NUMERICAL SOLUTION OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF MIXED TYPE*

by

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1 Introduction

The purpose of this paper is to review some recently developed numerical methods for the solution of nonlinear equations of mixed type. These methods have been used to calculate transonic flows with shock waves, and the discussion will be restricted to this topic, although some of the ideas could presumably be useful in other applications. Some typical transonic flow patterns are sketched in Figure 1. The type changes from elliptic in the region of subsonic flow to hyperbolic in the region of supersonic flow. If the flow is subsonic at infinity, the supersonic flow is confined to one or more bubbles standing above the profile. If the flow is supersonic at infinity, there is a subsonic pocket behind a detached bow wave, and oblique shock waves appear at the trailing edge, sometimes forming a fishtail pattern. Any proposed method should therefore be capable of handling a variety of complex flow patterns.

Three main elements can be recognized in the treatment of such a problem.

(1) The formulation of suitable mathematical model, such as a differential equation or variational principle.

(2) The construction of a discrete approximation to the continuous problem.

(3) The solution of the resulting set of nonlinear equations for the undetermined parameters (typically nodal values of the discrete model).

The first question will not be discussed at length in this paper. The emphasis will be on the numerical methods for solving the two equations which have chiefly been used in transonic flow calculations, the transonic potential flow equation and the transonic small disturbance equation.

The potential flow equation can be derived from the Euler equations for inviscid compressible flow by introducing the assumption that the flow is irrotational, so that we can define a potential ϕ. Then we find that in smooth regions of two dimensional flow ϕ satisfies the quasilinear equation

$$(a^2 - u^2)\phi_{xx} + 2uv\phi_{xy} + (a^2 - v^2)\phi_{yy} = 0 \tag{1.1}$$

where u and v are the velocity components

$$u = \phi_x, \quad v = \phi_y \tag{1.2}$$

and a is the local speed of sound. Given the ratio of specific heats γ and the stagnation speed of sound a_0, a can be determined from the energy relation

$$a^2 = a_0^2 - \frac{\gamma - 1}{2} q^2 \tag{1.3}$$

where q is the speed $\sqrt{u^2 + v^2}$. Equation (1.1) is hyperbolic when the local Mach number $M = q/a > 1$. On the profile the solution should satisfy the Neumann boundary condition

$$\frac{\partial \phi}{\partial n} = 0 \tag{1.4}$$

where n is the normal direction. At infinity the flow approaches a uniform stream with a Mach number M_∞. The density ρ and pressure p can be determined by the relations

$$\rho^{\gamma - 1} = M_\infty^2 a^2 \tag{1.5}$$

and

$$p = \frac{\rho^\gamma}{\gamma M_\infty^2}. \tag{1.6}$$

Multiplied by ρ/a^2, equation (1.1) is equivalent to the equation for conservation of mass

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0. \tag{1.7}$$

Multiplied by $\rho u/a^2$, on the other hand, it is equivalent to the equation for conservation of the x component of momentum

$$\frac{\partial}{\partial x}(p + \rho u^2) + \frac{\partial}{\partial y}(\rho u v) = 0. \tag{1.8}$$

For a flow in a finite region Ω the conservation law (1.7) can be derived from the Bateman variational principle that

$$I = -\iint_\Omega p \, dx \, dy$$

is stationary [1].

Smooth transonic flows are known to exist only in special cases [2]. In general shock waves appear. Thus we must admit weak solutions with suitable discontinuities. Since an irrotational flow is isentropic it is consistent to replace shock waves by jumps across which entropy is conserved. In this case it is not possible to conserve both mass and momentum. A fairly good approximation to shock waves of moderate strength is obtained by requiring the mass to be conserved. The corresponding momentum deficiency then yields an approximation to the wave drag [3]. Thus we seek a solution in which ϕ is continuous, and ϕ_x and ϕ_y are piecewise continuous, satisfying the conservation law (1.7) and the jump condition

$$[\rho\phi_y] - \frac{dy}{dx}[\rho\phi_x] = 0 \tag{1.9}$$

where [ ] denotes the jump condition, and dy/dx is the slope of the discontinuity.

If we construct a difference approximation in conservation form to the conservation law (1.7), then we can expect the jump condition (1.9) to be satisfied in the limit as the mesh width approaches zero [4]. Since the quasilinear form (1.1) does not distinguish between the conservation laws (1.7) and (1.8), difference approximations to (1.1) will not necessarily converge to a solution which satisfies the jump condition (1.9) unless a shock fitting procedure is used.

