Numerical semigroups, cyclotomic polynomials and Bernoulli numbers

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Abstract

We give two proofs of a folklore result relating numerical semigroups of embedding dimension two and binary cyclotomic polynomials and explore some consequences. In particular, we give a more conceptual reproof of a result of Hong et al. (2012) on gaps between the exponents of non-zero monomials in a binary cyclotomic polynomial.

The intent of the author with this paper is to better unify the various results within the cyclotomic polynomial and numerical semigroup communities.

1 Introduction

Let $a_1, \ldots, a_m$ be positive integers, and let $S = S(a_1, \ldots, a_m)$ be the set of all non-negative integer linear combinations of $a_1, \ldots, a_m$, that is,

$$S = \{x_1 a_1 + \cdots + x_m a_m \mid x_i \in \mathbb{Z}_{\geq 0}\}.$$

Then S is a *semigroup* (that is, it is closed under addition). The semigroup S is said to be *numerical* if its complement $\mathbb{Z}_{\geq 0} \backslash S$ is finite. It is not difficult to prove that $S(a_1, \ldots, a_m)$ is numerical if and only if $a_1, \ldots, a_m$ are relatively prime (see, e.g., [15, p. 2]). If S is numerical, then $\max\{\mathbb{Z}_{\geq 0} \backslash S\} = F(S)$ is the *Frobenius number* of S. Alternatively, by setting $d(k, a_1, \ldots, a_m)$ equal to the number of non-negative integer representations of k by $a_1, \ldots, a_m$, one can characterize $F(S)$ as the largest k such that $d(k, a_1, \ldots, a_m) = 0$. The value $d(k, a_1, \ldots, a_m)$ is called the *denumerant* of k. That $F(S(4, 6, 9, 20)) = 11$ is well-known to fans of Chicken McNuggets, as 11 is the largest number of McNuggets that cannot be exactly purchased; hence the notion of of the Frobenius number is less abstract than it might appear at first glance. A set of generators of a numerical semigroup is a minimal system of generators if none of its proper subsets generates the numerical semigroup. It is known that every numerical semigroup S has a unique minimal system of generators and also that this minimal system of generators is finite (see, e.g., [18, Theorem 2.7]). The cardinality of the minimal set of generators is called the *embedding dimension* of the numerical semigroup S and is denoted by $e(S)$. The smallest member in the minimal system of generators is called the

Mathematics Subject Classification (2000). 20M14, 11C08, 11B68 *multiplicity* of the numerical semigroup S and is denoted by $m(S)$. The *Hilbert series* of the numerical semigroup S is the formal power series

$$H_S(x) = \sum_{s \in S} x^s \in \mathbb{Z}[[x]].$$

It is practical to multiply this by $1-x$ as we then obtain a *polynomial*, called the *semigroup polynomial*:

$$P_S(x) = (1-x)H_S(x) = x^{F(S)+1} + (1-x) \sum_{\substack{0 \le s \le F(S) \\ s \in S}} x^s = 1 + (x-1) \sum_{s \notin S} x^s. \quad (1)$$

From P_S one immediately reads off the Frobenius number:

$$\deg(P_S(x)) = F(S) + 1. \quad (2)$$

The nth cyclotomic polynomial $\Phi_n(x)$ is defined by

$$\Phi_n(x) = \prod_{\substack{1 \le j \le n \\ (j,n)=1}} (x - \zeta_n^j) = \sum_{k=0}^{\varphi(n)} a_n(k) x^k,$$

with ζ_n a nth primitive root of unity (one can take $\zeta_n = e^{2\pi i/n}$). It has degree $\varphi(n)$, with φ Euler's totient function. The polynomial $\Phi_n(x)$ is irreducible over the rationals, see, e.g., Weintraub [22], and has integer coefficients. The polynomial $x^n - 1$ factors as

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \quad (3)$$

over the rationals. By Möbius inversion it follows from (3) that

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}, \quad (4)$$

where $\mu(n)$ denotes the Möbius function. From (4) one deduces that if $p|n$ is a prime, then

$$\Phi_{pn}(x) = \Phi_n(x^p). \quad (5)$$

A good source for further properties of cyclotomic polynomials is Thangadurai [19].

A purpose of this paper is to popularise the following folklore result and point out some of its consequences.

Theorem 1 *Let $p, q > 1$ be coprime integers, then*

$$P_{S(p,q)}(x) = (1-x) \sum_{s \in S(p,q)} x^s = \frac{(x^{pq}-1)(x-1)}{(x^p-1)(x^q-1)}.$$

In case p and q are distinct primes it follows from (4) and Theorem 1 that

$$P_{S(p,q)}(x) = \Phi_{pq}(x). \quad (6)$$

Already Carlitz [5] in 1966 implicitly mentioned this result without proof.

