Nikolaos I. Ioakimidis; Pericles S. Theocaris

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Aplikace matematiky, Vol. 23 (1978), No. 6, 439–452

Persistent URL: `http://dml.cz/dmlcz/103770`

NUMERICAL SOLUTION OF CAUCHY TYPE SINGULAR INTEGRAL EQUATIONS BY USE OF THE LOBATTO-JACOBI NUMERICAL INTEGRATION RULE

N. I. IOAKIMIDIS and P. S. THEOCARIS

(Received March 29, 1977)

INTRODUCTION

An effective method of numerical solution of singular integral equations with a kernel consisting of a regular part as well as a Cauchy type singular part consists in the reduction of such an equation to a system of linear equations. This method was applied by Theocaris and Ioakimidis [1] to the numerical solution of singular integral equations of the form

$$(1)\qquad A(x)\, w(x)\, \varphi(x) + B(x) \int_{-1}^{1} w(t) \frac{\varphi(t)}{t-x}\, dt + \int_{-1}^{1} w(t)\, k(t,x)\, \varphi(t)\, dt = f(x)\,, \quad -1 < x < 1\,,$$

where $A(x)$ and $B(x)$ are bounded continuous functions in the integration interval $[-1, 1]$, $w(x)$ is a weight function of the form

$$(2)\qquad w(x) = (1-x)^a (1+x)^b\,, \quad a, b > -1\,,$$

$\varphi(x)$ is the unknown function of the integral equation, supposed to vary regularly along the integration interval, $k(t, x)$ is a kernel bounded with respect to both its variables in the integration interval except for $x = \pm 1$, where it may present Cauchy type singularities behaving like $1/(t-x)$, and $f(x)$ is a known function which may present weak singularities near the ends $x = \pm 1$ of the integration interval. Of course, the constants a and b in the weight function $w(x)$ should be compatible with the behaviour of the known functions $A(x)$, $B(x)$, $k(t, x)$ and $f(x)$ near the end-points $x = \pm 1$ of the integration interval.

The results obtained in [1] for the reduction of Eq. (1) to a system of linear equations of the form

$$(3)\qquad \sum_{k=1}^{n} A_k \left\{ \frac{B(x_r)}{t_k - x_r} + k(t_k, x_r) \right\} \varphi(t_k) = f(x_r)\,, \quad r = 1, 2, \ldots, m\,,$$

where in most cases $m = n - 1$, n or $n + 1$, have been based on the use of the Gauss-Jacobi numerical integration rule [2] for the approximate expression of the integrals in Eq. (1) together with the application of the resulting approximate equation at a certain number of properly selected points x_r of the integration interval $(-1, 1)$.

In this paper it will be shown that the Lobatto-Jacobi numerical integration rule, well known for regular integrals [2], can also be used for the numerical solution of the Cauchy type singular integral equation (1). This method has been already applied to the numerical solution of Eq. (1) by the present authors in the special cases where $a = b = -1/2$ and $A(x) = 0$ [3], $a = b = 0$ [4] or $a + b = -\varkappa$ (where $\varkappa$ is an integer number) and $A(x)$ and $B(x)$ are constants [5]. In this last case the points x_r of application of Eq. (1), appropriately selected in order that the system of linear equations (3) result, were the roots of an appropriate Jacobi polynomial. In the general case where the singularities a and b in the weight function $w(x)$, given by Eq. (2), are arbitrary, these points x_r should be selected as the roots of a transcendental equation, as will be seen in the sequel.

For the numerical solution of Cauchy type singular integral equations of the form (1) arising in practical applications, the values of the unknown function $\varphi(x)$ at the end-points $x = \pm 1$ of the integration interval are in most cases of practical interest. Since the Lobatto-Jacobi numerical integration rule contains among the abscissae used these points, it is quite appropriate for the numerical solution of Eq. (1) by the method described previously. The advantages of the Lobatto-Jacobi numerical integration rule over the Gauss-Jacobi rule in the case considered are described in detail in [3]. Here we will apply the Lobatto-Jacobi numerical integration rule to the numerical solution of the Cauchy type singular integral equations in which the problem of a crack terminating perpendicularly at the interface of two isotropic elastic media under generalized plane stress, plane strain or antiplane shear conditions is reduced.

