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NUMERICAL SOLUTION OF A FREE BOUNDARY  
PROBLEM BY INTERCHANGING DEPENDENT  
AND INDEPENDENT VARIABLES

by

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## ABSTRACT

The classical problem of seepage of fluid through a porous dam is solved to illustrate a new approach to more general free boundary problems. The numerical method is based on the interchange of the dependent variable, representing velocity potential, with one of the independent space variables, which becomes the new variable to be computed. The need to determine the position of the whole of the free boundary in the physical plane is reduced to locating the position of the separation point on a fixed straight—line boundary in the transformed plane.

An iterative algorithm approximates within each single loop both a finite-difference solution of the partial differential equation and the position of the free boundary. The separation point is located by fitting a 'parabolic tail' to the finite-difference solution.



## 1. Introduction

A free boundary problem involves the solution of an elliptic partial differential equation subject to conditions on a boundary, part of which is unknown in position and shape. The most familiar model problem refers to the seepage of water through an earth dam, separating a high reservoir from a lower one. The upper surface of the water within the dam has to be determined as part of the solution.

Successive authors have approached the problem by solving a sequence of fixed boundary problems corresponding to successive, iteratively-computed positions of the free boundary. Relaxation methods, finite differences and finite elements have all been used to execute the solution. Key references are to be found in Cryer (1976), Aitchison (1972; 1977), and Furzeland (1977; 1979).

More recently, Aitchison (1977) and others referred to in Furzeland (1977;1979) have avoided the iterations by using the Baiocchi transformation (1972) to reformulate the problem as a variational inequality over a fixed domain.

The present paper transforms the region within the dam contained partly by the free boundary into a domain with fixed, known boundaries by interchanging the dependent variable with one of the independent, space variables.

This idea is well-tried in fluid flow problems and has recently been applied to moving boundary problems in heat flow (Crank & Phahle, 1973; Crank & Gupta, 1975; Crank & Crowley, 1978, 1979).

2.

In the transformed plane it is possible to solve the dam problem, for example, by an iterative algorithm which approximates within each single iterative loop both the solution of the partial differential- equation and the position of the free boundary.

## 2. The Seepage Problem

The mathematical formulation of the problem depicted in Figure 1, in terms of the velocity potential  $\phi$ , is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ in ABCDE,} \quad (1)$$

$$\phi = 1, \quad x = 0 \text{ on AB,} \quad (2)$$

$$\phi = \phi_d, \quad x = L \text{ on CD,} \quad (3)$$

$$\partial\phi/\partial n = 0, \quad y = 0 \text{ on BC,} \quad (4)$$

$$\phi = y, \quad x = L \text{ on DE} \quad (5)$$

$$\phi = y, \quad \partial\phi/\partial n = 0 \text{ on AE,} \quad (6)$$

where  $n$  is the outward normal on AE.

The double condition on AE is needed here as in all free boundary problems in order to determine the position of AE as well as to solve the differential equation (1).

We now replace  $\phi(x,y)$ , the usual dependent variable, by  $x = x(\phi, y)$  as the new dependent variable. It is easy to see that the partial differential equation (1) becomes

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 x}{\partial \phi^2} = \left( \frac{\partial x}{\partial \phi} \right)^{-3} = 0 \quad (7)$$

3.

In general, on a boundary  $y = g(x)$  we have

$$\phi_n = \frac{g' \phi_x - \phi_y}{[1 + (g')^2]^{\frac{1}{2}}}, \quad g' = dy/dx, \quad (8)$$

where, for example,  $\phi_n = \partial\phi/\partial n$  and  $n$  is the outward normal to  $y = g(x)$ .

Provided  $g' \neq \infty$ , we have on the free boundary AE

$$g' \frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = 0, \quad y = \phi. \quad (9)$$

But  $(\partial\phi/\partial y)_x = -(\partial x/\partial y)/(\partial x/\partial\phi)$  so that

$$g' + \left( \frac{\partial x}{\partial y} \right)_\phi = 0, \quad y = \phi. \quad (10)$$

On the impervious foundation BC we have

$$\frac{\partial x}{\partial y} = 0, \quad y = 0. \quad (11)$$

The other boundary conditions are

$$x = 0, \quad \phi = 1; \quad x = L, \quad \phi = \phi_d, \quad 0 \leq y \leq d, \quad (12)$$

$$\text{and } x = L, \quad \phi = y \text{ on DE.} \quad (13)$$

Thus, we now wish to solve (7) subject to conditions (10) to (13) inclusive in the region B'A'E'D'C in Figure 2 in the  $(y, \phi)$  plane. All the boundaries are fixed. The original free boundary AE has become the known straight boundary A'E',  $y = \phi$ . What we do not know, however, is the position of E' on A'D' corresponding to the separation point E in Fig. 1, at which the boundary condition on A'D' changes from  $x = L$  to the condition (9).

