - 1)$$

because the set-valued map F is $\ell-$Lipschitz. The map $x(t) := z(\tau - t)$ is a solution of the differential inclusion for F starting from $z(\tau)$ which reaches C before τ.

We now define $x_0 := z(\tau)$. Then clearly x_0 belongs to the domain of Θ_C and $\Theta_C(x_0) \leq \tau \leq \varepsilon_0 + Mh$. Thus $(x_0, \varepsilon_0 + Mh)$ belongs to $Viab_\Phi(\mathcal{K})$. So we have proved

Lemma 3.4 *Let us set $r := 2M(2e^{\ell T} - 1)$. Then*

$$(\boldsymbol{R}^N \times [0,T]) \cap \left(Viab_{\tilde{\Phi}_h}(\boldsymbol{R}^N \times [-Mh, +\infty))\right) \ \subset\ Viab_\Phi(\mathcal{K}) + (rB) \times [-Mh, Mh] .$$

It remains to show that this inclusion implies the desired rate of convergence. For that purpose, let us denote by Z_h the function whose epigraph is equal to $Viab_{\tilde{\Phi}_h}(\mathcal{K}_h)$. Then Lemma 3.4 states that, for any x such that $Z_h(x) \leq T$,

$$\inf_{\|y-x\| \leq rh} (\Theta_C(y) - Mh) \leq Z_h(x) .$$

On another hand, since $Viab_\Phi(\mathcal{K}) \subset \overrightarrow{\mathrm{Viab}}_{G_h}(\mathcal{K}) \subset Viab_{\tilde{\Phi}_h}(\boldsymbol{R}^N \times [-Mh, +\infty))$, we have, for any x belonging to the domain of Θ_C,

$$Z_h(x) \leq T_h^\infty(x) \leq \Theta_C(x) .$$

Now, if $\Theta_C(x) \leq T$, we have $Z_h(x) \leq T$, so that we obtain the desired rate of convergence:

$$\inf_{\|y-x\| \leq rh} (\Theta_C(y) - Mh) \leq T_h^\infty(x) \leq \Theta_C(x) .$$

Q.E.D.

Remark — It is proved in [12] that $\Theta_C(\cdot)$ is locally Lipschitz on a dense subset of its domain under suitable assumptions on f. This means that the previous scheme converges with a rate of h on a dense subset of the domain of $\Theta_C(\cdot)$. $\square$

Proof of Theorem 3.3 Part II — Set $T_h := T - Mh$ and

$$\forall x \in \boldsymbol{R}^N,\ \phi_h(x) := \inf_{\|y-x\|\le Mh} \phi(y) - Mh.$$

Let us also introduce the following dynamic:

$$\Psi_h(t,x,\varepsilon) := \begin{cases} \Psi(t,x) + M\ell h(\{0\}\times B_{N+1}), \text{ if } (t < T_h) \text{ or } (t \geq T_h,\ \varepsilon < \phi_h(y)), \\ \overline{Co}\{\{0\} \cup \Psi(t,x) + M\ell h(\{0\}\times B_{N+1})\} \text{ else} \end{cases}$$

where $B_{N+1} := B \times [-1,1]$. It is easy to check that Ψ_h satisfies conditions (10). Moreover, since f and g are bounded by M and $\phi(x) \geq -M$, one can verify that

$$\forall (t,x) \in [0,T]\times \boldsymbol{R}^N,\ V(t,x) \geq -M(T+1)\ .$$

In particular, if we set $\mathcal{K} := [0,T]\times \boldsymbol{R}^N \times [-M(T+1), +\infty)$,

$$\mathcal{E}pi(V) = Viab_\Psi([0,T]\times \boldsymbol{R}^N \times \boldsymbol{R}) = Viab_\Psi(\mathcal{K})\ .$$