THE LOBATTO-JACOBI RULE

The Lobatto-Jacobi numerical integration rule is associated with the Jacobi polynomials $P_n^{(a,b)}(z)$ and has, for regular integrals, the form [2]

$$\int_{-1}^{1} w(t)\, \varphi(t)\, \mathrm{d}t = \sum_{k=1}^{n} A_k\, \varphi(t_k) + E_n\,, \tag{4}$$

where the weight function $w(t)$ is given by Eq. (2), the abscissae t_k and the corresponding weights A_k are given by

$$(1 - t_k^2)\, P_{n-1}^{(a,b)\prime}(t_k) = 0\,, \quad t_1 = 1 > t_2 > \ldots > t_n = -1\,, \tag{5}$$

$$(6) \qquad A_1 = (1 + a)\, H_1\,, \quad A_n = (1 + b)\, H_n\,, \quad A_k = H_k \quad (k = 2, 3, \ldots, n - 1)\,,$$

$$H_k = \frac{2^{a+b+1}\, \Gamma(n + a)\, \Gamma(n + b)}{(n - 1)\, \Gamma(n)\, \Gamma(n + a + b + 1)} \frac{1}{\{P_{n-1}^{(a,b)}(t_k)\}^2} \quad (k = 1, 2, \ldots, n)\,,$$

and E_n is the error term, vanishing for integrands $\varphi(t)$ which are polynomiials of up to $(2n - 2)$ degree.

In accordance with the developments of Ioakimidis and Theocaris [6] who presented a general method of extending numerical integration rules for regular integrals to the case of Cauchy principal value integrals, the Lobatto-Jacobi numerical integration rule can also be used for the evaluation of Cauchy type principal value integrals. By taking into account the results obtained in [6], it can be easily found that

$$(7) \qquad \int_{-1}^{1} w(t) \frac{\varphi(t)}{t - x}\, dt = \sum_{k=1}^{n} A_k \frac{\varphi(t_k)}{t_k - x} - \frac{2q_n^{(a,b)}(x)\, \varphi(x)}{(1 - x^2)\, P_{n-1}^{(a,b)\prime}(x)} + E_n\,,$$

where the abscissae t_k and the weights A_k are given again by Eqs. (5) and (6) respectively, the function $q_n^{(a,b)}(z)$ is given by

$$(8) \qquad q_n^{(a,b)}(z) = -\frac{1}{2}\int_{-1}^{1} w(t) \frac{(1 - t^2)\, P_{n-1}^{(a,b)\prime}(t)}{t - z}\, dt$$

and the error term E_n may be computed by

$$(9) \qquad E_n = \frac{1}{\pi i}\int_C \frac{\varphi(\xi)}{\xi - x} \frac{q_n^{(a,b)}(\xi)}{(1 - \xi^2)\, P_{n-1}^{(a,b)\prime}(\xi)}\, d(\xi)\,,$$

where C denotes a contour surrounding the integration interval $[-1, 1]$. It may also be noted that $q_n^{(a,b)}(z)$, being a Cauchy type integral, is defined for the points x of the interval $(-1,1)$ in the principal value sense [7] and that the error term E_n vanishes for integrands $\varphi(t)$ which are polynomials of up to $(2n - 1)$ degree.

Now we will try to express the function $q_n^{(a,b)}(z)$, entering Eqs. (7) and (9), in terms of the derivative of the Jacobi function of the second kind $Q_n^{(a,b)}(z)$ [8]. First, we may note that $Q_n^{(a,b)}(z)$ is related to the Cauchy type integral $\Pi_n^{(a,b)}(z)$ defined by [9]

$$(10) \qquad \Pi_n^{(a,b)}(z) = -\int_{-1}^{1} w(t) \frac{P_n^{(a,b)}(t)}{t - z}\, dt$$

by the relation [8]

$$(11) \qquad \Pi_n^{(a,b)}(z) = 2(z - 1)^a\, (z + 1)^b\, Q_n^{(a,b)}(z)\,.$$

Next, by using the differentiation formula [10, p. 170]

$$(12)\qquad (2n + a + b)(1 - z^2)\begin{Bmatrix} P_n^{(a,b)\prime}(z) \\ Q_n^{(a,b)\prime}(z) \end{Bmatrix} =$$

$$= -n\{(2n + a + b)z + (b - a)\}\begin{Bmatrix} P_n^{(a,b)}(z) \\ Q_n^{(a,b)}(z) \end{Bmatrix} + 2(n + a)(n + b)\begin{Bmatrix} P_{n-1}^{(a,b)}(z) \\ Q_{n-1}^{(a,b)}(z) \end{Bmatrix},$$

valid both for the Jacobi polynomials $P_n^{(a,b)}(z)$ and the Jacobi functions of the second kind $Q_n^{(a,b)}(z)$, we can express the function $q_n^{(a,b)}(z)$, given by Eq. (8), as

$$(13)\qquad 2(2n + a + b)\, q_{n+1}^{(a,b)}(z) =$$
$$= -n\{(2n + a + b)z + (b - a)\}\, \Pi_n^{(a,b)}(z) + 2(n + a)(n + b)\, \Pi_{n-1}^{(a,b)}(z),$$

where Eq. (10) as well as the property of Jacobi polynomials $P_n^{(a,b)}(z)$ of being orthogonal along the interval $[-1, 1]$ with respect to the weight function $w(t)$, given by Eq. (2), have been taken into account. Furthermore, by combining Eqs. (11) and (12) we obtain

$$(14)\qquad 2(2n + a + b)(z - 1)^a (z + 1)^b (1 - z^2)\, Q_n^{(a,b)\prime}(z) =$$
$$= -n\{(2n + a + b)z + (b - a)\}\, \Pi_n^{(a,b)}(z) + 2(n + a)(n + b)\, \Pi_{n-1}^{(a,b)}(z).$$