We cover the region with a mesh of spacings  $\delta\phi$ ,  $\delta y$ , and denote  $x(i\delta\phi, j\delta y)$  by  $x_{i,j}$ . Equation (7) can be approximated in part by

$$\frac{\partial^2\phi}{\partial y^2} - 8\delta\phi \frac{(x_{i-1,j} - 2x_{i,j} + x_{i+1,j})}{(x_{i+1,j} - x_{i-1,j})^3} = 0, \quad (14)$$

In order to approximate  $\partial^2\phi/\partial y^2$  at the point  $(i\delta\phi, j\delta y)$  we need three values of  $\phi$  on the lines corresponding to equally-spaced values of  $y$ ,  $y_{j-1}$ ,  $y_j$ ,  $y_{j+1}$ , but chosen such that  $x = x_{i,j}$  at each point. We interpolate linearly on each of the mesh lines  $y = y_{j+1}$  and  $y = y_{j-1}$  as illustrated in Figure 3. Thus on the  $y = y_{j+1}$  line we obtain

$$\phi(x_{i,j}) = \frac{\phi_i(x_{i,j} - x_{i+j,j+1}) + \phi_{i+1}(x_{i,j+1} - x_{i,j})}{x_{i,j+1} - x_{i+1,j+1}}, \quad (15)$$

and similarly on  $y = y_{j-1}$ .

Substitution of the resulting two values of  $\phi$  together with  $\phi_i$  itself into the usual finite-difference replacement for  $\partial^2\phi/\partial y^2$  in (14) yields an approximation to (7) for the typical internal point  $(i\delta\phi, j\delta y)$ .

If we collect together terms in  $x_{i,j}$  we can write the difference equation in the simplest iterative form

$$\left\{ \frac{\phi_i - \phi_{i+1}}{x_{i,j+1}^n - x_{i+1,j+1}^n} + \frac{\phi_i - \phi_{i-1}}{x_{i,j-1}^n - x_{i-1,j-1}^n} + \frac{16\delta\phi(\delta y)^2}{(x_{i+1,j}^n - x_{i-1,j}^n)^3} + \right\} x_{i,j}^{n+1} =$$

$$= \frac{\phi_i x_{i+1,j+1}^n - \phi_{i+1} x_{i,j+1}^n}{x_{i,j+1}^n - x_{i+1,j+1}^n} + 2\phi_i + \frac{\phi_i x_{i-1,j-1}^n - \phi_{i-1} x_{i,j-1}^n}{x_{i,j-1}^n - x_{i-1,j-1}^n} +$$

$$+ \frac{8\delta\phi(\delta y)^2 (x_{i+1,j}^n + x_{i-1,j}^n)}{(x_{i+1,j}^n + x_{i-1,j}^n)^3}$$

where  $x_{i,j}^n$  is the  $n^{\text{th}}$  iterate of  $x_{i,j}$ . (16)



Correspondingly, on the boundary  $y = \phi$  we use either  $x = L$  or

$$x_{i,j}^{n+1} = x_{i,j-1}^n - \frac{2(\delta y)^2}{x_{i+1,j+1}^n - x_{i-1,j-1}^n} \quad (17)$$

from (10).

Application of (16) to points on the lower boundary,  $y = 0$ , i.e.  $j = 0$ , introduces fictitious points on the line  $j = -1$ , one step outside the region; these can be eliminated from (16) since (11) implies  $x_{i,-1} = x_{i,1}$  for all  $i$ .