The Bernoulli numbers B_n can be defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \ |z| < 2\pi. \tag{7}$$

One easily sees that $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30$ and $B_n = 0$ for all odd $n \geq 3$. The most basic recurrence relation is, for $n \geq 1$,

$$\sum_{j=0}^{n} \binom{n+1}{j} B_j = 0. \tag{8}$$

The Bernoulli numbers first arose in the study of power sums $S_j(n) := \sum_{k=0}^{n-1} k^j$. Indeed, one has, cf. Rademacher [14],

$$S_j(n) = \frac{1}{j+1} \sum_{i=0}^{j} \binom{j+1}{i} B_i n^{j+1-i}. \tag{9}$$

In Section 5, we consider an infinite family of recurrences for B_m of which the following is typical

$$\begin{aligned} B_m &= \frac{m}{4^m - 1}(1 + 2^{m-1} + 3^{m-1} + 5^{m-1} + 6^{m-1} + 9^{m-1} + 10^{m-1} + 13^{m-1} + 17^{m-1}) \\ &\quad + \frac{7^m}{4(1-4^m)} \sum_{r=0}^{m-1} \binom{m}{r} \left(\frac{4}{7}\right)^r (1 + 2^{m-r} + 3^{m-r}) B_r. \end{aligned}$$

The natural numbers $1, 2, 3, 5, 6, 9, 10, 13$ and 17 are precisely those that are not in the numerical semigroup $S(4, 7)$.

Let $f = c_1 x^{e_1} + \cdots + c_s x^{e_s}$, where the coefficients c_i are non-zero and $e_1 < e_2 < \cdots < e_s$. Then the *maximum gap* of f, written as $g(f)$, is defined by

$$g(f) = \max_{1 \leq i < s} (e_{i+1} - e_i), \ g(f) = 0 \text{ when } s = 1.$$

Hong et al. [9] studied $g(\Phi_n)$ (inspired by a cryptographic application [10]). They reduce the study of these gaps to the case where n is square-free and odd and established the following result for the simplest non-trivial case.

Theorem 2 [9]. *If p and q are arbitrary primes with $2 < p < q$, then $g(\Phi_{pq}) = p - 1$.*

In Section 6 a conceptual proof of Theorem 2 using numerical semigroups is given.

2 Inclusion-exclusion polynomials

It will turn out to be convenient to work with a generalisation of the cyclotomic polynomials, introduced by Bachman [1]. Let $\rho = \{r_1, r_2, \ldots, r_s\}$ be a set of natural numbers satisfying $r_i > 1$ and $(r_i, r_j) = 1$ for $i \neq j$, and put

$$n_0 = \prod_i r_i, \ n_i = \frac{n_0}{r_i}, \ n_{ij} = \frac{n_0}{r_i r_j}[i \neq j], \ldots \ .$$

For each such ρ we define a function Q_ρ by

$$Q_\rho(x) = \frac{(x^{n_0}-1)\cdot\prod_{i<j}(x^{n_{ij}}-1)\cdots}{\prod_i(x^{n_i}-1)\cdot\prod_{i<j<k}(x^{n_{ijk}}-1)\cdots}. \tag{10}$$

For example, if $\rho = \{p, q\}$, then

$$Q_{\{p,q\}}(x) = \frac{(x^{pq}-1)(x-1)}{(x^p-1)(x^q-1)}. \tag{11}$$

It can be shown that $Q_\rho(x)$ defines a polynomial of degree $d := \prod_i(r_i - 1)$. We define its coefficients $a_\rho(k)$ by $Q_\rho(x) = \sum_{k\geq 0} a_\rho(k)x^k$. Furthermore, $Q_\rho(x)$ is *selfreciprocal*; that is $a_\rho(k) = a_\rho(d-k)$ or, what amounts to the same thing,

$$Q_\rho(x) = x^d Q_\rho(\frac{1}{x}). \tag{12}$$

If all elements of ρ are prime, then comparison of (10) with (4) shows that

$$Q_\rho(x) = \Phi_{r_1r_2\cdots r_s}(x). \tag{13}$$

If n is an arbitrary integer and $\gamma(n) = p_1\cdots p_s$ its squarefree kernel, then by (5) and (13) we have $Q_{\{p_1,\ldots,p_s\}}(x^{n/\gamma(n)}) = \Phi_n(x)$ and hence inclusion-exclusion polynomials generalize cyclotomic polynomials. They can be expressed as products of cyclotomic polynomials.