Hence, Eq. (13) yields the following expression for the function $q_n^{(a,b)}(z)$:

$$(15)\qquad q_n^{(a,b)}(z) = (z - 1)^a (z + 1)^b (1 - z^2)\, Q_{n-1}^{(a,b)\prime}(z).$$

Equations (10–15) are valid for the points z of the complex plane outside the integration interval $[-1, 1]$. Thus, the expression (15) for the function $q_n^{(a,b)}(z)$ can be used for the computation of the error term E_n, given by Eq. (9), but it cannot be inserted into Eq. (7), except after a proper modification. Since any Cauchy type integral exists along the integration interval, its end-points excluded, only when interpreted in the principal value sense [7], the derivation of an expression similar to (15) for the function $q_n^{(a,b)}(z)$ when z coincides with a point x of the interval $(-1, 1)$, requires particular attention.

By using the second Plemelj formula [7] for the values $q_n^{(a,b)}(x)$ and $\Pi_n^{(a,b)}(x)$ of the functions $q_n^{(a,b)}(z)$ and $\Pi_n^{(a,b)}(z)$ along the integration interval $(-1, 1)$, considered as an open interval, we obtain

$$(16)\qquad q_n^{(a,b)}(x) = \tfrac{1}{2}\{q_n^{(a,b)}(x + 0i) + q_n^{(a,b)}(x - 0i)\},$$

$$(17)\qquad \Pi_n^{(a,b)}(x) = \tfrac{1}{2}\{\Pi_n^{(a,b)}(x + 0i) + \Pi_n^{(a,b)}(x - 0i)\},$$

where the right-hand side terms denote the limiting values of the functions $q_n^{(a,b)}(z)$ and $\Pi_n^{(a,b)}(z)$, the point z tending from the upper or the lower half-plane to the point x of the interval $(-1, 1)$. Hence, it is evident that Eq. (13) remains valid for the points x of the interval $(-1, 1)$.

Furthermore, by differentiating Eq. (17) with respect to x and taking into account the identity

$$\lim_{z \to x \pm 0i} \{\Pi_n^{(a,b)\prime}(z)\} = \frac{\mathrm{d}}{\mathrm{d}x} \{\Pi_n^{(a,b)}(x \pm 0i)\}, \tag{18}$$

which is generally valid for functions like $\Pi_n^{(a,b)}(z)$ defined as Cauchy type integrals [7], we obtain

$$\Pi_n^{(a,b)\prime}(x) = \tfrac{1}{2}\{\Pi_n^{(a,b)\prime}(x + 0i) + \Pi_n^{(a,b)\prime}(x - 0i)\}. \tag{19}$$

In this equation $\Pi_n^{(a,b)\prime}(x)$ denotes both the function $\Pi_n^{(a,b)\prime}(z)$ for $z \equiv x$ and the function $(\mathrm{d}/\mathrm{d}x)\{\Pi_n^{(a,b)}(x)\}$.

Now, by differentiating Eq. (11) with respect to z, we obtain

$$\begin{aligned} &(1 - z^2)\,\Pi_n^{(a,b)\prime}(z) = \\ &= 2(z - 1)^a (z + 1)^b \left[-\{(a + b)\,z + (a - b)\}\, Q_n^{(a,b)}(z) + (1 - z^2)\, Q_n^{(a,b)\prime}(z)\right]. \end{aligned} \tag{20}$$

Equation (20), combined with (11) and (12), gives for the derivative of the function $\Pi_n^{(a,b)}(z)$ outside the integration interval

$$\begin{aligned} &(2n + a + b)(1 - z^2)\,\Pi_n^{(a,b)\prime}(z) = -(n + a + b)\,. \\ &.\{(2n + a + b)\,z + (a - b)\}\,\Pi_n^{(a,b)}(z) + 2(n + a)(n + b)\,\Pi_{n-1}^{(a,b)}(z)\,. \end{aligned} \tag{21}$$

Because of Eqs. (17) and (19), it can be further seen that Eq. (21) remains valid even for the points x of the integration interval $(-1, 1)$.

