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Numerical Solution of Fractional Black-Scholes Equation by Using the Multivariate Padé Approximation

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In this study, a new application of multivariate Padé approximation method has been used for solving European vanilla call option pricing problem. Padé polynomials have occurred for the fractional Black-Scholes equation, according to the relations of "smaller than", or "greater than", between stock price and exercise price of the option. Using these polynomials, we have applied the multivariate Padé approximation method to our fractional equation and we have calculated numerical solutions of fractional Black-Scholes equation for both of two situations. The obtained results show that the multivariate Padé approximation is a very quick and accurate method for fractional Black-Scholes equation. The fractional derivative is understood in the Caputo sense.

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1. Introduction

Fractional calculus expresses the calculus of the integration and differentiation, the order of which is given by a fractional number. Many applications of fractional calculus amount for the replacing the time derivative in an evolution equation with a derivative of a fractional order.

Partial differential equations of fractional order, as generalizations of classical integer order partial differential equations, are increasingly used in model problems of fluid flow, physical and biological processes, control, engineering and systems [1–5]. Most fractional differential equations do not have exact analytical solutions, therefore, approximation and numerical techniques are used extensively for solving these equations [6]. Many powerful numerical and analytical methods have been presented in literature on finance. Among them, homotopy perturbation method with Sumudu transform and Laplace transform [7–9], homotopy analysis method [10], fractional variational iteration method [11], variational iteration method with Sumudu transform, finite difference method [12] and fractional diffusion models [13, 14] are relatively new approaches providing an analytical and numerical approximation to Black-Scholes option pricing equation. Methods described in [15, 16] are the other numerical methods, used in order to solve dynamic problems in elastic media and generalized semi-infinite programming.

In 1969, Fisher Black and Myron Scholes [17] got an idea that would change the world of finance forever. The central idea of their study revolved around the discovery that one did not need to estimate the expected return of a stock in order to price an option written on that stock. The Black-Scholes model for pricing stock options has been applied to many different commodities and payoff structures. The Black-Scholes model for value of an option is described in [15] by the following equation:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}+r\left(t\right)S\frac{\partial C}{\partial S}-r\left(t\right)C=0,$$
$$(S,t)\in\mathbb{R}^+\times(0,T)\,, \tag{1}$$

where $C=C(S,t)$ is the European option price at asset price S and time t. T is maturity, $r(t)$ is the risk-free interest rate and $\sigma(S,t)$ represents the volatility function of the underlying asset. In Eq. (1), we demonstrate that $C(0,t)=0$, $C(S,t)\sim S$ as $S\to\infty$ and $C(S,T)=\max(S-E,0)$. The Eq. (1) is clearly in backward form with final data given at $t=T$. The first thing to do in Eq. (1) is to get rid of the awkward S and S^2 terms multiplying $\partial C/\partial S$ and $\partial^2C/\partial S^2$. We set:

$$S=E\mathrm{e}^x,\quad t=T-\frac{2\tau}{\sigma^2},\quad C=E\nu(x,\tau).$$

This results in the Eq. (1) as

$$\frac{\partial^\alpha\nu}{\partial\tau^\alpha}=\frac{\partial^2\nu}{\partial x^2}+(k-1)\frac{\partial\nu}{\partial x}-k\nu,\quad 0<\alpha\le1, \tag{2}$$

with initial condition

$$\nu\left(x,0\right)=\max\left(\mathrm{e}^x-1,0\right). \tag{3}$$

The aim of this paper is to extend the application of multivariate Padé approximation (MPA) method to obtain approximate solution of fractional Black-Scholes Eq. (2) with initial condition (3).

2. Basic definitions

Definition 2.1. [18] The fractional derivative of $f\left(x\right)$ in the Caputo sense is defined as

$$D_*^\alpha f\left(x\right)=J^{m-\alpha}D^mf\left(x\right)=$$
$$\frac{1}{\Gamma\left(m-\alpha\right)}\int_0^x\left(x-t\right)^{m-\alpha-1}f^m\left(t\right)dt,$$

for $m-1<\alpha\le m$, $m\in\mathbb{N}$, $x>0$, $f\in C_{-1}^m$.

