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Research Article

Numerical Simulation of Fractional Control System Using Chebyshev Polynomials

Jun Zhang, [1,2,3] **Yugui Li**, [2,4] **and Jiaquan Xie** [5]

[1] *School of Mechanical Engineering, Taiyuan University of Science and Technology, Taiyuan 030024, China*
[2] *Collaborative Innovation Center of Taiyuan Heavy Machinery Equipment, Taiyuan 030024, China*
[3] *Taiyuan Institute of China Coal Technology Engineering Group, Taiyuan 03006, China*
[4] *School of Material Science and Engineering, Taiyuan University of Science and Technology, Taiyuan 030024, China*
[5] *College of Mechanical Engineering, Taiyuan University of Technology, Taiyuan 030024, China*

Correspondence should be addressed to Yugui Li; liyugui2018@126.com

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In the current study, a numerical scheme based on Chebyshev polynomials is proposed to solve the problem of fractional control system. The operational matrix of fractional derivative is derived and that is used to transform the original problem into a system of linear equations. Lastly, several numerical examples are presented to verify the effectiveness and feasibility of the given method.

1. Introduction

Fractional calculus has a long history and it has been widely used in various fields of engineering, sciences, applied mathematics, and economics [1–5]. Many real-world problems such as physics, chemistry, fluid mechanics, control, and mathematical biology can be modelled by building fractional constitutive models [6–9]. The typical fractional feedback control system is given in Figure 1. $G_c(t)$ is the fractional controller, $G_0(t)$ is the transfer function of fractional controller system, and $G_f(t)$ is the feedback loop transfer function of fractional system. $U(t)$ and $Y(t)$ are the input and output of the system.

The above fractional control system is a continuous system when the switch is always closed, and its time domain model can be established by the following formula [10]:

$$\begin{aligned} a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_0 D^{\alpha_0} y(t) \\ = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \cdots + b_0 D^{\beta_0} u(t) \end{aligned} \quad (1)$$

where $D^{\alpha} = {}_0^c D_t^{\alpha}$, and $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0 \geq 0$, $\beta_n > \beta_{n-1} > \cdots > \beta_0 \geq 0$, a_k, b_k are arbitrary real numbers. The S field is described by Laplace transform of (1) as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}} \quad (2)$$

So far, various numerical methods are presented to solve fractional differential equations. These methods include wavelets method [11, 12], Chebyshev and Legendre polynomials [13, 14], and collocation method [15–19]. In [20], N. I. Mahmudov utilized an approximate method to study partial-approximate controllability of semilinear nonlocal fractional evolution equations. In [21], Ali Lotf used Epsilon penalty and an extension of the Ritz method for solving a class of fractional optimal control problems with mixed boundary conditions. In this paper, we get the numerical solutions of fractional control system using Chebyshev polynomials.

The paper is organized as follows: in the next section, the definitions about fractional calculus are introduced. In Section 3, some relevant properties of Chebyshev polynomials are given. Numerical methods together with numerical examples are illustrated in Section 4. A conclusion is drawn in Section 5.

2. Preliminaries and Notations

Definition 1 (see [22]). The left-sided Riemann-Liouville fractional integral of order μ, $\mu \in R^+$, for a function $y(t)$, is defined as

$${}_0 I_t^{\mu} y(t) = \frac{1}{\Gamma(\mu)} \int_0^t y(\varsigma)(t-\varsigma)^{u-1} d\varsigma, \quad t > 0, \quad (3)$$

where $\varsigma \in R^+$ and $\Gamma(\cdot)$ denotes the gamma function.

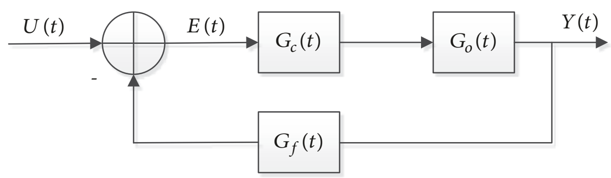

Figure 1: Fractional discrete control system.