Next, if we define the Jacobi function of the second kind $Q_n^{(a,b)}(x)$ on the integration interval $(-1, 1)$ by

$$Q_n^{(a,b)}(x) = \tfrac{1}{2}\Pi_n^{(a,b)}(x)/w(x), \tag{22}$$

a definition which is different from the analogous definition given in [10, p. 171], and take into account Eq. (21), we can easily see that $Q_n^{(a,b)}(x)$ satisfies Eq. (12) not only outside the integration interval $(-1, 1)$ but on this interval, too.

Finally, by combining Eqs. (12), (13) and (22), we obtain the following expression for the function $q_n^{(a,b)}(z)$ along the integration interval $(-1, 1)$

$$q_n^{(a,b)}(x) = w(x)\,(1 - x^2)\, Q_{n-1}^{(a,b)\prime}(x), \tag{23}$$

which is completely analogous to Eq. (15). It is also evident that such simple expressions of the function $q_n^{(a,b)}(z)$ like (15) and (23) cannot be obtained in terms of the function $\Pi_n^{(a,b)}(z)$.

Since the Jacobi function of the second kind $Q_n^{(a,b)}(z)$ can be expressed, outside the integration interval $[-1, 1]$, as a hypergeometric function [8], the same will be true for the function $q_n^{(a,b)}(z)$, given by Eq. (15) for the points of the complex plane outside the integration interval. For our problem, the values of the function $q_n^{(a,b)}(z)$ along the integration interval $(-1, 1)$ are of the greatest importance. Along this interval we can take into account that [11]

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\,\frac{P_n^{(a,b)}(t)}{t-x}\,\mathrm{d}t = \cot \pi a\, w(x)\, P_n^{(a,b)}(x) - \tag{24}$$

$$-\frac{2^{a+b}\,\Gamma(a)\,\Gamma(n+b+1)}{\pi\,\Gamma(n+a+b+1)}\,F\left(n+1,\ -n-a-b;\ 1-a;\ \frac{1-x}{2}\right),\quad a \neq 0, 1, \ldots .$$

Then, because of Eqs. (10) and (22), we obtain

$$Q_n^{(a,b)}(x) = -\frac{\pi}{2}\cot \pi a\, P_n^{(a,b)}(x) + \frac{2^{a+b-1}\,\Gamma(a)\,\Gamma(n+b+1)}{w(x)\,\Gamma(n+a+b+1)}\,. \tag{25}$$

$$.\,F\left(n+1,\ -n-a-b;\ 1-a;\ \frac{1-x}{2}\right),\quad a \neq 0, 1, \ldots .$$

At this stage we have to remark that Eq. (24) was considered in [11] to be valid only when the sum of the singularities $(a + b)$ of the weight function $w(x)$ is an integer number. Nevertheless, by taking into account the formula [10, p. 171]

$$Q_n^{(a,b)}(z) = -\frac{\pi}{2}\operatorname{cosec} \pi a\, P_n^{(a,b)}(z) + 2^{a+b-1}\,\frac{\Gamma(a)\,\Gamma(n+b+1)}{\Gamma(n+a+b+1)}\,. \tag{26}$$

$$.\,(z-1)^{-a}(z+1)^{-b}\,F\left(n+1,\ -n-a-b; 1-a;\ \frac{1-z}{2}\right)$$

as well as Eqs. (11),(17) and (22), we can see that Eqs. (24) and (25) remain valid for all values of a and b ($a \neq 0, 1, \ldots$). We can also mention that Eq. (25) differs from the corresponding equation given in [10, p. 171] because, as we have already mentioned, a definition of the function $Q_n^{(a,b)}(z)$ for the points x of the integration interval $(-1, 1)$ different from (22) was considered in this reference.

We can also remark that the functions $Q_n^{(a,b)}(z)$ satisfy both outside the integration interval $[-1, 1]$ and on this interval the same differential equation and the same reccurence relations as the Jacobi polynomials $P_n^{(a,b)}(z)$ [8, 1]. Hence, their evaluation is very easy. As regards the derivative $Q_n^{(a,b)\prime}(z)$, Eq. (12) can be used in the whole complex plane, too. Also, in the case when $a = 0, 1, \ldots$, an analogous formula to (24) can be easily obtained. Finally, for the numerical evaluation of $Q_o^{(a,b)}(x)$ which is necessary when using the recurrence relations of Jacobi functions for the evaluation of $Q_n^{(a,b)}(z)$, the developments of [1] which make use of a rapidly convergent infinite series along the whole interval $(-1, 1)$, can be taken into account.

Following these arguments, we will proceed to the application of the Lobatto-Jacobi numerical integration rule for both regular and Cauchy type principal value integrals, expressed by Eqs. (4) and (7) respectively, to the numerical solution of Eq. (1).