Theorem 3 [1]. *Given* $\rho = \{r_1, \ldots, r_s\}$ *and*

$$D_\rho = \{d : d|\prod_i r_i \text{ and } (d, r_i) > 1 \text{ for all } i\},$$

then $Q_\rho(x) = \prod_{d\in D_\rho}\Phi_d(x)$.

`Example`. We have $Q_{\{4,7\}} = \Phi_{28}\Phi_{14}$.

2.1 Binary inclusion-exclusion polynomials: a close-up

Lam and Leung [11] discuss binary cyclotomic polynomials Φ_{pq} in detail, with p and q primes (their results were anticipated by Lenstra [12]). Now, let $p, q > 1$ be positive coprime integers. All arguments in their paper easily generalize to this setting (instead of taking ξ to be a primitive pqth-root of unity as they do, one has to take ζ a pqth root of unity satisfying $\zeta^p \neq 1$ and $\zeta^q \neq 1$). One finds that

$$Q_{\{p,q\}}(x) = \sum_{i=0}^{\rho-1} x^{ip}\sum_{j=0}^{\sigma-1} x^{jq} - x^{-pq}\sum_{i=\rho}^{q-1} x^{ip}\sum_{j=\sigma}^{p-1} x^{jq}, \tag{14}$$

where ρ and σ are the (unique) non-negative integers for which $1 + pq = \rho p + \sigma q$. On noting that upon expanding the products in identity (14), the resulting monomials are all different, we arrive at the following result.

Lemma 1 *Let $p, q > 1$ be coprime integers. Let ρ and σ be the (unique) non-negative integers for which $1 + pq = \rho p + \sigma q$. Let $0 \leq m < pq$. Then either $m = \alpha p + \beta q$ or $m = \alpha p + \beta q - pq$ with $0 \leq \alpha \leq q - 1$ the unique integer such that $\alpha p \equiv m (\text{mod } q)$ and $0 \leq \beta \leq p - 1$ the unique integer such that $\beta q \equiv m (\text{mod } p)$. The inclusion-exclusion coefficient $a_{\{p,q\}}(m)$ equals*

$$\begin{cases} 1 & \text{if } m = \alpha p + \beta q \text{ with } 0 \leq \alpha \leq \rho - 1,\ 0 \leq \beta \leq \sigma - 1; \\ -1 & \text{if } m = \alpha p + \beta q - pq \text{ with } \rho \leq \alpha \leq q - 1,\ \sigma \leq \beta \leq p - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1 *The number of positive coefficients in $Q_{\{p,q\}}(x)$ equals $\rho\sigma$ and the number of negative ones equals $\rho\sigma - 1$. The number of non-zero coefficients equals $2\rho\sigma - 1$.*

This corollary (in case p and q are distinct primes) is due to Carlitz [5].

Lemma 1 can be nicely illustrated with an LLL-diagram (for Lenstra, Lam and Leung). Here is one such diagram for $p = 5$ and $q = 7$.

28	33	3	8	13	18	23
21	26	31	1	6	11	16
14	19	24	29	34	4	9
7	12	17	22	27	32	2
0	5	10	15	20	25	30

We start with 0 in the lower left and add p for every move to the right and q for every move upwards. Reduce modulo pq. Every integer $0, \ldots, pq - 1$ is obtained precisely once in this way (by the Chinese remainder theorem).

Lemma 1 can be reformulated in the following way.

Lemma 2 *Let $p, q > 1$ be coprime integers. The numbers in the lower left corner of the LLL-diagram are the exponents of the terms in $Q_{\{p,q\}}$ with coefficient 1. The numbers in the upper right corner are the exponents of the terms in $Q_{\{p,q\}}$ with coefficient -1. All other coefficients equal 0.*

3 Two proofs of the main (folklore) result

In terms of inclusion-exclusion polynomials we can reformulate Theorem 1 as follows.

Theorem 4 *If $p, q > 1$ are coprime integers, then $P_{S(p,q)}(x) = Q_{\{p,q\}}(x)$.*

Our first proof will make use of 'what is probably the most versatile tool in numerical semigroup theory' [18, p. 8], namely Apéry sets.