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Definition 2.2. [19] The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}.$$

3. Multivariate Padé approximation method

Let us consider the bivariate function $f(x,y)$ with Taylor series development $f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j$ around the origin. Now we define

$$p(x,y) = \begin{vmatrix} \sum\limits_{i+j=0}^{m} c_{ij}x^iy^j & \sum\limits_{i+j=0}^{m-1} c_{ij}x^iy^j & \cdots & \sum\limits_{i+j=0}^{m-n} c_{ij}x^iy \\ \sum\limits_{i+j=m+1} c_{ij}x^iy^j & \sum\limits_{i+j=m} c_{ij}x^iy^j & \cdots & \sum\limits_{i+j=m+1-n} c_{ij}x^iy^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i+j=m+n} c_{ij}x^iy^j & \sum\limits_{i+j=m+n-1} c_{ij}x^iy^j & \cdots & \sum\limits_{i+j=m} c_{ij}x^iy^j \end{vmatrix},$$

$$q(x,y) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum\limits_{i+j=m+1} c_{ij}x^iy^j & \sum\limits_{i+j=m} c_{ij}x^iy^j & \cdots & \sum\limits_{i+j=m+1-n} c_{ij}x^iy^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i+j=m+n} c_{ij}x^iy^j & \sum\limits_{i+j=m+n-1} c_{ij}x^iy^j & \cdots & \sum\limits_{i+j=m} c_{ij}x^iy^j \end{vmatrix}. \quad (4)$$

We call $p(x,y)$ and $q(x,y)$ Padé equations [18]. Thus the multivariate Padé approximant of order (m,n) for $f(x,y)$ is defined as the irreducible form [20], as

$$[m,n]_{(x,y)} = \frac{p(x,y)}{q(x,y)}. \quad (5)$$

4. Numerical solution of fractional Black-Scholes equation with MPA

In this section, we show the right relationship with the MPA solution and exact solution of the Black-Scholes option pricing equation. All the computations have been performed using Maple and Matlab Software. The solution of Eqs. (2) and (3) is given by

$$\nu(x,\tau) = \max(\mathrm{e}^x - 1, 0)\, E_\alpha(-k\tau^\alpha) + \mathrm{e}^x (1 - E_\alpha(-k\tau^\alpha)), \quad (6)$$

where $E_\alpha(z)$ is Mittag-Leffler function of one parameter [21]. For case $\alpha = 1$, we have

$$\nu(x,\tau) = \max(\mathrm{e}^x - 1, 0)\, \mathrm{e}^{-k\tau} + \mathrm{e}^x \left(1 - \mathrm{e}^{-k\tau}\right),$$

which is an exact solution of the classic Black-Scholes equation. Let us consider the Eq. (5) as

$$\nu(x,\tau) = \begin{cases} \mathrm{e}^x (1 - E_\alpha(-k\tau^\alpha)), \text{ if } S < E, \ (x < 0), \\ (\mathrm{e}^x - 1) E_\alpha(-k\tau^\alpha) + \mathrm{e}^x (1 - E_\alpha(-k\tau^\alpha)), \\ \quad \text{if } S > E, \ (x > 0). \end{cases} \quad (7)$$

4.1. Situation of $S < E$:

If we take that the stock price is smaller than the exercise price ($S < E$), we obtain that the expansion of the series in Eq. (6) for $\alpha = 1$ is

$$\nu(x,\tau) = k\tau + kx\tau - \frac{k^2\tau^2}{2} + \frac{k^3\tau^3}{6} - \frac{k^2x\tau^2}{2} + \frac{kx^2\tau}{2} + \frac{k^3x\tau^3}{6} - \frac{k^2x^2\tau^2}{4} + \cdots. \quad (8)$$

Now let us calculate the approximate solution of Eq. (8) for $m = 3$ and $n = 2$ by using MPA. We use the polynomials in Eq. (3) and we obtain:

$$p(x,\tau) = \frac{1}{36}k^7\tau^7 - \frac{1}{72}k^8\tau^8 + \frac{1}{216}k^9\tau^9 + \frac{1}{8}k^5x^2\tau^5 - \frac{1}{12}k^6x\tau^6 - \frac{1}{6}k^4x^3\tau^4 + \frac{1}{12}k^3x^4\tau^3 + \frac{1}{24}k^7x\tau^7 - \frac{1}{16}k^6x^2\tau^6 + \frac{7}{72}k^5x^3\tau^5 - \frac{1}{12}k^4x^4\tau^4 - \frac{1}{72}k^8x\tau^8 + \frac{1}{48}k^7x^2\tau^7 - \frac{1}{48}k^6x^3\tau^6 + \frac{1}{48}k^5x^4\tau^5 - \frac{1}{72}k^3x^6\tau^3, \quad (9)$$

$$q(x,\tau) = \frac{1}{24}k^3x^5\tau^7 - \frac{1}{6}k^3x^3\tau^3 + \frac{1}{12}k^2x^4\tau^2 + \frac{1}{8}k^4x^2\tau^4 - \frac{1}{12}k^5x\tau^5 + \frac{1}{72}k^6x^2\tau^6 - \frac{1}{36}k^6x\tau^6 + \frac{1}{8}k^3x^4\tau^3 + \frac{5}{144}k^4x^4\tau^4 - \frac{1}{36}k^5x^3\tau^5 - \frac{1}{9}k^4x^3\tau^4 + \frac{1}{12}k^5x^2\tau^5 + \frac{1}{36}k^6\tau^6 - \frac{1}{12}k^2x^5\tau^2 + \frac{1}{36}k^2x^6\tau^2. \quad (10)$$

In Eqs. (8) and (9), we consider the vanilla call option with parameter $r = 0.04$ and $\sigma = 0.2$ from [22]. Then $k = 2r/\sigma^2 = 2$, so we obtain the Padé approximant as $[3,2]_{(x,\tau)} = p(x,\tau)/q(x,\tau)$.