By applying the Lobatto-Jacobi numerical integration rule expressed by Eqs. (4) and (7) to the integrals of the left-hand side of Eq. (1) and nelgecting the error term E_n due to numerical integrations, we obtain the following approximate form of Eq. (1)

$$(27)\quad \left\{A(x)\,w(x) - 2\,B(x)\frac{q_n^{(a,b)}(x)}{(1-x^2)\,P_{n-1}^{(a,b)\prime}(x)}\right\}\varphi(x) + \sum_{k=1}^{n} A_k\left\{\frac{B(x)}{t_k - x} + k(t_k, x)\right\}\cdot \varphi(t_k) = f(x)\,, \quad -1 < x < 1\,.$$

Next, it is easy to see that if we choose the points x_r of application of Eq. (27) as the roots of the equation

$$(28)\qquad A(x)\,w(x) - 2\,B(x)\frac{q_n^{(a,b)}(x)}{(1-x^2)\,P_{n-1}^{(a,b)\prime}(x)} = 0\,,$$

or, because of Eq. (23), of

$$(29)\qquad F_{n-1}^{(a,b)}(x) = A(x)\,P_{n-1}^{(a,b)\prime}(x) - 2\,B(x)\,Q_{n-1}^{(a,b)\prime}(x) = 0\,,$$

which is in general a transcendental equation, then Eq. (1) can be approximated by the system of linear equations (3). As we will see in the sequel, the roots x_r of Eq. (29) are at least $(n-1)$. In this way, it is probable that one more linear equation, besides those of system (3), is required. Such an equation may result from a physical condition of the form

$$(30)\qquad \int_{-1}^{1} w(t)\,\varphi(t)\,\mathrm{d}t = C$$

(where C is a known constant), which after an application of the Lobatto-Jacobi rule (4) may be approximated by the following linear equation

$$(31)\qquad \sum_{k=1}^{n} A_k\,\varphi(t_k) = C\,,$$

or it may even be of the form

$$(32)\qquad \varphi(d) = C\,, \quad d = -1 \quad \text{or} \quad 0 \quad \text{or} \quad 1\,,$$

resulting also from physical considerations at the end-points or, perhaps, at the middle-point of the integration interval. More detail on this subject may be found in [1, 3, 4, 5, 11].

Furthermore, proceeding in a way analogous to that described in [1], where the Gauss-Jacobi method was applied to the numerical solution of Eq. (1), we can easily show that Eq. (29) has at least one root in each subinterval of the integration interval defined by two consecutive abscissae of the Lobatto-Jacobi numerical integration rule, the abscissae $t_k = \pm 1$ included, provided the restrictions $a, b > -1$ hold.

It can be also shown that in the case when the functions $A(x)$ and $B(x)$ reduce to constants, there is only one root of Eq. (29) in each of the subintervals considered, or, in another wording, Eq. (29) has $(n - 1)$ roots alternating with the n roots of the polynomial $(1 - x^2) P_{n-1}^{(a,b)\prime}(x)$, which are the abscissae of the Lobatto-Jacobi numerical integration rule. To achieve it, we take into account that both the Jacobi polynomials $P_n^{(a,b)}(x)$ and the Jacobi functions of the second kind $Q_n^{(a,b)}(x)$ satisfy the same second order differential equation [8, pp. 60, 74]. Then it can be seen that the same will be true for their derivatives $P_{n-1}^{(a,b)\prime}(x)$ and $Q_{n-1}^{(a,b)\prime}(x)$ as well as for the functions $F_{n-1}^{(a,b)}(x)$ defined by Eq. (29), the roots of which are sought. Then we can apply the following Sturm's type theorem reported by Porter [12]:

Theorem. *If within an interval (a, b) (excluding the ends) the coefficients of the differential equation*

$$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0$$

are continuous, and if there exists a solution y_1 which does not vanish between a and b such that its ratio to a linearly independent solution y_2 approaches zero as we approach each end of the interval, then y_2 vanishes once and only once between a and b,

by considering the differential equation satisfied by $P_{n-1}^{(a,b)\prime}(x)$, $Q_{n-1}^{(a,b)\prime}(x)$ and $F_{n-1}^{(a,b)}(x)$, easily resulting from the developments of [8, p. 60]

$$(33) \qquad (1 - x^2)\,y'' + \{b - a - (a + b + 4)\,x\}\,y' + \\ + \{(n - 1)(n + a + b) - (a + b + 2)\}\,y = 0$$

and assuming that the functions y_1 and y_2 coincide with the functions $P_{n-1}^{(a,b)\prime}(x)$ and $F_{n-1}^{(a,b)}(x)$, while the interval (a, b) is each one of the $(n - 1)$ subintervals to which the roots of $P_{n-1}^{(a,b)\prime}(x)$ divide the integration interval $[-1, 1]$. By the direct application of the foregoing theorem we can find that in each of these subintervals there exists one and only one root of $F_{n-1}^{(a,b)}(x)$.