A useful simplification is provided by the small disturbance theory. Suppose that the profile is given in the form $y = \tau f(x)$ and τ is small. If we expand the solution in powers of τ under the assumption that $1 - M_\infty^2 \sim \tau^{2/3}$, and retain only the leading term, we obtain the transonic small disturbance equation [5]. Let K be the similarity constant $(1 - M_\infty^2)/\tau^{2/3}$. Then a typical form is

$$A\,\phi_{xx} + \phi_{yy} = 0 \tag{1.10}$$

where

$$A = K - (\gamma + 1)\phi_x \tag{1.11}$$

In this equation the y coordinate has been scaled by the factor $\tau^{1/3}$, and ϕ is the disturbance potential, scaled by the factor $\tau^{-2/3}$. The Neumann boundary condition is now transferred to the axis.

$$\phi_y = \frac{df}{dx} \quad \text{at} \quad y = 0. \tag{1.12}$$

If the small disturbance equation is written in the conservation form

$$\frac{\partial}{\partial x}\left(K\phi_x - \frac{\gamma+1}{2}\phi_x^2\right) + \phi_{yy} = 0 \tag{1.13}$$

then the corresponding jump condition

$$[\phi_y] - \frac{dy}{dx}\left[\phi_x - \frac{\gamma+1}{2}\phi_x^2\right] = 0 \tag{1.14}$$

yields a consistent approximation to shock waves [5].

2 Formulation of Finite Difference Methods

The methods proposed in this paper use finite difference approximations to the differential equation. Their formulation is based on an idea introduced by Murman and Cole [6]. That is to use central difference formulas in the subsonic zone, where the governing equation is elliptic, and upwind differemce formulas in the supersonic zone, where it is hyperbolic. Thus the numerical scheme has a directional bias. This corresponds to the upwind region of dependence of the flow in the supersonic zone, and also serves the purpose of enforcing the entropy condition that discontinuous expansions must be excluded. If we consider the transonic flow past a profile with fore and aft symmetry such as an ellipse, the desired solution of the potential equation is not symmetric. Instead it exhibits a smooth acceleration over the front half of the profile followed a discontinuous recompression through a shock wave. In the absence of a directional bias in the numerical scheme the fore and aft symmetry would be preserved in any solution which could be obtained, resulting in the appearance of improper discontinuities. It is not so easy to introduce the desired bias in a finite element formulation when there is no particular coordinate direction that can be treated separately as the time-like direction. Thus the construction of a unified finite element method for the subsonic and supersonic zones appears difficult.

The dominant term in the discretization error introduced by the upwind differencing acts like an artificial viscosity. We can turn this idea around. Instead of using a switch in the difference scheme to introduce a viscosity, we can explicitly add a viscosity which produces an upwind bias in the difference scheme at supersonic points. This simplifies the construction of difference schemes in conservation form. Suppose that we have a central difference approximation in conservation form. Then the conservation form will be preserved as long as the added viscosity has a divergence form. The effect of the viscosity is simply to alter the conserved quantities by terms proportional to the mesh width Δx, which vanish in the limit as Δx approaches zero. By including a switching function in the viscosity to make it vanish in the subsonic zone we continue to obtain the sharp representation of shock waves which results from switching the difference scheme.

The finite difference approximation produces a set of nonlinear difference equations. There remains the problem of finding a convergent iterative scheme for solving these equations. Suppose that in the $(n+1)^{st}$ cycle the residual R_{ij} at the point $i\Delta x$, $j\Delta y$ is evaluated by inserting the result $\phi_{ij}^{(n)}$ of the n^{th} cycle in the difference approximation. Then the correction $C_{ij} = \phi_{ij}^{(n+1)} - \phi_{ij}^{(n)}$ is to be calculated by solving an equation of the form

$$NC + \sigma R = 0 \tag{2.1}$$

where N is a discrete linear operator, and σ is a scaling function. In a relaxation method N is restricted to a lower triangular or block triangular form so that the elements of C can be determined sequentially. In the analysis of such a scheme it is helpful to introduce a time dependent analogy. The vector R is an approximation to $L\phi$, where L is the operator appearing in the differential equation. If we consider C as representing $\Delta t\,\phi_t$, where t is an artificial time coordinate, and N is an approximation to a differential operator $(1/\Delta x)F$, then equation (2.1) is an approximation to

$$F\phi_t + \sigma\frac{\Delta x}{\Delta t}L\phi = 0 \tag{2.2}$$

Thus we should choose N so that this is a convergent time dependent process.

With this approach the formulation of a relaxation method for solving an equation of mixed type is reduced to three main steps:

(1) Construct a central difference approximation to the differential equation.

(2) Add a numerical viscosity to produce the desired directional bias in the hyperbolic region.