First proof of Theorem 4. The *Apéry set* of S with respect to a nonzero $m \in S$ is defined as

$$\text{Ap}(S; m) = \{s \in S : s - m \notin S\}.$$

Note that

$$S = \text{Ap}(S; m) + m\mathbb{Z}_{\geq 0}$$

and that $\mathrm{Ap}(S;m)$ consists of a complete set of residues modulo m. Thus we have

$$H_S(x) = \sum_{w\in \mathrm{Ap}(S;m)} x^w \sum_{i=0}^{\infty} x^{mi} = \frac{1}{1-x^m} \sum_{w\in \mathrm{Ap}(S;m)} x^w. \tag{15}$$

Note that if $S = \langle a_1, \ldots, a_n \rangle$, then $\mathrm{Ap}(S;a_1) \subseteq \langle a_2, \ldots, a_n \rangle$. It follows that $\mathrm{Ap}(S(p,q);p)$ consists of multiples of q. The latter set equals the minimal set of multiples of q representing every congruence class modulo p and hence $\mathrm{Ap}(S(p,q);p) = \{0, q, \ldots, (p-1)q\}$ (see [16, Proposition 1] or [18, Example 8.22]). Hence

$$H_{S(p,q)}(x) = \frac{1 + x^q + \cdots + x^{(p-1)q}}{1-x^p} = \frac{1-x^{pq}}{(1-x^q)(1-x^p)}.$$

Using this identity and (11) easily completes the proof. □

`Remark`. This proof is an adaptation of the arguments given in [16]. Indeed, once we know the Apéry set of a numerical semigroup S, by using [16, (4)], we obtain an expression for $H_S(x)$ and consequently for $P_S(x)$. Theorem 4 is a particular case of [16, Proposition 2], with $\{p,q\} = \{a, a+d\}$ and $k = 1$.

Our second proof uses the denumerant (see [15, Chapter 4] for a survey) and the starting point is the observation that

$$\frac{1}{(1-x^p)(1-x^q)} = \sum_{j\ge 0} r(j)x^j, \tag{16}$$

where $r(j)$ denotes the cardinality of the set $\{(a,b) : a \ge 0, b \ge 0, ap + bq = j\}$. In the terminology of the introduction, we have $r(j) = d(j;p,q)$. Concerning $r(j)$ we make the following observation.

Lemma 3 *Suppose that $k \ge 0$, then $r(k+pq) = r(k) + 1$.*

Proof. Put $\alpha \equiv kp^{-1} (\mathrm{mod}\ q)$, $0 \le \alpha < q$ and $\beta \equiv kq^{-1} (\mathrm{mod}\ p)$, $0 \le \beta < p$ and $k_0 = \alpha p + \beta q$. Note that $k_0 < 2pq$. We have $k \equiv k_0 (\mathrm{mod}\ pq)$. Now if $k \notin S$, then $k < k_0$ and $k + pq = k_0 \in S$ (since $k_0 < 2pq$). It follows that if $r(k) = 0$, then $r(k+pq) = 1$. If $k \in S$, then $k = k_0 + tpq$ for some $t \ge 0$ and we have $r(k) = 1 + t$, where we use that

$$k = (\alpha + tq)p + \beta q = (\alpha + (t-1)q)p + (\beta + 1)p = \cdots = \alpha p + (\beta + tq)p.$$

We see that $r(k+pq) = 1 + t + 1 = r(k) + 1$. □

`Remark`. It is not difficult to derive an explicit formula for $r(n)$ (see, e.g., [2, Section 1.3] or [13, pp. 213-214]). Let p^{-1}, q^{-1} denote inverses of p modulo q, respectively q modulo p. Then we have

$$r(n) = \frac{n}{pq} - \left\{\frac{p^{-1}n}{q}\right\} - \left\{\frac{q^{-1}n}{p}\right\} + 1,$$

where $\{x\}$ denote the fractional-part function. Note that Lemma 3 is a corollary of this formula.

Second proof of Theorem 4. From Lemma 3 we infer that

$$\begin{aligned}(1-x^{pq})\sum_{j\ge 0} r(j)x^j &= \sum_{j=0}^{pq-1} r(j)x^j + \sum_{j=pq}^{\infty}(r(j)-r(j-pq))x^j \\ &= \sum_{j=0}^{pq-1} r(j)x^j + \sum_{j\ge pq} x^j = \sum_{j\in S(p,q)} x^j,\end{aligned}$$

where we used that $r(j)\le 1$ for $j<pq$ and $r(j)\ge 1$ for $j\ge pq$. Using this identity and (16) easily completes the proof. □

4 Symmetric numerical semigroups

A numerical semigroup S is said to be *symmetric* if

$$S\cup(F(S)-S)=\mathbb{Z},$$

where $F(S)-S=\{F(S)-s|s\in S\}$. Symmetric semigroups occur in the study of monomial curves that are complete intersections, Gorenstein rings, and the classification of plane algebraic curves, see, e.g. [15, p. 142]. For example, Herzog and Kunz showed that a Noetherian local ring of dimension one and analytically irreducible is a Gorenstein ring if and only if its associate value semigroup is symmetric.

