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Full Length Research Paper

Numerical solution of Fredholm integral equations of the second kind by using fuzzy transforms

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In this paper, we introduce an approach by using inverse fuzzy transforms based on the fuzzy partition with combination in collocation technique for the numerical solution of Fredholm integral equations of the second kind. The main advantage of this approach is to reduce the problem to the linear system of equations. We present the convergence theorem for this method. Finally, we give two examples to illustrate the efficiency of the proposed method.

Key words: Fredholm integral equation, basic function, fuzzy transforms.

INTRODUCTION

Integral equations are studied in relation to vehicular traffic, biology, economics, etc. In classic mathematics, various kinds of transforms are used as powerful methods for the construction of approximation models and for numerical solution of differential and integral-differential equations. To date, many different basis functions were introduced to obtain the solution of integral equations. Maleknejad and Kajani (2002) proposed piecewise constant basis functions to obtain numerical solution of integro-differential equations. Ezzati and Najafalizadeh (2011) applied Chebyshev polynomials for nonlinear Volterra-Fredholm integral equations. In Maleknejad et al. (2003), Legendre wavelet was used to solve linear Fredholm and Volterra integral equations of the second kind. Solving nonlinear Volterra integral equation of the second kind using Chebyshev polynomials was done in the work of Maleknejad et al. (2007). Several numerical methods based on different wavelets, such as Legendre and Walsh were designed for analysis of control systems and various related applications (Maleknejad et al., 2003; Yousefi and Razzaghi, 2005), and approximating the solution of integral equations (Deb et al., 2007; Yousefi and Razzaghi, 2005).

Fuzzy transform (F-transform) was proposed by Perfilieva (2006) and studied in several papers (Perfilieva, 2004, 2009; Stepnicka and Polakovic, 2009; Dankova and Stepnicka, 2006; Plskova, 2006). Perfilieva (2003) and Perfilieva and Chaldeeva (2001) in their work showed how ordinary differential equations can be approximately solved with the help of the F-transform. In the work of Stepnicka and Valasek (2005), this technique has been further elaborated and applied to the solution of partial differential equations. Moreover, a function obtained by the inverse F-transform has nice filtering properties which can be used to remove noise from images or from any other kind of data. Also, F-transforms can be used for data compression (Perfilieva, 2005). Applications of F-transforms were proposed recently in image processing (Di Martino et al., 2010, 2008; Stepnicka and Polakovic, 2009). Approximation properties of the F-transforms were studied in Perfilieva (2006) and Perfilieva (2004). The success of these applications is due to the fact that F-transforms are capable to accurately approximate any continuous function.

This study shows how this technique can be applied to solve numerical solution of Fredholm integral equation of the second kind. In the structure of Fredholm integral equation, we use inversion form of F-transform instead of precise representation of the original function. This process terminates to solving a linear system of equations that can be found to be unknown.

Here, we will apply the F-transform to approximate the unknown function, $f(x)$, of which this function is the solution of Fredholm integral equation of the second kind

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as follows:

$$f(x)=g(x)+\int_a^b k(x,t).f(t)dt, \qquad x\in[a,b] \tag{1}$$

Some basic definitions for F-transforms are presented subsequently. A fuzzy partition is created for each input domain. The related number of fuzzy sets was established by an expert. This method is based on approximating the solution of Equation 1 using the concept of F-transform; investigation of mathematical formulation of the proposed method; presentation of error analysis; provision of some numerical results; conclusion.

PRELIMINARIES

Consider the Fredholm integral equation of the second kind defined in Equation 1, in which the function $g(x)$ and kernel $k(x,t)$ are given and the function $f(x)$ is unknown. We take an interval $[a,b]$ as a universe. The fuzzy partition of the universe is given by fuzzy subsets of the universe $[a_i,b_i]$ determined by their membership function which must have the properties described in the following definition.