(3) Add time dependent terms to embed the steady state equation in a convergent time dependent process.

Methods constructed along these lines have proved extremely reliable. Their main shortcoming is a rather slow rate of convergence.

In order to speed up the convergence we can extend the permissible class of operators N. In particular, if N is taken as the Laplacian, we can solve the resulting discrete Poisson equation by a fast direct method at each iteration. This method converges rapidly in subsonic flow but diverges for transonic flows. However, if we use several relaxation steps after each Poisson step, the two methods in combination give fast convergence.

3 The Relaxation Method for the Small Disturbance Equation

The treatment of the small disturbance equation is simplified by the fact that the characteristics are locally symmetric about the x direction. Thus the desired directional bias can be introduced simply by switching to upwind differencing in the x direction at all supersonic points. To preserve the conservation form some care must be exercised in the method of switching. Let p_{ij} be a central difference approximation to the x derivatives at the point $i\Delta x$, $j\Delta y$:

$$\begin{aligned} p_{ij} &= K\frac{(\phi_{i+1,j}-\phi_{ij})-(\phi_{ij}-\phi_{i-1,j})}{\Delta x^2} - (\gamma+1)\frac{(\phi_{i+1,j}-\phi_{ij})^2-(\phi_{ij}-\phi_{i-1,j})^2}{2\Delta x^3} \\ &= A_{ij}\frac{\phi_{i+1,j}-2\phi_{ij}+\phi_{i-1,j}}{\Delta x^2} \end{aligned} \tag{3.1}$$

where

$$A_{ij} = K - (\gamma+1)\frac{\phi_{i+1,j}-\phi_{i-1,j}}{2\Delta x} \tag{3.2}$$

Also let q_{ij} be a central difference approximation to ϕ_{yy}

$$q_{ij} = \frac{\phi_{i,j+1}-2\phi_{ij}+\phi_{i,j-1}}{\Delta y^2}. \tag{3.3}$$

Define a switching function μ with the value unity at supersonic points and zero at subsonic points

$$\mu_{ij} = \begin{cases} 0 & \text{if} \quad A_{ij} > 0 \\ 1 & \text{if} \quad A_{ij} < 0 \end{cases}. \tag{3.4}$$

Then we approximate equation (1.13) by

$$p_{ij} + q_{ij} - \mu_{ij}\, p_{ij} + \mu_{i-1,j}\, p_{i-1,j} = 0. \tag{3.5}$$

This is equivalent to Murman's conservative scheme [7]. In the supersonic zone p_{ij} is replaced by the upwind formula $p_{i-1,j}$. At points where the flow enters and leaves the supersonic zone μ_{ij} and $\mu_{i-1,j}$ have different values, giving special parabolic and shock point operators.

The added term $-\mu_{ij}\, p_{ij} + \mu_{i-1,j}\, p_{i-1,j}$ are an approximation to $\partial P/\partial x$ where

$$P = \mu\, \Delta x \frac{\partial}{\partial x}\left(K\phi_x - \frac{\gamma+1}{2}\phi_x^2\right).$$

This may be regarded as an artificial viscosity of order Δx. The use of a divergence form for the viscosity preserves the conservation form of the difference scheme, and it can be shown that the difference approximation converges to a solution satisfying the correct jump condition [7].

The nonlinear difference equations (3.1)–(3.5) may be solved by a generalization of the line relaxation method for elliptic equations. At each point we calculate the coefficient A_{ij} and the residual R_{ij} by substituting the result $\phi_{ij}^{(n)}$ of the previous cycle in the difference equations. Then we set $\phi_{ij}^{(n+1)} = \phi_{ij}^{(n)} + C_{ij}$ where the correction C_{ij} is determined by solving the linear equations

$$\frac{C_{i,j+1} - 2C_{ij} + C_{i,j-1}}{\Delta y^2} + (1 - \mu_{ij})\, A_{ij} \frac{-\frac{2}{\omega}\, C_{ij} + C_{i-1,j}}{\Delta x^2} + \mu_{i-1,j}\, A_{i-1,j} \frac{C_{ij} - 2C_{i-1,j} + C_{i-2,j}}{\Delta x^2} + R_{ij} = 0 \quad (3.6)$$