We will now show that the selfreciprocity of $Q_{\{p,q\}}(x)$ implies that $S(p,q)$ is symmetric (a well-known result, see, e.g., [18, Corollary 4.7]).

Theorem 5 *Let S be a numerical semigroup. Then S is symmetric if and only if $P_S(x)$ is selfreciprocal.*

Proof. If $s\in S\cap(F(S)-S)$, then $s=F(S)-s_1$ for some $s_1\in S$. This implies that $F(S)\in S$, a contradiction. Thus S and $F(S)-S$ are disjoint sets. Since every integer $n\ge F(S)+1$ is in S and every integer $n\le -1$ is in $F(S)-S$, the assertion is equivalent to showing that

$$\sum_{\substack{0\le j\le F(S)\\ j\in S}} x^j+\sum_{\substack{0\le j\le F(S)\\ j\in S}} x^{F(S)-j}=1+x+\cdots+x^{F(S)}, \tag{17}$$

if and only if $P_S(x)$ is selfreciprocal. On noting by (1) that

$$x^{F(S)+1}P_S(\frac{1}{x})-P_S(x)=1-x^{F(S)+1}+(x-1)\Big(\sum_{\substack{0\le j\le F(S)\\ j\in S}} x^j+\sum_{\substack{0\le j\le F(S)\\ j\in S}} x^{F(S)-j}\Big),$$

we see that $x^{F(S)+1}P_S(1/x)=P_S(x)$ if and only if (17) holds. Clearly (17) holds if and only if S is symmetric. □

Using the latter result and Theorem 4 we infer the following classical fact.

Theorem 6 *A numerical semigroup of embedding dimension* 2 *is symmetric.*

Theorem 4 together with Theorem 3 shows that if $e(S) = 2$, then $P_S(x)$ can be written as a product of cyclotomic polynomials. This leads to the following problem.

Problem 1 *Characterize the numerical semigroups* S *for which* $P_S(x)$ *can be written as a product of cyclotomic polynomials.*

Since $P_S(0) \neq 0$, the product cannot involve $\Phi_1(x) = x - 1$ and so it is selfreciprocal. Therefore, by Theorem 5 such an S must be symmetric. Ciolan et al. [6] make some progress towards solving this problem and show, e.g., that $P_S(x)$ can be written as a product of cyclotomic polynomials also if $e(S) = 3$ and S is symmetric.

5 Gap distribution

The non-negative integers not in S are called the *gaps* of S. E.g., the gaps in $S(4,7)$ are $1, 2, 3, 5, 6, 9, 10, 13$ and 17. The number of gaps of S is called the *genus* of S, and denoted by $N(S)$. The set of gaps is denoted by $G(S)$. The following well-known result holds, cf. [15, Lemma 7.2.3] or [18, Corollary 4.7].

Theorem 7 *We have* $2N(S) \geq F(S) + 1$ *with equality if and only if* S *is symmetric.*

Proof. The proof of Theorem 5 shows that $2\#\{0 \leq j \leq F(S) : j \in S\} \leq F(S)+1$ with equality if and only if S is symmetric. Now use that $\#\{0 \leq j \leq F(S) : j \in S\} = F(S) + 1 - N(S)$. □

From (2) and Theorem 1 we infer the following well-known result due to Sylvester:

$$F(S(p,q)) = pq - p - q. \tag{18}$$

From Theorem 6, Theorem 7 and (18), we obtain another well-known result of Sylvester:

$$N(S(p,q)) = (p-1)(q-1)/2. \tag{19}$$

For four different proofs of (18) and more background see [15, pp. 31-34]; the shortest proof of (18) and (19) the author knows of is in the book by Wilf [23, p. 88].

Additional information on the gaps is given by the so-called *Sylvester sum*

$$\sigma_k(p,q) := \sum_{s \notin S(p,q)} s^k.$$

By (19) we have $\sigma_0(p,q) = (p-1)(q-1)/2$. By (1) and Theorem 4 we infer that

$$\sum_{j \notin S(p,q)} x^j = \frac{1 - Q_{\{p,q\}}(x)}{1-x}. \tag{20}$$

It is not difficult to derive a formula for $\sigma_k(p,q)$ for arbitrary k. On substituting $x=e^z$ and recalling the Taylor series expansion $e^z=\sum_{k\ge 0} z^k/k!$, we obtain from (20) and (11) the identity

$$\sum_{k=0}^{\infty} \sigma_k(p,q)\frac{z^k}{k!} = \frac{e^{pqz}-1}{(e^{pz}-1)(e^{qz}-1)} - \frac{1}{e^z-1}. \tag{21}$$