Definition 2 (see [22]). The left-sided Caputo fractional-order derivative of order μ, $\mu \in R^+$, is defined as

$$ {}_0^cD_t^\mu y(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t \frac{y^{(m)}(\varsigma)\, d\varsigma}{(t-\varsigma)^{\mu+1-m}}, \quad m-1 < \mu \le m, \tag{4}$$

where $m \in N^+$.

3. Chebyshev Polynomials

3.1. The Properties of Chebyshev Polynomials. The analytical form of the Chebyshev polynomials $T_i(t)$ of degree i is given by [23].

$$T_i(t) = i\sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!} t^k, \quad t \in [0,1] \tag{5}$$

where $T_i(0) = (-1)^i$ and $T_i(1) = 1$.

The orthogonality is

$$\int_0^1 T_j(t) T_k(t) w(t)\, dt = h_k, \tag{6}$$

where the weight function $w(t) = 1/\sqrt{t-t^2}$ and

$$h_k = \begin{cases} \frac{b_k}{2}\pi, & k = j, \\ 0, & k \ne j, \end{cases} \quad b_0 = 2,\ b_k = 1,\ k \ge 1. \tag{7}$$

3.2. Function Approximation. Suppose that $y(t) \in L^2[0,1]$; it may be expanded in terms of the Chebyshev polynomials as

$$y(t) = \sum_{i=0}^{\infty} c_i T_i(t), \tag{8}$$

where the coefficient c_i is given by

$$c_i = \frac{1}{h_1}\int_0^1 y(t) T_i(t) w(t)\, dt, \quad i = 0,1,2,\ldots \tag{9}$$

If we consider the truncated series in (5), then we have

$$y(t) \approx y_M(t) = \sum_{i=0}^{M} c_i T_i(t) = C^T \Phi(t), \tag{10}$$

where

$$C = [c_0, c_1, \ldots, c_M]^T, \quad \Phi(t) = [T_0(t), T_1(t), \ldots, T_M(t)]^T. \tag{11}$$

Then the derivative of vector $\Phi(t)$ can be expressed by

$$\frac{d\Phi(t)}{dt} = \mathbf{P}^{(1)}\Phi(t), \tag{12}$$

where $\mathbf{P}^{(1)}$ is the $(M+1)\times(M+1)$ operational matrix of derivative given by

$$\mathbf{P}^{(1)} = (p_{ij}) = \begin{cases} \frac{4i}{b_j}, & j = 0,1,\ldots, i = j+k, \begin{cases} k = 1,3,5,\ldots,M, & \text{if } M \text{ is odd}, \\ k = 1,3,5,\ldots,M-1, & \text{if } M \text{ is even}, \end{cases} \\ 0, & \textit{otherwise} \end{cases} \tag{13}$$

Similarly, the operational matrix $\mathbf{P}^n$ of n*–times* differentiation of $\Phi(t)$ can be expressed as

$$\frac{d^n\Phi(t)}{dt^n} = \mathbf{P}^n\Phi(t), \tag{14}$$

where $\mathbf{P}^n = (\mathbf{P}^{(1)})^n$.

3.3. Operational Matrix of Fractional-Order Derivative. The main objective of this section is to prove the following theorem for the fractional derivatives of the Chebyshev polynomials [23].