### 3. The iterative cycle

We start by assuming the separation point  $E'$  to be at one of the mesh points on  $A'D'$  in Figure 2. Then we know that  $x = L$  along  $C'D'$  and  $D'E'$  and also that  $x = 0$  on  $B'A'$ . For every other mesh point within the region and on the remaining parts of the boundary we have derived an equation. We carry out one iterative cycle by sweeping along successive  $j$ -lines from left to right in Figure 2 in the order  $j = 0, 1, 2, \dots$  where  $y = j = 0$  is the lower boundary. The new values of  $x_{i,j}$  are retained for use in the next cycle subject to the proviso that on the boundary  $A'D'$  we take the new value  $x_{i,i}^{n+1}$  to be

$$x_{i,i}^{n+1} = \min (L, x_{i,i}^{n+1}),$$

since we know that  $x_{i,j} \leq L$ . We proceed with successive loops, iterating values of the solution and the position of the separation point  $E'$  along  $A'D'$  in the same loop by using (18). The iteration proceeds until the difference between successive iterates at each point of the mesh is less than some prescribed amount. The highest mesh point on the boundary  $A'D'$  (Fig. 2) at which  $x = L$  is the best approximation to the separation

point that can be obtained directly from the set of finite-difference equations (16) and (17). In general, however, the true separation point will lie between two neighbouring mesh points and the finite-difference solution itself will be least accurate near the separation point. In particular, this solution will not satisfy the condition that the gradient of the free boundary  $dx/dy$ , should approach zero at the separation point. We therefore fit a "parabolic tail" to the finite-difference approximation to the free boundary.

#### 4. Improved position of separation point

Suppose the true position of the separation point is denoted by  $E'(x = L, y = y_s)$ , (Fig. 2) where  $y_s < y_j$ , and  $(j,j)$  is the lowest mesh point on the free boundary for which  $x < L$  in the finite-difference solution.

Since  $dx/dy = 0$  at  $E'(x = L, y = y_s)$  we write near  $E'$ , for  $x \leq L$ ,

$$x = L - A(y - y_s)^2, \quad (19)$$

where  $A$  is a constant to be determined. Then

$$x_{j,j} = L - A(y_j - y_s)^2 \quad (20)$$

and

$$\left. \frac{dy}{dx} \right)_{y_j} = g'(y_j) = -\frac{1}{2A(y_j - y_s)} \quad (21)$$

But from (10) we have

$$g'(y_j) = -\left. \frac{\partial x}{\partial y} \right)_{y_j} = -\left( \frac{x_{j,j} - x_{i,j-1}}{\partial y} \right) \quad (22)$$

approximately

By inserting in (20), (21) and (22) the relevant values of  $x$  and  $y$  from the final stage of the iterative solution, we obtain an interpolated value for  $y_s$ . If  $y_s < y_{j-1}$ , (19) also yields a revised value for  $x_{j-1,j-1}$ .

In order to estimate the effect of the "parabolic tail" (19) on the finite-difference solution elsewhere in the domain, a further stage of the iterative process was performed using the value of  $x_{j-1,j-1}$  and of  $g'$  at that point from the "parabolic tail". If a significant change was found in  $x_{j,j}$  a new tail was fitted to update  $y_s$ .

### Numerical results

In order to facilitate comparisons with the results quoted by Aitchison (1977) and Elliott (1976) we take  $L = 2/3$ ,  $d = 1/6$ ,  $H = 1$  and  $h_d = 1/6$ .

Calculations have been carried out on an  $18 \times 18$  mesh ( $\delta\phi = \delta y = 0.0556$ ) and  $30 \times 30$  mesh ( $\delta\phi = \delta y = 0.0333$ ). In order to start the iterative process the separation point was first assumed to be at the point  $D'$  in Fig. 2, corresponding to  $D$  in Fig. 1. Starting at mesh points further along the line  $D'A'$  was later found to yield the same final results.

The initial values of  $x$  along the free boundary  $D'A'$  were linear interpolates between  $x = 2/3$  at  $D'$  and  $x = 0$  at  $A'$ . On each line,  $y = \text{constant}$ , initial values of  $x$  at internal mesh points were obtained by linear interpolation between the end values on  $C'D'A'$  and  $B'A'$ .

It has not been possible to carry out a formal study of convergence of the iterative process for solving the non-linear difference equations (16). Instead, Table 1 demonstrates the convergence of some of the numerical values obtained on the line  $y = 1/6$  at selected stages of the  $18 \times 18$  mesh iterative solution. Essentially the same behaviour was found for the  $30 \times 30$  mesh. The iteration process ceases when  $\left| x_{ij}^{n+1} - x_{ij}^n \right| < 10^{-4}$  at all points of the mesh.