Let us define by induction the following nonincreasing sequence of closed sets:

$$\begin{cases} A_h^0 & := [0,T]\times \boldsymbol{R}^N \times [-M(T+1), +\infty) \\ A_h^{n+1} & := \{(t,x,\varepsilon) \in A_h^n \mid G_h(t,x,\varepsilon) \cap A_h^n \neq \emptyset\} \end{cases}$$

where $G_h(t,x,\varepsilon) := (t,x,\varepsilon) + h\Psi_h(t,x,\varepsilon)$. A verification by induction shows that we have

$$\forall n \geq 1\,,\ A_h^n = Epi(V_h^n)\ .$$

Therefore the sequence $V_h^n(x)$ is nondecreasing for any x and h. Proposition 2.14 of [11] states that the nonincreasing sequence (A_h^n) converges to the discrete viability kernel of $\mathcal{K}$ for G_h. This implies that the nondecreasing sequence V_h^n converges pointwisely to some lower semicontinuous function V_h^∞ whose epigraph is exactly the viability kernel of $\mathcal{K}$ for G_h (see for instance the proof of Proposition 5.4 in [10]):

$$\bigcap_n A_h^n = \overrightarrow{\mathrm{Viab}}_{G_h}(\mathcal{K}) = Epi(V_h^\infty)\ .$$

Let us now introduce the following dynamic $\tilde{\Psi}_h$:

$$\tilde{\Psi}_h(t,x,\varepsilon) := \begin{cases} \Psi(t,x) + 2M\ell h(\{0\}\times B_{N+1}), \text{ if } (t < T_{2h}) \text{ or } (t \geq T_{2h},\ \varepsilon < \phi_{2h}(y)), \\ \overline{Co}\{\{0\} \cup \Psi(t,x) + 2M\ell h(\{0\}\times B_{N+1})\} \text{ otherwise} \end{cases}$$

(i.e., $\tilde{\Psi}_h = \Psi_{2h}$). It is easy to check that $\tilde{\Psi}_h$ satisfies (15). Thus Proposition 3.2 implies that

$$\overrightarrow{\mathrm{Viab}}_{G_h}(\mathcal{K}) \subset Viab_{\tilde{\Psi}_h}(\mathcal{K} + MhB_{N+2})$$

where $B_{N+2} := [-1,1]\times B \times [-1,1]$. The sequel of the proof of PART II involves the same kind of estimations than the proof of Lemma 3.4. Therefore, we give the result without proof:

Lemma 3.5 *There is some positive constant $r = r(M,T,\ell)$ such that*

$$Viab_{\tilde{\Psi}_h}(\mathcal{K} + MhB_{N+2}) \subset Viab_\Psi(\mathcal{K}) + rhB_{N+2}.$$

Proof of Theorem 3.3 Part III — Let us remark that the numerical scheme to compute W can be found in [9], [2] and [14]. Set

$$\Lambda_h(x,\varepsilon) := \Lambda(x,\varepsilon) + chB_{N+1}$$

where $B_{N+1} := B \times [-1,1]$ and $c = M(\ell + \lambda)$.

Then Λ_h satisfies clearly conditions (10). Set $G_h(x,\varepsilon) := (x,\varepsilon) + h\Lambda_h(x,\varepsilon)$.

Lemma 3.6 *The unique fixed point W_h of the operator T_h is related with the discrete viability kernel of $\big(\boldsymbol{R}^N \times [-T,T]\big)$ for G_h by the following formula:*

$$Epi^T(W_h) = \overrightarrow{\mathrm{Viab}}_{G_h}\,\big(\boldsymbol{R}^N \times [-T,T]\big)$$

where $T := \frac{2M}{\lambda}$.

The idea of the proof can be found in [2] and [14], so we omit this proof.