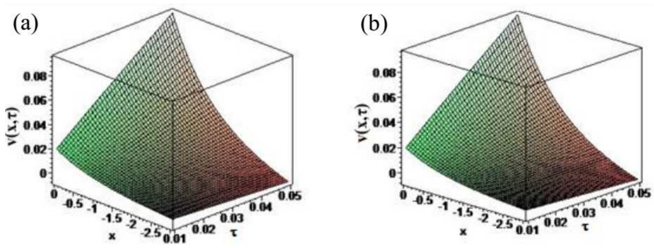

Fig. 1. $S < E$ solutions for $\alpha = 1$, (a) exact solution, (b) MPA solution.

TABLE I

Numerical values for $\alpha = 1$ and $S < E$.

x	τ	ν_{exact}	ν_{MPA}	(error)
−0.030459207	0.01	0.0192072869	0.0192072927	0.0000000057
−0.072570693	0.03	0.0541589841	0.0541594194	0.0000004353
−0.139262067	0.05	0.0827914463	0.0827937732	0.0000023269
−0.198450939	0.07	0.1071262470	0.1071326101	0.0000063631
−0.261364764	0.09	0.1268419372	0.1268521246	0.0000101874

The expansion of the series in Eq. (6) for $\alpha = 0.5$ is

$$\begin{aligned}\nu(x,\tau) &= 1.12837916k\tau^{0.5} - k^2\tau + 0.75225277k^3\tau^{1.5}\\ &+1.12837916kx\tau^{0.5} - k^2x\tau + 0.75225277k^3x\tau^{1.5}\\ &+0.56418958kx^2\tau^{0.5} - 0.5k^2x^2\tau\\ &+0.376126381k^3x^2\tau^{1.5} + 0.18806319kx^3\tau^{0.5}\\ &-0.1667k^2x^3\tau + 0.12537546k^3x^3\tau^{1.5}\\ &+0.047015798kx^4\tau^{0.5} - \cdots . \end{aligned} \tag{11}$$

In Fig. 2 and Table II we compare MPA of order $(5,2)$, calculated using Eq. (11), with the exact solution, for $k = 2$.

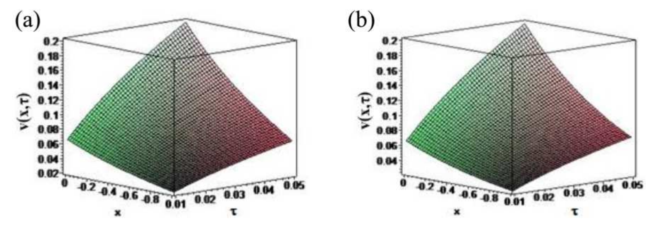

Fig. 2. $S < E$ solutions for $\alpha = 0.5$, (a) exact solution (b) MPA solution.

4.2. Situation of $S > E$

If we consider another situation, when the stock price is bigger than the exercise price ($S > E$), we obtain the expansion of the series in Eq. (6) for $\alpha = 1$ as

TABLE II

Numerical values for $\alpha = 0.5$ and $S < E$.

x	τ	ν_{exact}	ν_{MPA}	(error)
-0.182321557	0.005	0.1179336530	0.1180861820	0.0001525294
-0.885423230	0.007	0.0676492994	0.0675087033	0.0001405961
-0.235675801	0.009	0.1443018610	0.1447560420	0.0004541815
-0.709253762	0.011	0.0977995954	0.0980183133	0.0002187179
-0.650946216	0.015	0.1178105405	0.1184099336	0.0005993931

$$\nu(x,\tau) = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + k\tau - \frac{k^2\tau^2}{2} + \frac{k^3\tau^3}{6} - \frac{k^4\tau^4}{24} + \cdots . \tag{12}$$