Lemma 3. *Let $T_i(t)$ be a Chebyshev polynomial, then*

$$D^\mu T_i(t) = 0, \quad i = 0,1,2,\ldots,\lceil\mu\rceil - 1,\ \mu > 0. \tag{15}$$

Theorem 4. *Let $\Phi(t)$ be the Chebyshev vector defined in (13) and suppose $\mu > 0$, then*

$${}_0^cD^\mu \Phi(t) \approx \mathbf{P}^{(\mu)}\Phi(t), \tag{16}$$

where $\mathbf{P}^{(\mu)}$ is the $(M+1)\times(M+1)$ differential operational matrix of order μ in the Caputo sense and it is defined as follows:

$$\mathbf{P}^{(\mu)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ S_\mu(\lceil\mu\rceil,0) & S_\mu(\lceil\mu\rceil,1) & S_\mu(\lceil\mu\rceil,2) & \cdots & S_\mu(\lceil\mu\rceil,M) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\mu(i,0) & S_\mu(i,1) & S_\mu(i,2) & \cdots & S_\mu(i,M) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\mu(M,0) & S_\mu(M,1) & S_\mu(M,2) & \cdots & S_\mu(M,M) \end{pmatrix} \tag{17}$$

where

$$S_\mu(i,j) = \sum_{k=\lceil\mu\rceil}^{i} \frac{(-1)^{i-k}\, 2i\,(i+k-1)!\,\Gamma(k-\mu+1/2)}{b_j \Gamma(k+1/2)\,(i-k)!\,\Gamma(k-\mu-j+1)\,\Gamma(k-\mu+j+1)} \quad (18)$$

$$i = \lceil\mu\rceil, \lceil\mu\rceil+1, \ldots, M.$$

4. Numerical Experiments

In this section, we utilize the Chebyshev polynomials to carry out the numerical simulation of fractional control system. Firstly, each term of (1) can be expressed by the Chebyshev polynomials basis as

$$D^{\alpha_n} y(t) \approx {}_0^c D_t^{\alpha_n}\left(C^T \Phi(t)\right) \approx C^T \mathbf{P}^{(\alpha_n)} \Phi(t), \quad (19)$$

$$D^{\alpha_{n-1}} y(t) \approx {}_0^c D_t^{\alpha_{n-1}}\left(C^T \Phi(t)\right) \approx C^T \mathbf{P}^{(\alpha_{n-1})} \Phi(t), \quad (20)$$

$$\vdots \quad (21)$$

$$D^{\alpha_1} y(t) \approx {}_0^c D_t^{\alpha_1}\left(C^T \Phi(t)\right) \approx C^T \mathbf{P}^{(\alpha_1)} \Phi(t), \quad (22)$$

and

$$D^{\beta_n} u(t) \approx {}_0^c D_t^{\beta_n}\left(U^T \Phi(t)\right) \approx U^T \mathbf{P}^{(\beta_n)} \Phi(t), \quad (23)$$

$$D^{\beta_{n-1}} u(t) \approx {}_0^c D_t^{\beta_{n-1}}\left(U^T \Phi(t)\right) \approx U^T \mathbf{P}^{(\beta_{n-1})} \Phi(t), \quad (24)$$

$$\vdots \quad (25)$$

$$D^{\beta_1} u(t) \approx {}_0^c D_t^{\beta_1}\left(U^T \Phi(t)\right) \approx U^T \mathbf{P}^{(\beta_1)} \Phi(t), \quad (26)$$

where C and U can be obtained from (12). Substituting (19)-(26) into (1), we have

$$\begin{aligned} &a_n C^T \mathbf{P}^{(\alpha_n)} \Phi(t) + a_{n-1} C^T \mathbf{P}^{(\alpha_{n-1})} \Phi(t) + \cdots \\ &\quad + a_0 C^T \mathbf{P}^{(\alpha_1)} \Phi(t) \\ &= b_m U^T \mathbf{P}^{(\beta_n)} \Phi(t) + b_{m-1} U^T \mathbf{P}^{(\beta_{n-1})} \Phi(t) + \cdots \\ &\quad + b_0 U^T \mathbf{P}^{(\beta_1)} \Phi(t) \end{aligned} \quad (27)$$

Test Problem 4.1. Consider the following fractional Relaxation-Oscillation equation system

$$\begin{aligned} &D^{0.5} y(t) + y(t) = u(t), \\ &y(0) = 0, \\ &t \in [0, 50] \end{aligned} \quad (28)$$

If the input function of the system is $u(t) = t^2 + 2t^{1.5}/\Gamma(2.5)$, the analytical solution of this system is $y(t) = t^2$. When $M = 5$, the output solutions by analytical method and our proposed method are shown in Figure 2, and the absolute errors for the analytical and numerical solutions are shown in Figure 3.