Fig. 4 shows an extract from the results obtained on the 18 x 18 grid and Fig. 5 shows all the mesh points in the region of the separation point.

In Table 2 the positions of the free boundary calculated by the present method using the 18 x 18 and 30 x 30 meshes are compared with corresponding results obtained by Aitchison (1977) and Elliott (1976).

Table 3 compares various values obtained for the y coordinate of the separation point E. It is worth recalling that Cryer (1976) considered the most reliable value to be 0.5297.

### Conclusions

The primary aim of this paper is to demonstrate that the idea of interchanging the dependent with one of the independent variables can form the basis of a method which iterates simultaneously the position of the free boundary and the solution of the partial differential equation. The results compare favourably with those obtained recently by mathematically more sophisticated techniques. For a given mesh, this method locates the position of the free boundary more precisely than do fixed-domain methods based on a minimisation or similar formulation. We are conscious that we have used only the very simplest iteration algorithm and that the convergence would very probably be improved by using alternative algorithms. We have not felt it worthwhile to explore this aspect in relation to the already over-studied dam problem. Equally, the method could be developed using finite elements instead of finite differences.

TABLE 1

Convergence of values of  $10^4x$  at selected internal points  
of an 18x18 mesh on  $y=1/6$

| $\phi$<br>Iteration | 4/18 | 8/18 | 12/18 | 17/18 |
|---------------------|------|------|-------|-------|
| 0                   | 6222 | 4444 | 2667  | 444   |
| 50                  | 6374 | 4753 | 2833  | 465   |
| 250                 | 6404 | 4995 | 3181  | 550   |
| 289                 | 6405 | 4995 | 3186  | 552   |

TABLE 2

Comparison of values of  $10^4x$  at chosen  $y$ -values on the free boundary.

| $y$    | (18 x 18) mesh | (30 x 30) mesh | Aitchison<br>(24 x 24) mesh. | Linear<br>element | Quadratic<br>element |
|--------|----------------|----------------|------------------------------|-------------------|----------------------|
| 0.5330 | 6669           | 6667           | 6667                         | -                 | -                    |
| 0.5333 | 6667           | 6667           | 6665                         | -                 | -                    |
| 0.5667 | 6549           | 6562           | 6423                         | -                 | -                    |
| 0.6000 | 6432           | 6458           | 6180                         | -                 | -                    |
| 0.6333 | 6149           | 6120           | 5960                         | 5957              | 5975                 |
| 0.6667 | 5806           | 5787           | 5652                         | 5631              | 5648                 |
| 0.7000 | 5462           | 5448           | 5314                         | 5283              | 5285                 |
| 0.7333 | 5089           | 5064           | 4943                         | 4908              | 4914                 |
| 0.7667 | 4667           | 4647           | 4425                         | 4494              | 4519                 |
| 0.8000 | 4205           | 4192           | 4095                         | 4072              | 4132                 |
| 0.8333 | 3722           | 3695           | 3611                         | 3583              | 3653                 |
| 0.8667 | 3174           | 3151           | 3081                         | 3050              | 3243                 |
| 0.9000 | 2549           | 2549           | 2498                         | 2440              | 2460                 |
| 0.9333 | 1860           | 1871           | 1835                         | 1735              | 1904                 |
| 0.9667 | -              | 1078           | -                            | -                 | -                    |

TABLE 3COMPARISON OF SEPARATION POINTS

| Mesh Size | $y_s$  | Elliott | Aitchison             |
|-----------|--------|---------|-----------------------|
| 18 x 18   | 0.5338 | -       | Values in the range * |
| 24 x 24   | -      | 0.5338  |                       |
| 30 x 30   | 0.5337 | -       | 0.5289 - 0.5426       |

\* The range comes from fitting an analytic expression through  $r$  points of the numerical solution. For  $r = 13$  to  $16$  inclusive  $y_s$  is effectively constant at  $y_s = 0.5289$ . Cryer (1976) quotes  $y_s = 0.5297$  evaluated from the analytic solution by Polubarinova-Kochina (1962).