Although this fact is evident for all subintervals (a, b) for which $a, b \neq \pm 1$, it is necessary to show that $F_{n-1}^{(a,b)}(x)$ becomes unbounded as $x \to \pm(1 - 0)$, in order that the proof for the subintervals $(-1, t_{n-1})$ and $(t_2, 1)$ be complete. Since $P_{n-1}^{(a,b)\prime}(x)$ is bounded for $x \to \pm 1$, we have to show, because of Eq. (29), that $Q_{n-1}^{(a,b)\prime}(x)$ becomes unbounded as $x \to \pm(1 - 0)$. This fact can be easily established for $x \to 1 - 0$ if Eq. (25) and the assumption $a > -1$ are taken into account with the exception of the case when a is a non-negative integer number. However, even in this case it can be seen that $Q_{n-1}^{(a,b)\prime}(x)$ becomes unbounded for $x \to 1 - 0$, if the developments of Szegö [8, pp. 78–79] are taken into account. In accordance with these developments, we have the following behaviour of $Q_n^{(a,b)}(x)$ for $x \to 1 - 0$

$$(34) \qquad Q_n^{(a,b)}(x) \sim \begin{cases} (1 - x)^{-a}, & a = 1, 2, \ldots \\ \ln(1 - x), & a = 0 \end{cases}$$

and, therefore, $Q_n^{(a,b)\prime}(x)$ tends to infinity for $x \to 1 - 0$. In an analogous way it can be shown that $Q_n^{(a,b)\prime}(x)$ tends to infinity for $x \to -1 + 0$ under the assumption that $b > -1$.

Thus, it is seen that under the original assumptions $a, b > -1$ (Eq. (2)), necessary for the existence of the integrals with a weight function $w(t)$, the function $Q_{n-1}^{(a,b)\prime}(x)$ becomes unbounded as $x \to \pm(1 - 0)$. Thus it was proved, for the case when $A(x)$ and $B(x)$ are constants, that the number of roots x_r of the function $F_{n-1}^{(a,b)}(x)$ (that is the points of application of Eq. (1) used for its reduction to the system of linear equations (3)) is $(n - 1)$ and the roots alternate with the n roots of the function $(1 - x^2)$. $P_{n-1}^{(a,b)\prime}(x)$, which are the abscissae used in the Lobatto-Jacobi numerical integration rule.

AN APPLICATION

As an application we consider the problem of a crack of a length c perpendicular to the interface of two semiinfinite isotropic elastic media S_1 and S_2 as shown in Fig. 1. These media are characterized by their shear moduli μ_1 and μ_2 and their

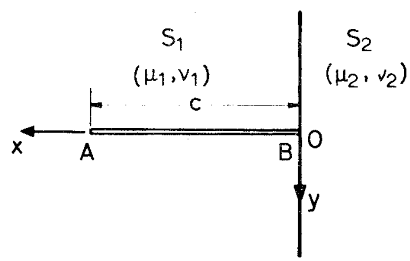

Fig. 1. A straight crack terminating perpendicularly at the interface of two media.

Poisson ratios ν_1 and ν_2, respectively. The whole elastic plane is supposed to be under a generalized plane stress, a plane strain or antiplane shear conditions. These problems have been considered by Erdogan and Cook [11, 13, 14], who showed that they can be reduced to a Cauchy type singular integral equation along the crack.

If in the cases of a generalized plane stress or a plane strain the loading on the edges of the crack is supposed to be a uniform compressive loading of a constant intensity σ, then, in accordance with the developments of [11,13], the whole problem can be reduced to the following Cauchy type singular integral equation, written here under a slightly modified form

$$\frac{1}{\pi}\int_0^1 w(t)\,\varphi(t)\left\{\frac{1}{t-x}+k(t,x)\right\}\mathrm{d}t = 1\,, \tag{35}$$

where

$$w(t) = (1-t)^a\, t^b \tag{36}$$

with $a = -1/2$ and b determined as the smallest root of the equation

$$2d_1 \cos \pi(b+1) - d_2(b+1)^2 - d_3 = 0\,, \tag{37}$$

the constants d_1, d_2 and d_3 given by

$$\begin{aligned} d_1 &= \quad (m+\varkappa_2)(1+m\varkappa_1)\,,\\ d_2 &= -4(1-m)(m+\varkappa_2)\,,\\ d_3 &= \quad (1-m)(m+\varkappa_2)+(1+m\varkappa_1)(m+\varkappa_2)-m(1+\varkappa_1)(1+m\varkappa_1)\,, \end{aligned} \tag{38}$$

where

$$m = \mu_2/\mu_1 \tag{39}$$

and

$$\varkappa_i = (3-v_i)/(1+v_i)\,, \quad i = 1, 2 \tag{40}$$

for generalized plane stress conditions and

$$\varkappa_i = 3 - 4v_i\,, \quad i = 1, 2 \tag{41}$$

for plane strain conditions. As regards the kernel $k(t, x)$ in Eq. (35), it is given by

$$k(t,x) = \frac{1}{2d_1(t+x)}\left\{d_2 + d_3 - 3d_2\,\frac{x}{t+x} + 2d_2\left(\frac{x}{t+x}\right)^2\right\}. \tag{42}$$