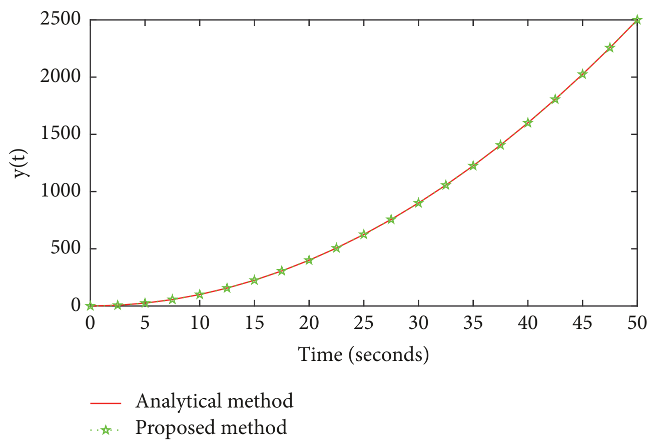

Figure 2: The output solutions by analytical method and our proposed method.

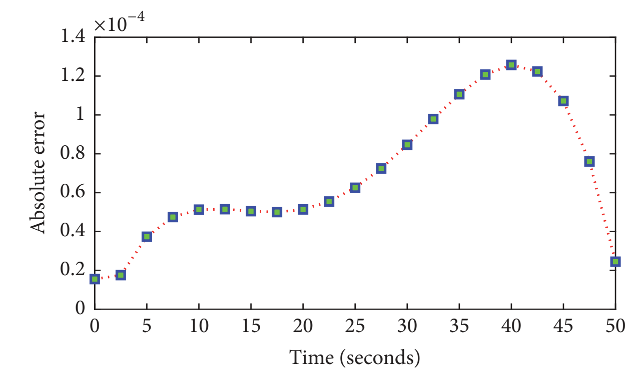

Figure 3: The absolute errors between the analytical and numerical results.

Test Problem 4.2. Consider the following fractional control system:

$$\begin{aligned} &y''(t) + D^{1.8} y(t) + y(t) = u(t), \\ &y'(0) = y(0) = 0, \\ &t \in [0, 4] \end{aligned} \quad (29)$$

where $u(t) = t^2(t-3) + 6t^{1.2}/\Gamma(2.2) - 6t^{0.2}/\Gamma(1.2) + 6t - 6$, the analytical solution of this system is $y(t) = t^2(t-3)$. When $M = 3, 4, 5$, the output solutions by analytical method and our proposed method are shown in Figure 4. Figure 4 shows that the numerical solutions approximate to analytical solutions as M increases. As M increases, the resulting coefficient matrix becomes large and may be singular [24].

Test Problem 4.3. Consider the following fractional control system:

$$\begin{aligned} &D^{1.8} y(t) + D^{1.5} y(t) + y'(t) + y(t) = u(t), \\ &y'(0) = 0, \\ &y(0) = -1, \\ &t \in [0, 1] \end{aligned} \quad (30)$$

Table 1: Absolute errors for the numerical and analytical results.