Definition 1

Let $x_1<\ldots<x_n$ be fixed nodes within $[a,b]$ such that $x_1=a, x_n=b$ and $n\geq 2$. We say that fuzzy sets $A_1,\ldots,A_n$, identified with their membership functions $A_1(x),\ldots,A_n(x)$ defined on $[a,b]$, form a fuzzy partition of $[a,b]$ if they fulfill the following conditions for $k=1,\ldots,n$ (Perfilieva, 2006):

1. A_k : $[a,b]\rightarrow[0,1]$, $A_k(x_k)=1$;
2. $A_k(x)=0$ if $x\notin(x_{k-1},x_{k+1})$ where for the uniformity of denotation, we put $x_0=a,\ x_{n+1}=b$;
3. $A_k(x)$ is continuous;
4. $A_k(x), k=2,\ldots,n$, strictly increases on $[x_{k-1},.x_k]$ and $A_k(x)$, $k=1,\ldots,n-1$, strictly decreases on $[x_k,\ x_{k+1}]$;
5. for all $x\in[a,b]$

$$\sum_{k=1}^{n}A_k(x)=1.$$

The membership functions $A_1,\ldots,A_n$ are called basic functions.

Definition 2

Let $A_1,\ldots,A_n$ be basic functions which form a fuzzy partition of $[a,b]$ and f be any function from $C([a,b])$ (Bade and Ruda, 2011). We say that the *n*-tuple of real numbers $[F_1,\ldots,F_n]$ given by

$$F_k=\frac{\int_a^b f(x)A_k(x)dx}{\int_a^b A_k(x)dx},\ k=1,\ldots,n$$

is the (integral) F-transform of f with respect to $A_1,\ldots,A_n$. Let us remark that this definition is correct, because for each $k=1,\ldots,n$ the product $f\,A_k$ is an integrable function on $[a,b]$ (Perfilieva, 2006).

Denote the F-transform of a function f with respect to $A_1,\ldots,A_n$ by $F_n[f]$. Then, according to Definition 2, we can write:

$$F_n[f]=[F_1,\ldots,F_n].$$

The elements $F_1,\ldots,F_n$ are called components of the F-transform.

Lemma 1

Let f be a continuous function on $[a,b]$ and $A_1,\ldots,A_n$, $n\geq 3$ be basic functions which form a uniform fuzzy partition of $[a,b]$, but function f be twice continuously differentiable in (a,b). Then, for each $k=1,\ldots,n$:

$$F_k=f(x_k)+O(h^2).$$

Proof: Perfilieva (2006).

Definition 3

Let $A_1,\ldots,A_n$ be basic functions which form a fuzzy partition of $[a,b]$ and f be a function from $C([a,b])$ (Bade and Ruda, 2011). Let $F_n[f]=[F_1,\ldots,F_n]$ be the integral F-transform of f with respect to $A_1,\ldots,A_n$. Then, the function:

$$f_{F,n}(x)=\sum_{k=1}^{n} F_k A_k(x),$$

is called the inverse F-transform (Perfilieva, 2006).

The following theorem shows that the inverse F-transform, $f_{F,n}$, can approximate the original continuous function f with an arbitrary precision.

Theorem 1

Let *f* be a continuous function on $[a,b]$. Then, for any $\varepsilon>0$ there exists n_ε and a fuzzy partition $A_1,\ldots,A_{n\varepsilon}$ of $[a,b]$ such that for all $x\in[a,b]$:

$$\left|f(x)-f_{F,n_\varepsilon}(x)\right|\le\varepsilon,$$

where $f_{F,n_\varepsilon}(x)$ is the inverse F-transform of f with respect to the fuzzy partition $A_1,\ldots,A_{n\varepsilon}$.

Proof: Perfilieva (2006).

Definition 4

Suppose that function f is given at nodes $p_1,\ldots,p_l\in[a,b]$ and $A_1,\ldots,A_n$, $n<l$, be basic functions which form a fuzzy partition of $[a,b]$. We say that the *n*-tuple of real numbers $[F_1,\ldots,F_n]$ is the discrete F-transform of f with respect to $A_1,\ldots,A_n$ if

$$F_k=\frac{\sum_{j=1}^{l} f(p_j)A_k(p_j)}{\sum_{j=1}^{l} A_k(p_j)},$$ (Perfilieva, 2006).