From Theorem 3.1, the functions V_h converge epigraphically to V. To establish the rate of this convergence, let us define $\tilde{\Lambda}_h$ by setting:

$$\tilde{\Lambda}_h(x,\varepsilon) := \Lambda(x,\varepsilon) + 2chB_{N+1}$$

(let us notice again that $\tilde{\Lambda}_h = \Lambda_{2h}$). Then $\tilde{\Lambda}_h$ satisfies assumption (15), so that we can apply Proposition 3.2:

$$\overrightarrow{\mathrm{Viab}}_{G_h}\,(\mathcal{K}) \subset Viab_{\tilde{\Lambda}_h}\,(\mathcal{K} + MhB_{N+1})$$

To complete the proof, we have to estimate the distance between $Viab_\Lambda\,(\mathcal{K})$ and the set $Viab_{\tilde{\Lambda}_h}\,(\mathcal{K} + MhB_{N+1})$. We already know that

$$Viab_\Lambda\,(\mathcal{K}) \subset Viab_{\tilde{\Lambda}_h}\,(\mathcal{K} + MhB_{N+1})\ .$$

Let us recall that

$$\tilde{\Lambda}_h(x,\varepsilon) = \{(f(x,u) + 2chb, \theta)\,|\,(u,b) \in U \times B,\ \lambda\varepsilon - M - 2ch \le \theta \le \lambda\varepsilon - m(x,u) + 2ch\}$$

From Theorem 2.1, $Viab_{\tilde{\Lambda}_h}\,(\mathcal{K} + MhB_{N+1})$ is the $(T+Mh)-$epigraph of the value function Z_h of the following minimization problem

$$Z_h(x_0) := \min_{u(\cdot),b(\cdot)} \int_0^{+\infty} e^{-\lambda t} m(x_h(t), u(t))dt - \frac{2ch}{\lambda}$$

where $x_h(\cdot)$ is the solution to

$$x_h'(t) = f(x_h(t), u(t)) + 2chb(t)\ ,\ u(t) \in U\ ,\ b(t) \in B\ \&\ x(0) = x_0.$$

Let us now estimate $W(x_0) - Z_h(x_0)$. We already know that this quantity is non-negative. Let $\bar{u}(\cdot)$ and $\bar{b}(\cdot)$ be optimal for Z_h. Then

$$W(x_0) - Z_h(x_0) \le \int_0^{+\infty} e^{-\lambda t}|m(x_h(t), \bar{u}(t)) - m(x(t), \bar{u}(t))|dt + \frac{2ch}{\lambda}\ .$$

where $x(\cdot)$ is the solution to

$$x'(t) = f(x(t), \bar{u}(t))\ \&\ x(0) = x_0.$$

From Gronwall Lemma, one has $\|x_h(t) - x(t)\| \le 2ch(e^{\ell t} - 1)$. Thus, for any $\tau \ge 0$,

$$W(x_0) - Z_h(x_0) \le 2ch \int_0^{\tau} e^{-\lambda t}(e^{\ell t} - 1)dt + 2M \int_\tau^{+\infty} e^{-\lambda t}dt + \frac{2ch}{\lambda}$$

Now, if $\lambda > \ell$, one chooses $\tau = +\infty$ and the previous inequality becomes

$$W(x_0) - Z_h(x_0) \leq \frac{2c\lambda h}{(\lambda - \ell)}$$

If $\lambda < \ell$, choose $\tau := \frac{1}{\ell}\log(1 + \frac{M}{ch})$. Then the inequality becomes

$$W(x_0) - Z_h(x_0) \leq r h^{\frac{\lambda}{\ell}}$$

for some constant r depending on M, ℓ and λ.

If $\lambda = \ell$, setting $\tau := \frac{1}{\lambda}\log(\frac{ch+M}{ch})$ yields the following inequality

$$W(x_0) - Z_h(x_0) \leq \frac{2ch}{\lambda}\left(\log(\frac{ch+M}{ch}) + 1\right)$$

Since, $Z_h(x_0) \leq W_h(x_0) \leq W(x_0)$, the previous inequalities imply Theorem 3.3 Part III.