on each sucessive vertical line. In these equations ω is the over-relaxation factor for subsonic points with a value in the range $1 \le \omega \le 2$. In a typical line relaxation scheme for an elliptic equation, provisional values $\tilde{\phi}_{ij}$ are determined on the line $x = i\Delta x$ by solving the difference equations with the latest available values $\phi_{i-1,j}^{(n+1)}$ and $\phi_{i+1,j}^{(n)}$ inserted at points on the adjacent lines. Then new values $\phi_{ij}^{(n+1)}$ are determined by the formula $\phi_{ij}^{(n+1)} = \phi_{ij}^{(n)} + \omega(\tilde{\phi}_{ij} - \phi_{ij}^{(n)})$. Equation (3.6) is derived by modifying this process. New values $\phi_{ij}^{(n+1)}$ are used instead of provisional values $\tilde{\phi}_{ij}$ to evaluate ϕ_{yy} at both supersonic and subsonic points. At supersonic points ϕ_{xx} is also evaluated using new values. At subsonic points ϕ_{xx} is evaluated from $\phi_{i-1,j}^{(n+1)}$, $\phi_{i+1,j}^{(n)}$ and a linear combination of $\phi_{ij}^{(n+1)}$ and $\phi_{ij}^{(n)}$ equivalent to $\tilde{\phi}_{ij}$. In the subsonic zone the scheme acts like a line relaxation scheme, with a comparable rate of convergence. In the supersonic zone it is equivalent to a marching scheme, once the coefficients A_{ij} have been evaluated. Since the supersonic difference scheme is implicit, no limit is imposed on the step length Δx as A_{ij} approaches zero near the sonic line. The transition at the sonic line is effected smoothly because ϕ_{yy} is treated in the same manner throughout the flow. If provisional values $\tilde{\phi}_{ij}$ were used to evaluate ϕ_{yy} at subsonic points, there would be a discontinuity at the sonic line in the treatment of ϕ_{yy}.

To illustrate the application of the Murman difference formulas consider uniform flow in a parallel channel. Then $\phi_{yy} = 0$, and with a suitable normalization $K = 0$, so that the equation reduces to

$$-\frac{\partial}{\partial x}\left(\frac{\phi_x^2}{2}\right) = 0$$

with ϕ and ϕ_x given at $x = 0$, and ϕ is given at $\phi = L$. Since ϕ_x^2 is constant, ϕ_x simply reverses sign at a jump. Provided we enforce the entropy condition that ϕ_x decreases through a jump, there is a unique solution with a single jump whenever $\phi_x(0) > 0$ and $\phi(0) + L\phi_x(0) \ge \phi(L) \ge \phi(0) - L\phi_x(0)$. Let $u_{i+1/2} = (\phi_{i+1} - \phi_i)/\Delta x$ and $u_i = (u_{i+1/2} + u_{i-1/2})/2$. Then the difference equations can be written as

$$
\begin{array}{llll}
u_{i+1/2}^2 = u_{i-1/2}^2 & \text{when} & u_i \leq 0, \quad u_{i-1} \leq 0 & \text{(elliptic)} \\
u_{i-1/2}^2 = u_{i-3/2}^2 & \text{when} & u_i > 0, \quad u_{i-1} > 0 & \text{(hyperbolic)} \\
u_{i+1/2}^2 = u_{i-3/2}^2 & \text{when} & u_i \leq 0, \quad u_{i-1} > 0 & \text{(shockpoint)} \\
0 = 0 & \text{when} & u_i > 0, \quad u_{i-1} \leq 0 & \text{(parabolic)}
\end{array}
$$

These admit the correct solution, illustrated in Figure 3a, with a constant slope on the two sides of the shock. The shock point operator allows a single link with an intermediate slope, corresponding to the shock lying in the middle of a mesh cell.

The difference equations also admit, however, various improper solutions. Figure 3b illustrates a sawtooth solution with u^2 constant everywhere except in one cell ahead of a shock point. Figure 3c illustrates another improper solution in which the shock is too far forward. At the last interior point there is then an expansion shock which is admitted by the parabolic operator. Since the difference equations have more than one root we must depend on the iterative scheme to find the desired root. The scheme should ideally be designed so that the correct solution is stable under a small perturbation, and improper solutions are unstable. Using a scheme similar to (3.6), the instability of the sawtooth solution has been confirmed in numerical experiments. The solutions with an expansion shock at the downstream boundary are stable, on the other hand, if the compression shock is more than the width of a mesh cell too far forward. Thus there is a continuous range of stable improper solutions, while the correct solution is an isolated stable equilibrium point.