We obtain from (21), on multiplying by z and using the Taylor series expansion (7), that

$$\sum_{m=1}^{\infty} m\sigma_{m-1}(p,q)\frac{z^m}{m!} = \sum_{i=0}^{\infty} B_i p^i \frac{z^i}{i!}\sum_{j=0}^{\infty} B_j q^j \frac{z^j}{j!}\sum_{k=0}^{\infty}\frac{(pqz)^k}{(k+1)!} - \sum_{m=0}^{\infty} B_m \frac{z^m}{m!}.$$

Equating coefficients of z^m then leads to the following result.

Theorem 8 [17]. *For $m \ge 1$ we have*

$$m\sigma_{m-1}(p,q) = \frac{1}{m+1}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\binom{m+1}{i,j,m+1-i-j}B_iB_jp^{m-j}q^{m-i} - B_m.$$

Using this formula we find e.g. that $\sigma_1(p,q)=\frac{1}{12}(p-1)(q-1)(2pq-p-q-1)$ (this result is due to Brown and Shiue [3]) and $\sigma_2(p,q)=\frac{1}{12}(p-1)(q-1)pq(pq-p-q)$. The proof we have given here of Theorem 8 is due to Rødseth [17], with the difference that we gave a different proof of the identity (21).

By using the formula (9) for power sums we obtain from Theorem 8 the identity

$$m\sigma_{m-1}(p,q) = \sum_{r=0}^{m}\binom{m}{r}p^{m-r-1}B_{m-r}q^rS_r(p) - B_m,$$

giving rise to the following recursion formula for B_m:

$$B_m = \frac{m}{p^m-1}\sigma_{m-1}(p,q) + \frac{q^m}{p(1-p^m)}\sum_{r=0}^{m-1}\binom{m}{r}\left(\frac{p}{q}\right)^r B_rS_{m-r}(p).$$

On taking $p=4$ and $q=7$, we obtain the recursion for B_m stated in the introduction.

Tuenter [20] established the following characterization of the gaps in $S(p,q)$. For every finite function f,

$$\sum_{n\notin S}(f(n+p)-f(n)) = \sum_{n=1}^{p-1}(f(nq)-f(n)),$$

where p and q are interchangeable. He shows that by choosing f appropriately one can recover all earlier results mentioned in this section and in addition the identity

$$\prod_{n\notin S(p,q)}(n+p) = q^{p-1}\prod_{n\notin S(p,q)} n.$$

Wang and Wang [21] obtained results similar to those of Tuenter for the *alternate Sylvester sums* $\sum_{s\notin S(p,q)}(-1)^s s^k$.

6 A reproof of Theorem 2

As mentioned previously, the gaps for $S(4,7)$ are given by $1,2,3,5,6,9,10,13$ and 17. One could try to break this down in terms of *gap blocks*, that is blocks of consecutive gaps, (also known in the literature as *deserts* [7, Definition 16])): $\{1,2,3\}$, $\{5,6\}$, $\{9,10\}$, $\{13\}$, and $\{17\}$. It is interesting to compare this with the distribution of the *element blocks*, that is finite blocks of consecutive elements in S. For $S(4,7)$ we get $\{0\}$, $\{4\}$, $\{7,8\}$, $\{11,12\}$ and $\{14,15,16\}$. The longest gap block we denote by $g(G(S))$ and the longest element block by $g(S)$.

The following result gives some information on gap blocks and element blocks in a numerical semigroup of embedding dimension 2. Recall that the smallest positive integer of S is called the *multiplicity* and denoted by $m(S)$.

Lemma 4
1) *The longest gap block,* $g(G(S))$*, has length* $m(S)-1$.
2) *The longest element block,* $g(S)$*, has length not exceeding* $m(S)-1$.
3) *If* S *is symmetric, then* $g(S)=m(S)-1$.

Proof. 1) Let $S=\{s_0,s_1,s_2,s_3,\ldots\}$ be the elements of S written in ascending order, i.e., $0=s_0<s_1<s_2<\cdots$. Since $s_0=0$ and $s_1=m(S)$ we have $g(G(S))\ge m(S)-1$. Since all multiples of $m(S)$ are in S, it follows that actually $g(G(S))=m(S)-1$.
2) If $g(S)\ge m(S)$, it would imply that we can find $k,k+1,\ldots,k+m(S)-1$ all in S such that $k+m(S)\notin S$. This is clearly a contradiction.
3) If S is symmetric, then we clearly have $g(S)=g(G(S))=m(S)-1$. □

`Remark`. The second observation was made by my intern Alexandru Ciolan. It allows one to prove Theorem 10.