Time (seconds)	Analytical results	$M = 4$	$M = 6$	$M = 8$
0.1	-0.9900	4.2619269e-4	8.7631391e-6	1.9898239e-7
0.2	-0.9600	5.3938108e-4	7.8927198e-6	1.7497499e-7
0.3	-0.9100	6.8319833e-4	5.3619996e-6	2.4727493e-7
0.4	-0.8400	5.87172528-4	3.6391397e-6	2.8979387e-7
0.5	-0.7500	6.8719873e-5	2.3701310e-5	4.9839739e-7
0.6	-0.6400	7.3264873e-4	2.8748927e-5	5.3773098e-6
0.7	-0.5100	7.7838719e-4	3.9210200e-5	7.3937910e-6
0.8	-0.3600	8.3719731e-4	3.7101706e-5	5.3793700e-7
0.9	-0.1900	9.3871937e-4	4.2171077e-5	6.3897193e-6
1.0	0	8.6152715e-4	5.8973427e-5	7.6386329e-7

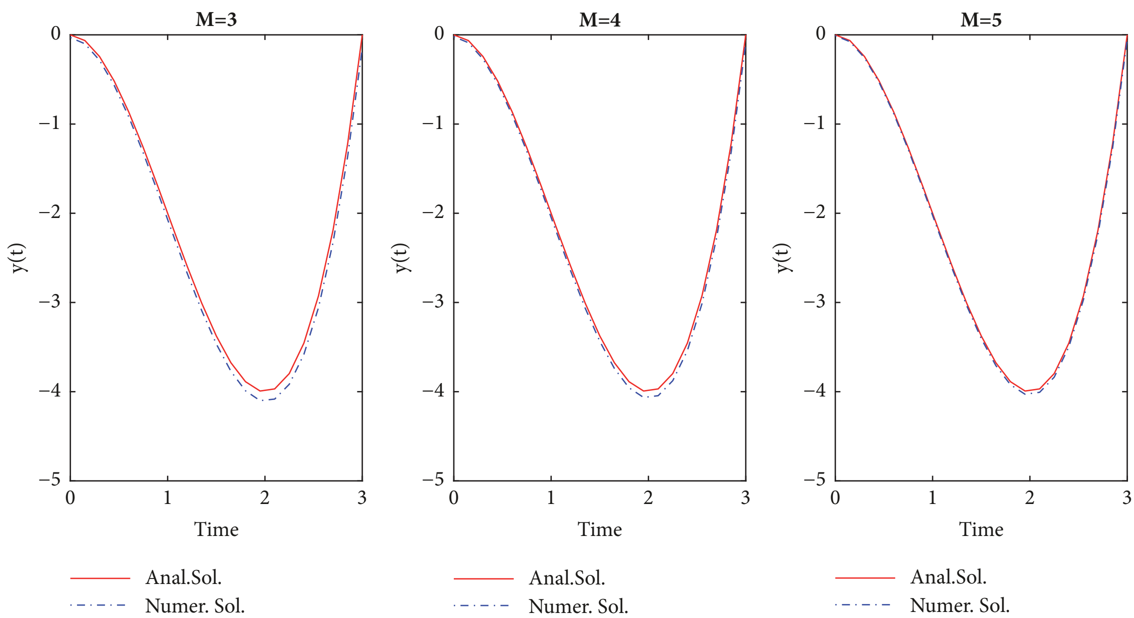

Figure 4: The analytical and numerical solutions for some different values of M.

where $u(t) = t^2 + (2/\Gamma(1.2))t^{0.2} + (2/\Gamma(1.5))t^{0.5} + 2t - 1$. The analytical solution of this system is $y(t) = t^2 - 1$. When $M = 4, 6, 8$, the absolute errors for the numerical and analytical results are listed in Table 1. Table 1 shows that the numerical solutions are in agreement with the analytical solutions well as M grows. With M increases, the coefficient matrix may be ill conditioned. The discussion on the ill-conditioned matrix is presented in the literature [25–27].

5. Conclusions

This paper presents a numerical approach for solving the fractional control system using Chebyshev polynomials. The derived operational matrix of fractional derivative is used to transfer the original problem into a system of linear algebra equations which can be easily solved. Numerical results show that the numerical solutions converge to the analytical solutions well as M grows.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally contribute to this paper.

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