Q.E.D.

Remark : We can compare with [9] where, under the same assumptions, the rate of convergence is proved to be of order $\frac{\gamma}{2}$. We do not know if the improvement comes just from the method or if it is related with functional T_h which differs slightly from that given in [9].

4 On Approximation of Optimal Controls

This section is devoted to the approximation of the optimal controls. We shall compare between optimal controls of the continuous optimal time Θ_C and of the discrete optimal time

$$\vartheta_h(x_0) := \inf\{n \in \boldsymbol{N} \mid \exists \ \overrightarrow{x} \in \overrightarrow{S}_{G_h}(x_0),\, x_n \in C_h\}$$

where $G_h(x) := F(x) + M\ell h B$ and $C_h := C + MhB$.

We are going to prove that the discrete optimal controls associated with $\vartheta_h(x_0)$ converge to an optimal control associated with $\Theta_C(x_0)$.

Proposition 4.1 *Suppose that $U \subset \boldsymbol{R}^N$ and*[1]

$$(17) \qquad \begin{cases} i) & f(x,u) = \varphi(x) + g(x)u \text{ where } g(x) \in \mathcal{L}(\boldsymbol{R}^N, \boldsymbol{R}^N) \\ ii) & U \text{ is strictly convex, compact and with a nonempty interior} \\ iii) & \varphi \text{ and } g \text{ are } \ell\text{-Lipschitz continuous} \\ iv) & \forall\, x \in K,\ g(x) \text{ is invertible and} \\ & x \mapsto g(x) \text{ and } x \mapsto g(x)^{-1} \text{ are of class } C^1 \\ & \forall\, x \in K,\ 0 < \frac{1}{\ell} \leq \|Dg(x)\| \leq \ell \\ v) & f \text{ is bounded by } M \end{cases}$$

Consider $x_0 \in \boldsymbol{R}^N$ such that $\Theta_C(x_0) < +\infty$ and for which there exists a unique optimal control $u(\cdot)$. Let $\overrightarrow{u_h} = (u_h^n)_n$ be a discrete optimal control associated with $\vartheta_h(x_0)$ and let

$$v_h(t) := u_h^n,\ \forall t \in [nh, (n+1)h)$$

be the piecewise constant function associated with $(u_h^n)_n$. Then

$$\lim_{h \to 0} \|v_h(\cdot) - u(\cdot)\|_{L^1([0,\Theta_C(x_0)],U)} = 0\ .$$

[1] For sake of simplicity, we denote by ℓ all constants in (17).

Remark: If there is no uniqueness of the optimal control, one can obtain the same convergence property but only up to subsequence.

Proof — Let $\overrightarrow{x_h} \in \overrightarrow{G_h}\ (x_0)$, $\overrightarrow{u_h}$ and v_h be respectively an optimal solution, an optimal control for $\vartheta_h(x_0)$ and the piecewise constant function associated with $\overrightarrow{u_h}$. If we set $N_h = \frac{\vartheta_h(x_0)}{h}$, then $x_{N_h} \in C_h$.

We define the piecewise linear interpolation $x_h(\cdot)$ given by

$$x_h(t) := x_h^n + \frac{t - nh}{h}(x_h^{n+1} - x_h^n),\ \forall t \in\ [nh, (n+1)h)$$

Let us notice that

$$\|x_h'(t) - f(x_h(t), v_h(t))\| \leq M\ell h \text{ for almost every } t \geq 0.$$

From Filippov's Theorem, there exists a solution $\overline{x_h}(\cdot) \in S_F(x_0)$ satisfying

$$(18) \qquad \begin{array}{lcl} \|x_h(t) - \overline{x_h}(t)\| & \leq & Mh(e^{\ell t} - 1),\ \ \forall t \in [0, \Theta_C(x_0)] \\ \|x_h'(t) - \overline{x_h}'(t)\| & \leq & M\ell h e^{\ell t},\ \ \text{a.e. } \ t \in [0, \Theta_C(x_0)] \end{array}$$