In the discrete case, we define the inverse F-transform only at nodes where the original function is given. Through this paper, we suppose $l=n$.

Definition 5

Let function f be given at nodes $p_1,\ldots,p_l\in[a,b]$ and $F_n[f]=[F_1,\ldots,F_n]$ be the discrete F-transform of f with respect to $A_1,\ldots,A_n$ (Bade and Ruda, 2011). Then, the function:

$$f_n^F(p_j)=\sum_{k=1}^{n} F_k A_k(p_j),$$

defined at the same nodes, is the inverse discrete F-transform (Perfilieva, 2006).

PROPOSED METHOD

Here, we consider Equation 1. For solving this kind of integral equation numerically, we approximate the unknown function $f(x)$ by $f_n^F(x)$. By substituting $f_n^F(x)$ instead of the unknown function $f(x)$ inside the integral in Equation 1, we have:

$$f(x)\cong g(x)+\int_a^b k(x,t)\left(\sum_{i=1}^{n} F_i A_i(t)\right)dt,$$

$$f(x)\cong g(x)+\int_a^b k(x,t)\left(\sum_{i=1}^{n}\left(\frac{\sum_{k=1}^{n} f(x_k)A_i(x_k)}{\sum_{k=1}^{n} A_i(x_k)}\right)A_i(t)\right)dt,$$

$$f(x)\cong g(x)+\sum_{k=1}^{n} f(x_k)\left(\sum_{i=1}^{n}\frac{A_i(x_k)}{\sum_{k=1}^{n} A_i(x_k)}\int_a^b k(x,t)A_i(t)dt\right).$$

If we set $x=x_j$, $j=1,\ldots,n$, we conclude the linear system of equations as follows:

$$f(x_j)\cong g(x_j)+\sum_{k=1}^{n} f(x_k)\left(\sum_{i=1}^{n}\frac{A_i(x_k)}{\sum_{k=1}^{n} A_i(x_k)}\int_a^b k(x_j,t)A_i(t)dt\right)$$

Equivalently, the following linear system of equations can be solved:

$$BX = y \tag{2}$$

where $B = (b_{ij})$, $i, j = 1, \ldots, n$, and

$$b_{ii} = 1 - \sum_{m=1}^{n} \frac{A_m(x_i)}{\sum_{k=1}^{n}(A_m(x_k))}\left(\int_a^b k(x_i,t)A_m(t)dt\right)$$
$$b_{ij} = -\sum_{m=1}^{n} \frac{A_m(x_j)}{\sum_{k=1}^{n}(A_m(x_k))}\left(\int_a^b k(x_i,t)A_m(t)dt\right), \quad i \neq j \tag{3}$$

$$X = [f(x_1), f(x_2), \ldots, f(x_n)]^T, \quad y = [g(x_1), g(x_2), \ldots, g(x_n)]^T. \tag{4}$$

By solving the system $BX = y,$ we can obtain $f(x_i),$ $i = 1, \ldots, n$. Substituting $f(x_i),$ $i = 1, \ldots, n$, in F_k defined in Definition 4, we can compute $f_n^F(x) = \sum_{i=1}^{n} F_i A_i(x)$ as an approximation of $f(x)$. For solving the system $BX = y$, it is clear that matrix B must be invertible. So, we give the following theorem.

Theorem 2

Let B be an $n \times n$ matrix as introduced in Equation 3, and let $M = \sup_{x,t \in [a,b]} |k(x,t)|$. If $M < \frac{1}{(b-a)}$, then B is invertible.