Equation (35) should also be supplemented by the condition of single-valuedness of displacements

$$\int_0^1 w(t)\,\varphi(t)\,\mathrm{d}t = 0\,. \tag{43}$$

For the numerical solution of Eq. (35) supplemented by condition (43), Erdogan and Cook [11, 13] have used the Gauss-Jacobi numerical integration rule, of course after a transformation of the integration interval [0, 1] to the interval [−1, 1]. Although the results obtained by Erdogan and Cook were correct, it was shown by Theocaris and Ioakimidis [1] that the selection of points x_r of application of Eq. (35) was not the proper one and this fact resulted in a very slow convergence of the results with increasing values of the number n of abscissae used. Since in the problem under consideration the values of the stress intensity factors at the crack tips A and B (Fig. 1) are the quantities of the greatest importance and these factors are related to the unknown function $\varphi(t)$ by [11, 13]

$$(44) \qquad k_A = 2^{1/2}\,\varphi(1)\,\sigma c^{-a},$$

$$k_B = -2^{1/2}\frac{m(1+\varkappa_1)\{(3+2b)(1+m\varkappa_1)-(1+2b)(m+\varkappa_2)\}}{2d_1 \sin \pi b}\,\varphi(0)\,\sigma c^{-b},$$

it is evident from the arguments of [3] that the use of a Lobatto-type numerical integration rule for the numerical solution of Eq. (35) presents considerable advantages over the use of a Gauss-type rule. Thus, the Lobatto-Jacobi numerical integration rule has been used for the numerical solution of Eq. (35), together with condition (43), in accordance with the developments of the present paper. The elastic materials have been assumed to be epoxy (S_1) and aluminum (S_2). Then we have [11, 13]: $\nu_1 = 0{\cdot}35$, $\nu_2 = 0{\cdot}30$, $m = \mu_2/\mu_1 = 23{\cdot}077$.

The results obtained after a reduction of Eq. (35) and condition (43) to a system of linear equations and the numerical solution of the latter are presented in Table 1 both for a generalized plane stress and plane strain conditions. The number of abscissae used, equal to the number of linear equations, was obtained as $n = 3(3)\,15$. Also an appropriate transformation of variables has been made for the reduction of the integration interval [0, 1] to [−1, 1]. The results, shown in Table 1, are in

Table 1

Convergence of the numerically obtained reduced values of the stress intensity factors at the tips of a crack terminating perperdicularly at the interface of two media ($\nu_1 = 0{\cdot}35$, $\nu_2 = 0{\cdot}30$, $\mu_2/\mu_1 = 23{\cdot}077$).

n	Plane stress		Plane strain		Antiplane shear	
	$k_A\sigma^{-1}c^a$ $a = -0{\cdot}5$	$k_B\sigma^{-1}c^b$ $b = -0{\cdot}288977$	$k_A\sigma^{-1}c^a$ $a = -0{\cdot}5$	$k_B\sigma^{-1}c^b$ $b = -0{\cdot}338113$	$k_A\sigma^{-1}(c/2)^a$ $a = -0{\cdot}5$	$k_B\sigma^{-1}(c/2)^b$ $b = -0{\cdot}130657$
3	0·872674	4·157028	0·878736	2·865473	0·896674	8·769577
6	0·878576	4·161217	0·882556	2·789663	0·906129	11·583040
9	0·878637	4·146768	0·882547	2·777447	0·906559	12·598343
12	0·878650	4·141488	0·882545	2·775064	0·906631	13·160276
15	0·878655	4·139260	0·882544	2·774655	0·906652	13·528842
Refs.*) [13, 14]	0·8789	4·1760	0·8827	2·7845	0·90709	13·13303

*) The results of these references may have been obtained for a slightly different value of $m = \mu_2/\mu_1$ (considered here equal to 23·077).

accordance with those given in [11, 13] for the same problem. Nevertheless, they are seen to converge to their correct values much more rapidly if a comparison of the speed of convergence with Table VII of [11] is made*). It can also be seen that the results of column 3 of Table 1 converge faster than the corresponding results obtained in [1] by using the Gauss-Jacobi numerical integration rule after a proper selection of the points of application of Eq. (35).