¿From the Measurable Selection Theorem and assumption (17-iv), there is a unique control $\overline{u_h}(\cdot)$ such that

$$\overline{x_h'}(\cdot) = \varphi(\overline{x_h}(\cdot)) + g(\overline{x_h}(\cdot))\overline{u_h}(\cdot)\ .$$

¿From standard arguments, there exists a subsequence again denoted $\overline{x_h}$ such that $\overline{x_h}$ converges strongly to $\overline{x}$ in $L^1([0, \Theta_C(x_0)], \boldsymbol{R}^N)$ and $\overline{x_h}'$ converges weakly to $\overline{x}'$ in $L^1([0, \Theta_C(x_0)], \boldsymbol{R}^N)$. Moreover, $\overline{x}$ belongs to $\mathcal{S}_F(x_0)$ and the Measurable Selection Theorem together with assumption (17-iv) yield the existence of a unique control $\overline{u}$ such that

$$\overline{x}'(\cdot) = \varphi(\overline{x}(\cdot)) + g(\overline{x}(\cdot))\overline{u}(\cdot)\ .$$

Since φ and g are Lipschitz continuous, we have on one hand that $\varphi(\overline{x_h}(\cdot)) \to \varphi(\overline{x}(\cdot))$ as well as $g(\overline{x_h}(\cdot)) \to g(\overline{x}(\cdot))$ in $L^1([0, \Theta_C(x_0)], \boldsymbol{R}^N)$.
On another hand, the sequence $g(\overline{x_h}(\cdot))\overline{u_h}(\cdot) = \overline{x_h}'(\cdot) - \varphi(\overline{x_h}(\cdot))$ converges weakly to $g(\overline{x}(\cdot))\overline{u}(\cdot) = \overline{x}'(\cdot) - \varphi(\overline{x}(\cdot))$. Hence, by (17-iv), $\overline{u_h}(\cdot) \rightharpoonup \overline{u}(\cdot)$. Then (18) together with assumption (17-iv) yields that

$$(19) \qquad \overline{v_h}(\cdot) \rightharpoonup \overline{u}(\cdot)\ .$$

From Part I of Theorem 3.3, $\overline{x}$ is an optimal solution starting from x_0. Since we have supposed that the optimal control $u(\cdot)$ is unique, we have necessarily that $\overline{u}(t) = u(t)$ for almost every $t \geq 0$.

We claim that for almost every t, $u(t)$ is an extremal point of U. Indeed, otherwise the Lebesgue measure of the set

$$\{t \in [0, \Theta_C(x_0)],\ |\, u(t) \in\ Int(U)\,\}$$

is positive. Then there exists $\beta > 0$ such that the Lebesgue measure of the set

$$A := \{t \in [0, \Theta_C(x_0)],\ |\, (1+\beta)f(\overline{x}(t), u(t)) \in\ F(\overline{x}(t))\,\}$$

is positive. Let $\eta : \boldsymbol{R} \mapsto \boldsymbol{R}$ be the unique absolutely continuous function solution to

$$\eta'(t) = 1 + \beta \mathbf{1}_A(\eta(t))\ \&\ \eta(0) = 0\ ,$$

where $\mathbf{1}_A$ denotes the indicatrix function of A. Then one can easily check that $t \mapsto x(\eta(t))$ is a solution to $S_F(x_0)$ which reaches C at a time strictly smaller then $\Theta_C(x_0)$. This is a contradiction with the very definition of $\Theta_C(x_0)$ and our claim is proved.

So $\bar{u}(\cdot)$ is an extremal point of the bounded convex set $L^1([0, \Theta_C(x_0)], U)$. By Visitin Theorem [21], the convergence in (19) is strong. This completes the proof.

Q.E.D.

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