Proof

We prove that $\|I - B\|_\infty = \max_{1 \le i \le n}\left(\sum_{j=1}^{n}|b_{ij}|\right) < 1$. We have:

$$\|I - B\|_\infty = \max_{1 \le i \le n}\left(\sum_{j=1}^{n}\left| -\sum_{m=1}^{n} \frac{A_m(x_j)}{\sum_{k=1}^{n}(A_m(x_k))}\left(\int_a^b k(x_i,t)A_m(t)dt\right)\right|\right)$$

$$\le \max_{1 \le i \le n}\left(\sum_{m=1}^{n}\left(\sum_{j=1}^{n} \frac{|A_m(x_j)|}{\left|\sum_{k=1}^{n}(A_m(x_k))\right|}\left(\int_a^b |k(x_i,t)|A_m(t)dt\right)\right)\right)$$

$$\le M(b-a) < 1.$$

By using Neuman series, we conclude that $(I-(I-B))^{-1} = B^{-1}$ exists. This completes the proof.

Error analysis

Now, we present the convergence theorem that justifies the proposed method for approximating the solution of Equation 1. Let $(C([a,b]), \|\cdot\|)$ be the space of all continuous functions on interval $[a,b]$ with the norm $\|f\|_\infty = \max_{x\in[a,b]}|f(x)|$. We suppose that $g(x) \neq 0,$ and also $|k(x,t)| \le M, \forall x,t \in [a,b]$. Moreover, we define $\alpha = M(b-a)$. With these conditions, we present the following theorem.

Theorem 3

Let $f(x)$ be an exact solution of Equation 1, which is twice continuously differentiable in (a,b). Let $f_n^F(x)$ be the approximate solution of Equation 1 as Equation 5. The solution of linear Fredholm integral equation by using F- transform is convergence if $0 < \alpha < 1$.

Proof

Clearly, we have:

$$\left\| f - f_n^F \right\|_\infty = \max_{x\in[a,b]}\left| f(x) - f_n^F(x)\right|$$
$$= \max_{x\in[a,b]}\left| g(x) + \int_a^b k(x,t)f(t)dt - \left(g(x) + \int_a^b k(x,t)\sum_{i=1}^{n} F_i A_i(t)dt\right)\right|$$
$$\le \int_a^b |k(x,t)|\left| f(t) - \sum_{i=1}^{n} F_i A_i(t)\right| dt$$
$$\le \int_a^b |k(x,t)|\left\| f - \sum_{i=1}^{n} F_i A_i\right\|_\infty dt$$
$$\le M(b-a)\left\| f - f_n^F \right\|_\infty.$$

Hence, we have:

$$\left\| f - f_n^F \right\|_\infty \le \alpha \left\| f - f_n^F \right\|_\infty$$

Then, if $0 < \alpha < 1$, we will have:

$$\lim_{n\to\infty}\left\| f - f_n^F \right\|_\infty = 0$$

Table 1. Absolute errors for Examples 1 and 2.

n	Absolute error for Example 1	Absolute error for Example 2
5	3.18444×10^{-4}	4.54923×10^{-2}
10	3.94423×10^{-5}	8.69262×10^{-3}
15	1.42839×10^{-5}	3.57449×10^{-3}
20	7.32311×10^{-6}	1.9376×10^{-4}

NUMERICAL EXAMPLES

In order to show the accuracy of the proposed method, we present two examples. Software "Mathematica7" is applied for computing the examples.

Example 1

Consider the equation:

$$f(x)=-\frac{2}{7}x^2+\sqrt{x}+\int_0^1 x^2y^2.f(y)dy$$

The exact solution of this equation is as follows:

$$f(x)=\sqrt{x}\ .$$

Table 1 shows the comparison of numerical and exact solution.

Example 2

Consider the following Fredholm integral equation of the second kind:

$$f(x)=\sin(\pi x^2)-\frac{x^2}{\pi}+\int_0^1 x^2y\ f(y)dy.$$

The exact solution of this equation is as follows:

$$f(x)=\sin(\pi\, x^2).$$

Table 1 shows the comparison of numerical and exact solution.

Conclusion

In this paper, we used direct and inverse form of fuzzy transform to approximate the solution of Fredholm integral equation of the second kind. The properties of basic functions and F-transform are caused to reduce Fredholm integral equation to a system of linear equations.

The numerical and exact solutions are computed and compared by absolute error in two examples. Results show that the proposed method can be an effective method to solve Fredholm integral equations of the second kind.

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