Finally, we will generalize a result of Hong et al. [9].

Theorem 9 *If* $p,q>1$ *are coprime integers, then* $g(Q_{\{p,q\}}(x))=\min\{p,q\}-1$.

Proof. Note that $g(Q_{\{p,q\}}(x))$ equals the maximum of the longest gap block length and the longest element block length and hence by Lemma 4 equals $m(S(p,q))-1=\min\{p,q\}-1$. □

This result can be easily generalized further.

Theorem 10 *We have* $g(P_S(x))=m(S)-1$.

Proof. Using that $P_S(x)=(1-x)H_S(x)$ and Lemma 4 we infer that $g(P_S(x))=\max\{g(S),g(G(S))\}=m(S)-1$. □

7 The LLL-diagram revisited

It is instructive to indicate (we do this in boldface) the gaps of $S(p,q)$ in the LLL-diagram. They are those elements $\alpha p+\beta q$ with $0\le\alpha\le q-1$, $0\le\beta\le p-1$ for which $\alpha p+\beta q>pq$. Note that the Frobenius number equals $(q-1)p+(p-1)q-pq$ and so appears in the top right hand corner of the LLL-diagram. We will demonstrate this (again) for $p=5$ and $q=7$.

28	33	**3**	**8**	**13**	**18**	**23**
21	26	31	**1**	**6**	**11**	**16**
14	19	24	29	34	**4**	**9**
7	12	17	22	27	32	**2**
0	5	10	15	20	25	30

As a check we can verify that $N(S(p,q)) = (p-1)(q-1)/2$ integers appear in boldface.

On comparing coefficients in the identity $(1-x)\sum_{j\in S(p,q)} x^j = \sum_{j\ge 0} a_{\{p,q\}}(j)x^j$ we get the following reformulation of Theorem 4 at the coefficient level.

Theorem 11 *If $p,q>1$ are coprime integers, then*

$$a_{\{p,q\}}(k) = \begin{cases} 1 & \text{if } k\in S(p,q),\ k-1\notin S(p,q); \\ -1 & \text{if } k\notin S(p,q),\ k-1\in S(p,q); \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2 *The non-zero coefficients of $Q_{\{p,q\}}$ alternate between 1 and -1.*

The next result gives an example where an existing result on cyclotomic coefficients yields information about numerical semigroups.

Theorem 12 *Let p,q,ρ and σ be as in Lemma 1. If $S=S(p,q)$, then there are $\rho\sigma-1$ gap blocks and $\rho\sigma-1$ element blocks.*

Proof. In view of Theorem 11 we have $a_{\{p,q\}}(k)=1$ if and only if k is at the start of an element block (including the infinite block $[F(S)+1,\infty)\cap\mathbb{Z}$). Moreover, $a_{\{p,q\}}(k)=-1$ if and only if k is at the end of a gap block. The proof is now completed by invoking Corollary 1. □

Using Lemma 2 and Theorem 11 our folklore result can now be reformulated in terms of the LLL-diagram.

Theorem 13 *Let $p,q>1$ be coprime integers and denote $S(p,q)\cap\{0,\ldots,pq-1\}$ by T. The integers $k\in T$ such that $k-1\notin T$ are precisely the integers in the lower left corner of the LLL-diagram. The integers $k\notin T$ such that $k-1\in T$ are precisely the integers in the upper right corner. If k is not in the lower left or upper right corner, then either $k\in T$ and $k-1\in T$ or $k\notin T$ and $k-1\notin T$.*

Denote $S(p,q)$ by S. Note that the upper right integer in the lower left corner of the LLL-diagram equals $F(S)+1$ and that the remaining integers in the lower left corner are all $<F(S)$. This observation together with (19) then leads to the following corollary of Theorem 13.

Corollary 3 *If $p,q>1$ are coprime integers, then*

$$\begin{cases} \{0\le k\le F(S): k\in S, k-1\in S\} = (p-1)(q-1)/2-\rho\sigma+1; \\ \{0\le k\le F(S): k\in S, k-1\notin S\} = \rho\sigma-1; \\ \{0\le k\le F(S): k\notin S, k-1\in S\} = \rho\sigma-1; \\ \{0\le k\le F(S): k\notin S, k-1\notin S\} = (p-1)(q-1)/2-\rho\sigma-1. \end{cases}$$

The distribution of the quantity $\rho\sigma$ that appears at various places in this article has been recently studied using deep results from analytic number theory by Bzdęga [4] and Fouvry [8]. In particular they are interested in counting the number of integers $m = pq \leq x$ with p, q distinct primes such that $\theta(m)$, the number of non-zero coefficients of Φ_m, satisfies $\theta(m) \leq m^{1/2+\gamma}$, with $\gamma > 0$ fixed. (Note that by Corollary 1 we have $\theta(m) = 2\rho\sigma - 1$.)