Now let us calculate the approximate solution of Eq. (12) for $m = 2$ and $n = 2$ by using MPA. Considering Eq. (4) we have

$$\begin{aligned} p(x,\tau) &= \frac{x^5}{12} - \frac{k\tau x^4}{12} - \frac{xk^4\tau^4}{12} + \frac{k^5\tau^5}{12} + \frac{k^5\tau^5 x}{12}\\ &-\frac{x^5k\tau}{12} - \frac{2k^2\tau^2x^3}{3} - \frac{2k^3\tau^3x^2}{3} + \frac{k^4\tau^4x^2}{6}\\ &-\frac{k^2\tau^2x^4}{6}, \end{aligned} \tag{13}$$

$$\begin{aligned} q(x,\tau) &= \frac{k^4\tau^4}{12} - \frac{k^2\tau^2x^2}{2} + \frac{x^4}{12} - \frac{k\tau x^3}{6} - \frac{k^3\tau^3x}{6}\\ &+\frac{k^2\tau^2x^3}{12} - \frac{x^5}{24} + \frac{x^6}{144} + \frac{k^3\tau^3x^3}{18} + \frac{k^5\tau^5}{24} - \frac{k^3\tau^3x^2}{12}\\ &+\frac{k^6\tau^6}{144} + \frac{k\tau x^4}{24} + \frac{k^2\tau^2x^4}{48} - \frac{xk^4\tau^4}{24} + \frac{k^4\tau^4x^2}{48}. \end{aligned} \tag{14}$$

Assuming $r = 0.2$ and $\sigma = 0.5$ in Eqs. (13) and (14), we get $k = 2r/\sigma^2 = 1.6$. In Fig. 3 and Table III we show Padé approximant of order $(2,2)$ for $\nu(x,\tau)$.

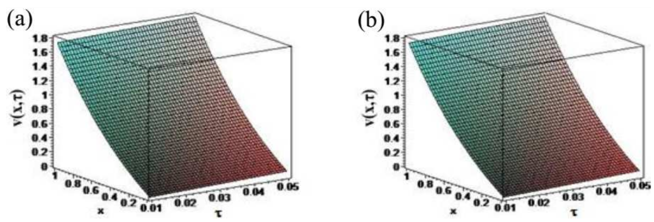

Fig. 3. $S > E$ solutions for $\alpha = 1$, (a) exact solution, (b) MPA solution.

The expansion of the series in Eq. (6) for $\alpha = 0.5$ is

$$\begin{aligned}\nu(x,\tau) &= x + 0.5x^2 + 0.166667x^3 + 0.0416667x^4\\ &+1.1283792k\tau^{0.5} - k^2\tau + 0.7522528k^3\tau^{1.5}\\ &-0.5k^4\tau^2. \end{aligned} \tag{15}$$

By choosing $k = 1.6$, we obtained Fig. 4 and Table IV for MPA of order $(3,2)$ using Eq. (15).

TABLE III

Numerical values for $\alpha = 1$ and $S > E$.

x	τ	ν_{exact}	ν_{MPA}	(error)
0.061875404	0.01	0.0797024666	0.0797022906	0.0000001761
0.287682072	0.03	0.3801995456	0.3801802601	0.0000192855
0.510825624	0.05	0.7435503209	0.7431672251	0.0003830958
0.245839283	0.07	0.3846497928	0.3846644396	0.0000146468
0.216533816	0.09	0.3758773283	0.3758865244	0.0000091961

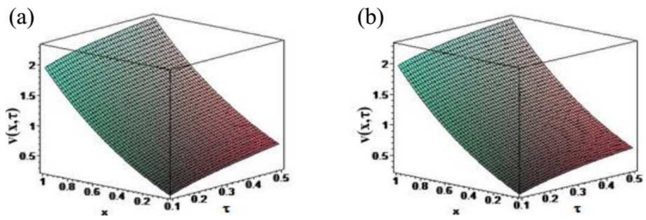

Fig. 4. $S > E$ solutions for $\alpha = 0.5$, (a) exact solution, (b) MPA solution.

TABLE IV

Numerical values for $\alpha = 0.5$ and $S > E$.

x	τ	v_{exact}	v_{MPA}	(error)
0.154150680	0.01	0.3243898510	0.3243627730	0.0000270776
0.467769214	0.03	0.8458217946	0.8451944819	0.0006273127
0.240139484	0.05	0.5748540320	0.5739498130	0.0009042194
0.139761942	0.07	0.4927793411	0.4918048113	0.0009745298
0.223143551	0.09	0.6239103446	0.6229615523	0.0009487923

5. Conclusions

In this study, a numerical solution of Black-Scholes option pricing equation has been achieved. A numerical approach, which is based on MPA method, is successfully applied to fractional Black-Scholes European option pricing equation. This study is one of the few studies about numerical solution of FBSE. The FBSE is considered as two parts, according to stock price S and the exercise price E. It is a fact that if $S = E$ then the value of the option is zero. We have discussed the cases of $S < E$ and $S > E$. For the values of $\alpha = 1$ and $\alpha = 0.5$, the numerical results obtained using MPA are compared with the exact solutions. The data used in the comparison is the real life data of the vanilla call option. When looking at error values for the values of $\alpha = 1$ and $\alpha = 0.5$, numerical results in tables and figures show that the results of MPA are in excellent agreement with the analytical results. Thus, it is observed that MPA is a successful method for determination of the European option pricing.

Acknowledgments

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