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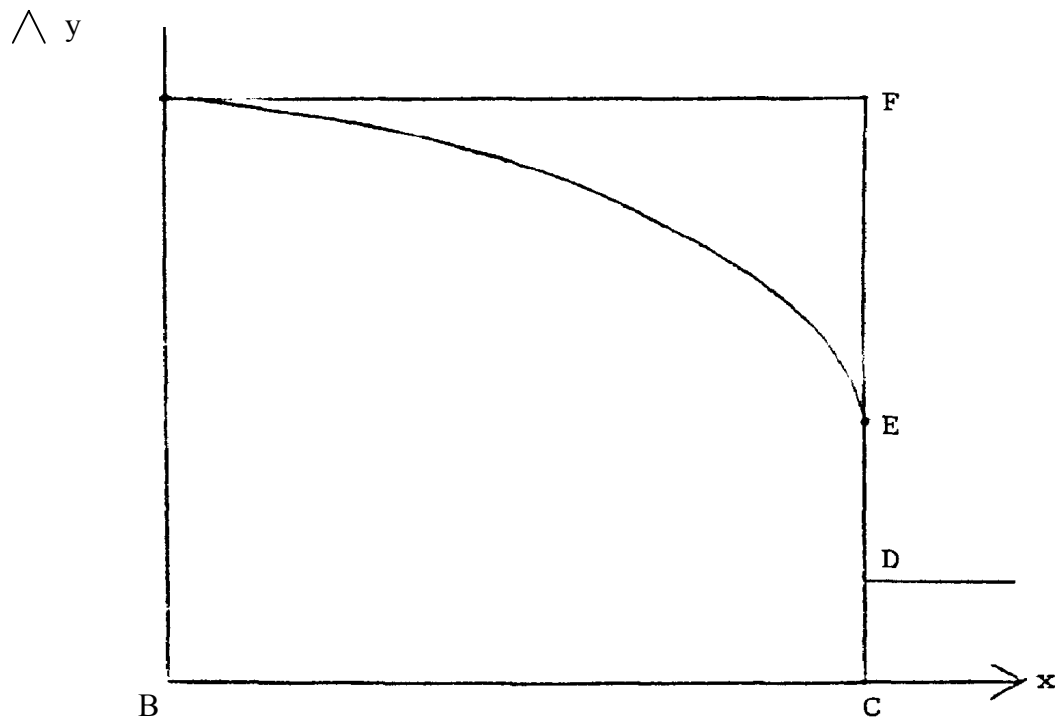


Figure 1. The dam problem.

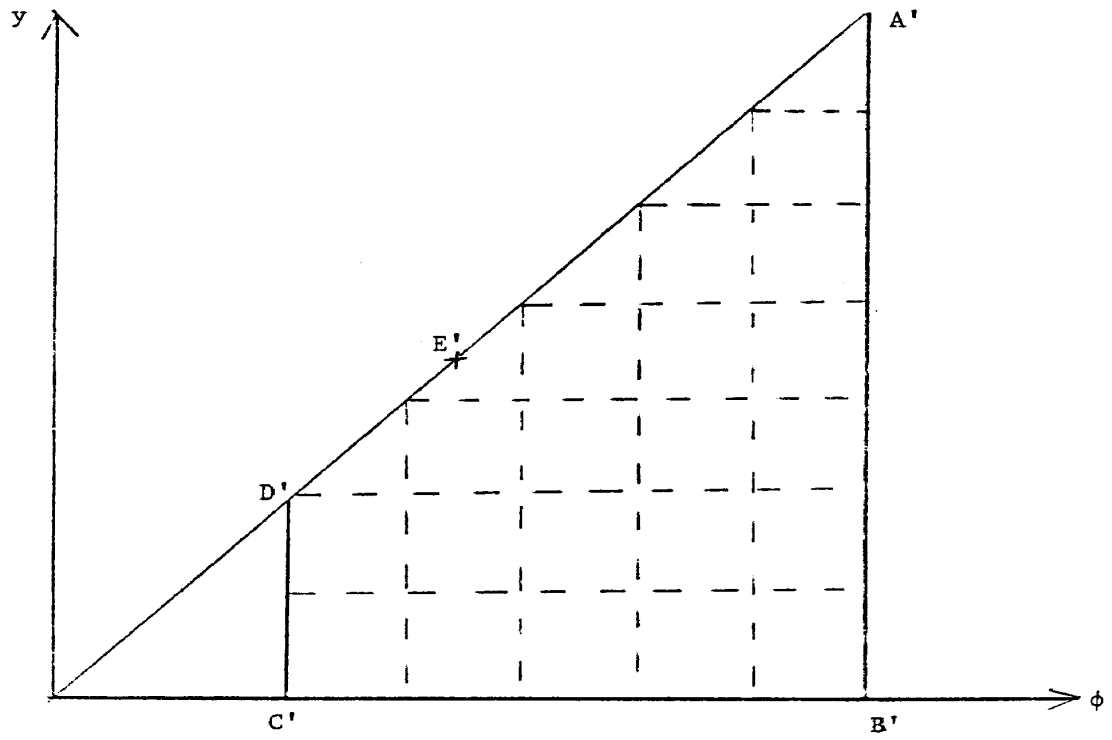


Figure 2. Transformed plane.