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References

[1] G. Bachman, On ternary inclusion-exclusion polynomials, *Integers* **10** (2010) A48 623–638.

[2] M. Beck and S. Robins, *Computing the continuous discretely. Integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics. Springer, New York, 2007.

[3] T.C. Brown and P. J.-S. Shiue, A remark related to the Frobenius problem, *Fibonacci Quart.* **31** (1993) 32–36.

[4] B. Bzdęga, Sparse binary cyclotomic polynomials, *J. Number Theory* **132** (2012) 410–413.

[5] L. Carlitz, The number of terms in the cyclotomic polynomial $F_{pq}(x)$, *Amer. Math. Monthly* **73** (1966) 979-981.

[6] A. Ciolan, P.A. García-Sánchez and P. Moree, Cyclotomic numerical semigroups, in preparation.

[7] J.I. Farrán and C. Munuera, Goppa-like bounds for the generalized Feng-Rao distances, International Workshop on Coding and Cryptography (WCC 2001) (Paris), *Discrete Appl. Math.* **128** (2003) 145–156.

[8] É. Fouvry, On binary cyclotomic polynomials, *Algebra Number Theory* **7** (2013) 1207–1223.

[9] H. Hong, E. Lee, H.-S. Lee and C.-M. Park, Maximum gap in (inverse) cyclotomic polynomial, *J. Number Theory* **132** (2012) 2297–2315, available at `http://dx.doi.org/10.1016/j.jnt.2012.04.008`.

[10] H. Hong, E. Lee, H.-S. Lee and C.-M. Park, Simple and exact formula for minimum loop length in Ate_i pairing based on Brezing-Weng curves, *Des. Codes Cryptogr.* **67** (2013) 271–292.

[11] T.Y. Lam and K.H. Leung, On the cyclotomic polynomial $\Phi_{pq}(x)$, *Amer. Math. Monthly* **103** (1996) 562–564.

[12] H. W. Lenstra, Vanishing sums of roots of unity, Proceedings, Bicentennial Congress Wiskundig Genootschap (Vrije Univ., Amsterdam, 1978), Part II, 249–268, Math. Centre Tracts **101**, Math. Centrum, Amsterdam, 1979.

[13] I. Niven, H.S. Zuckerman and H.L. Montgomery, *An introduction to the theory of numbers*, 5th edition, John Wiley & Sons, Inc., New York, 1991.

[14] H. Rademacher, *Topics in analytic number theory*, Die Grundlehren der mathematischen Wissenschaften **169**, Springer-Verlag, New York-Heidelberg, 1973.

[15] J.L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Oxford Lecture Series in Mathematics and its Applications **30**, Oxford University Press, Oxford, 2005.

[16] J.L. Ramírez Alfonsín and Ø.J. Rødseth, Numerical semigroups: Apéry sets and Hilbert series, *Semigroup Forum* **79** (2009) 323–340.

[17] Ø.J. Rødseth, A note on Brown and Shiue's paper on a remark related to the Frobenius problem, *Fibonacci Quart.* **32** (1994) 407–408.

[18] J.C. Rosales and P.A. García-Sánchez, *Numerical semigroups*, Developments in Mathematics **20**, Springer, New York, 2009.

[19] R. Thangadurai, On the coefficients of cyclotomic polynomials, *Cyclotomic fields and related topics* (Pune, 1999), 311–322, Bhaskaracharya Pratishthana, Pune, 2000.

[20] H.J.H. Tuenter, The Frobenius problem, sums of powers of integers, and recurrences for the Bernoulli numbers, *J. Number Theory* **117** (2006), 376–386.

[21] W. Wang and T. Wang, Alternate Sylvester sums on the Frobenius set, *Comput. Math. Appl.* **56** (2008) 1328–1334.

[22] S.H. Weintraub, Several proofs of the irreducibility of the cyclotomic polynomials, *Amer. Math. Monthly* **120** (2013) 537–545.

[23] H.S. Wilf, *Generatingfunctionology*, Academic Press, Inc., Boston, MA, 1990.

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