Furthermore, the antiplane shear problem for the same crack AB (shown in Fig. 1) was considered. This problem was reduced by Erdogan and Cook [14] to the following Cauchy type singular integral equation, written here in a slightly modified form

$$\frac{1}{\pi}\int_0^1 w(t)\,\varphi(t)\left\{\frac{1}{t-x}+\frac{\lambda}{t+x}\right\}\mathrm{d}t = 1\,, \tag{45}$$

where a constant antiplane shear loading $(-\sigma)$ was assumed on the crack edges and

$$\lambda = (\mu_1 - \mu_2)/(\mu_1 + \mu_2)\,. \tag{46}$$

The weight function $w(t)$ is given again by Eq. (36) where [14]

$$a = -1/2\,, \quad \cos \pi b = -\lambda\,. \tag{47}$$

The results of the numerical solution of Eq. (45) supplemented also by condition (43) and under exactly the same conditions considered for the numerical solution of Eq. (35), are shown in the form of the reduced values of the stress intensity factors at the crack tips determined by [14]

$$k_A = 2^{1/2}\,\varphi(1)\,\sigma c^{-a}\,, \quad k_B = -(2m)^{1/2}\,\varphi(0)\,\sigma c^{-b} \tag{48}$$

in the last two columns of Table 1. These results can be seen to be in accordance with the corresponding results of [14].

Finally, we observe from the results of Table 1 that the convergence of the reduced values of the stress intensity factors at the tip A, far from the interface, is much faster, especially in the antiplane shear case, than the convergence of the corresponding values at the tip B lying on the interface. This fact can be easily explained if the form of the kernels in Eqs. (35) and (45) is taken into account. Since these kernels contain, besides the Cauchy type term, terms which have poles in the complex plane and not simple polynomials, it is evident that these terms will contribute significantly to the error terms E_n neglected during the application of the Lobatto-Jacobi numerical integration rule for the approximation of the integrals in the integral equations. When a point x of application of the integral equations is near the crack tip B, the

*) The results of the second column of this Table seem to be multiplied by a constant. The correct result is given in [13] and may be seen to be in agreement with the results obtained here (Table 1 column 4).

poles of the kernels lie near this crack tip and their contribution may become significant. The opposite holds for the crack tip A; near this crack tip no poles of the kernels exist except the one due to the Cauchy type principal value term. Nevertheless, this pole has been taken into account through a proper selection of the points of application of the integral equations.

Although it seems very difficult to prove theoretically the convergence of the numerical results obtained from the numerical solution of Eqs. (35) and (45) to their correct values, nevertheless, it seems that this occurs. Firstly, it is seen that the results presented in Table 1 converge very fast with the exception of the results of the last column of this Table. Secondly, the results of columns 1 and 2 (corresponding to the case of a generalized plane stress) are seen to be in agreement with the results obtained by Lin and Mar [15], which are 0·855 and 4·240, respectively. Also the results of columns 5 and 6 (corresponding to the case of antiplane shear) are seen to converge to their theoretical values, which can be found if the developments of Smith [16] of Chou [17] are taken into consideration together with Eqs. (48). Thus, in the special case considered it results in

$$k_A = \frac{b}{\sin \frac{1}{2}\pi b}\, \sigma c^{-a}\,, \quad k_B = \frac{1-b}{\sin \frac{1}{2}\pi b}\, m^{1/2}\, 2^{b-1/2} \sigma c^{-b}\,. \tag{49}$$

The numerical values obtained from Eqs. (49) are 0·90667 and 14·48995 in the reduced form for the tips A and B respectively. These values are in accordance with the results of columns 5 and 6 of Table 1, respectively.

Finally, it can be mentioned that the influence of the pole due to the second term of the kernel of Eq. (45) was seen, through numerical experiments, not to be the main reason for the slow convergence of the numerical results in the last column of Table 1. It seems that this slow convergence is due to the fact that $\varphi(t)$ has not a regular behaviour for $t \to 0$ [16, 17].

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Souhrn

NUMERICKÉ ŘEŠENÍ SINGULÁRNÍCH INTEGRÁLNÍCH ROVNIC CAUCHYOVA TYPU LOBATTOVOU-JACOBIOVOU METODOU NUMERICKÉ INTEGRACE

N. I. IOAKIMIDIS, P. S. THEOCARIS

Lobattovu-Jacobiovu metodu numerické integrace lze rozšířit na numerický výpočet hlavní hodnoty integrálu Cauchyova typu a na numerické řešení singulárních integrálních rovnic s jádrem Cauchyova typu jejich převedením na soustavu lineárních rovnic. Za tím účelem jsou integrály v takové singulární integrální rovnici nahrazeny součty stejně jako v případě regulárních integrálů, při vhodné volbě bodů v integračním intervalu. Metoda je aplikována na problém z rovinné teorie pružnosti.

Authors' addresses: Dr. *P. S. Theocaris* and Dr. *N. I. Ioakimidis*, Department of Theoretical and Applied Mechanics, National Technical University of Athens, 5 K. Zographou Street, Zographou, Athens (625), Greece.