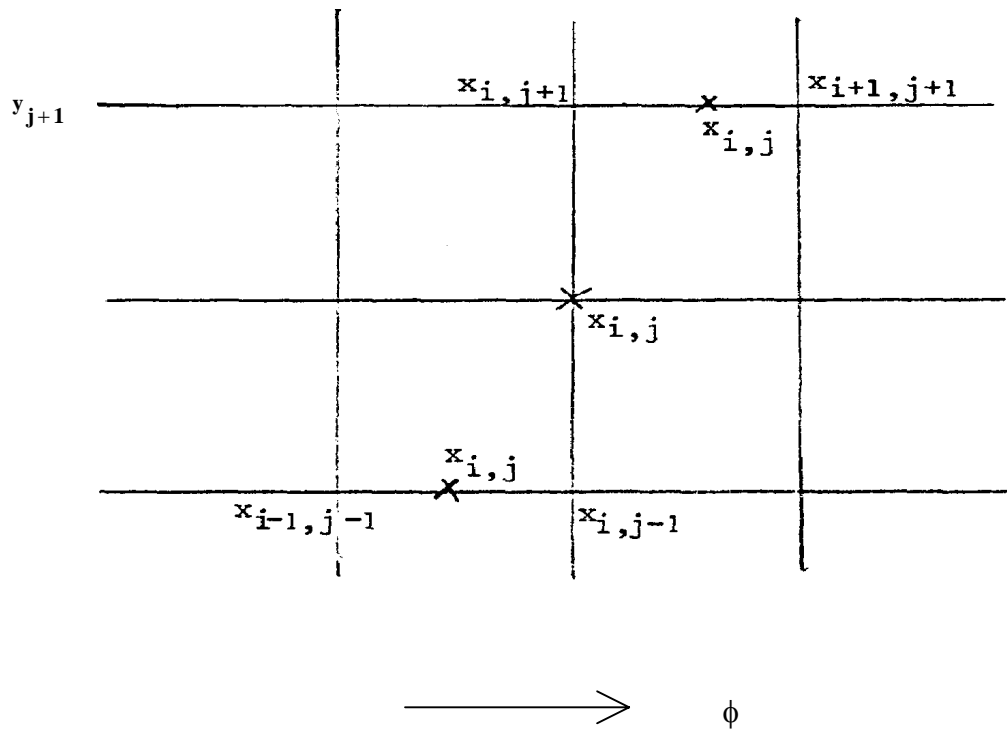


Figure 3

Interpolation for points at which  $x = x_{i,j}$ .

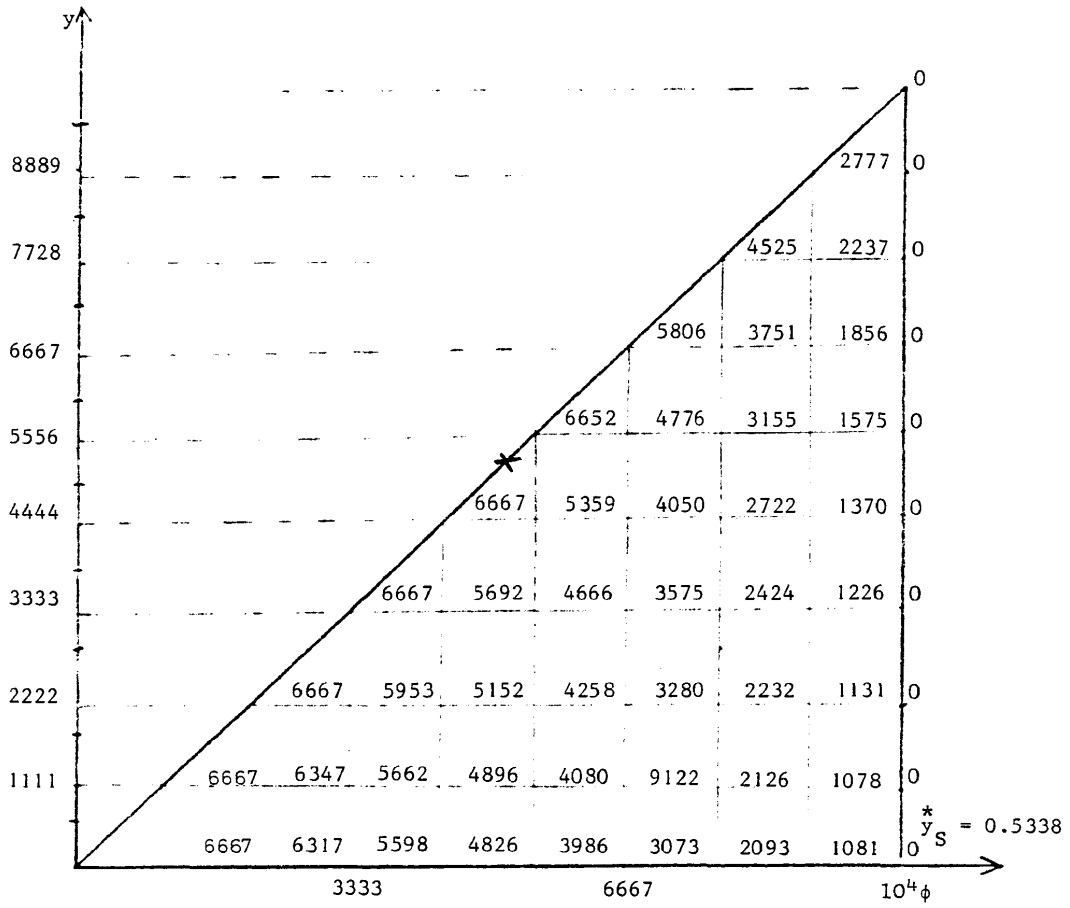


Figure 4. Selected values of  $10^4 x$  from 18 x 18 mesh.

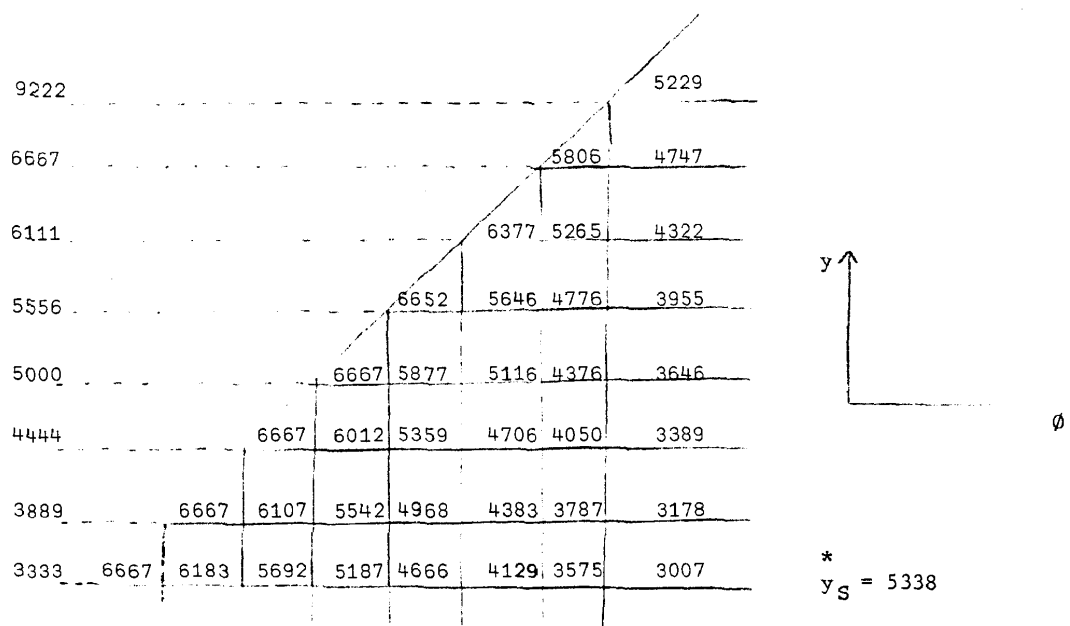


Figure 5. All values of  $10^4 x$  on section of 18 x 18 mesh. around separation point.

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