4 Difference Schemes for the Potential Flow Equation in Quasilinear Form

It is less easy to construct difference approximations to the potential flow equation with a correct directional bias because the upwind direction is not known in advance. If, however, the supersonic flow is confined to a bubble above the profile, it may be possible to use a coordinate system in which the x coordinate is more or less aligned with the flow in the supersonic zone. For this purpose we can use a conformal mapping to make the profile coincide with an x coordinate line [8, 9]. A simple difference approximation to the quasilinear form (1.1) can then be constructed in the following manner. The velocity components u and v

are evaluated throughout the flow field by central difference formulas, and the speed of sound is determined by equation (1.3). Then at subsonic points we use central difference formulas for ϕ_{xx}, ϕ_{xy} and ϕ_{yy}, while at supersonic points we switch to upwind difference formulas for ϕ_{xx} and ϕ_{xy}. The upwind difference formulas can be regarded as approximations to $\phi_{xx} - \Delta x \phi_{xxx}$ and $\phi_{xy} - (\Delta x/2)\phi_{xxy}$. Thus they introduce an effective artificial viscosity

$$\Delta x \left\{ (u^2 - a^2)\phi_{xxx} + uv\phi_{xxy} \right\} = \Delta x \left\{ (u^2 - a^2)u_{xx} + uvv_{xx} \right\}.$$

When the flow is not perfectly aligned with the x coordinate there exist supersonic points at which $u^2 < a^2 < u^2 + v^2$. One characteristic lies ahead of the y coordinate line at such a point, so that the difference scheme does not have the correct region of dependence. Also the artificial viscosity $\Delta x(u^2 - a^2)\phi_{xxx}$ is introduced by the upwind difference formula for ϕ_{xx} is then negative. Despite this fact, schemes of this type have proved quite satisfactory in practice for flows with supersonic zones of moderate size.

To treat more general flows it is necessary to derive a method of rotating the upwind differencing to conform with the flow direction [10]. For this purpose suppose that s and n are streamwise and normal Cartesian coordinates in a reference frame locally aligned with the flow. Then equation (1.1) is equivalent to

$$(a^2 - q^2)\phi_{ss} + a^2\phi_{nn} = 0. \tag{4.1}$$

Since u/q and v/q are the local direction cosines

$$\phi_{ss} = \frac{1}{q^2}\left(u^2\phi_{xx} + 2uv\phi_{xy} + v^2\phi_{yy}\right) \tag{4.2}$$

and

$$\phi_{nn} = \frac{1}{q^2}\left(v^2\phi_{xx} - 2uv\phi_{xy} + v^2\phi_{yy}\right). \tag{4.3}$$

Now we use central differencing at subsonic points as before, but at supersonic points we switch to upwind differencing for ϕ_{ss}. Thus if $u > 0$, $v > 0$, ϕ_{ss} is approximated at a point $i\Delta x$, $j\Delta y$ in the supersonic zone by using the formulas $(\phi_{ij} - 2\phi_{i-1,j} + \phi_{i-2,j})/\Delta x^2$ and $(\phi_{ij} - \phi_{i-1,j} - \phi_{i,j-1} + \phi_{i-1,j-1})/\Delta x \Delta y$ to represent ϕ_{xx} and ϕ_{xy}, and a similar formula to represent ϕ_{yy}. This reduces exactly to the Murman scheme when either $u = 0$ or $v = 0$. Also the upwind differencing introduces an effective artificial viscosity

$$\left(1 - \frac{a^2}{q^2}\right)\left\{\Delta x\left(u^2 u_{xx} + uvv_{xx}\right) + \Delta y\left(uvu_{yy} + v^2 v_{yy}\right)\right\}$$

which is symmetric in x and y.

5 Difference Schemes for the Potential Flow Equation in Conservation Form

In the construction of a discrete approximation to the conservation form (1.7) of the potential flow equation, it is convenient to accomplish the switch to upwind differencing by the explicit addition of an artificial viscosity in the manner proposed in Section 2. Let S_{ij} be a central difference approximation to the left-hand side of equation (1.7)

$$S_{ij} = \frac{(\rho u)_{i+1/2,j} - (\rho u)_{i-1/2,j}}{\Delta x} + \frac{(\rho v)_{i,j+1/2} - (\rho v)_{i,j-1/2}}{\Delta y}. \tag{5.1}$$

In forming S_{ij} we have to evaluate the velocities at the midpoints of the mesh intervals, using formulas such as $u_{i+1/2,j} = (\phi_{i+1,j} - \phi_{ij})/\Delta x$ and $u_{i,j+1/2} = (\phi_{i+1,j} + \phi_{i+1.j+1} - \phi_{i-1,j} - \phi_{i-1,j+1})/4\Delta x$. The density is evaluated from equation (1.5). Then we shall solve an equation of the form

$$S_{ij} + T_{ij} = 0 \tag{5.2}$$

where T_{ij} is the artificial viscosity. The viscosity will be constructed in the a divergence form

$$T = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

to preserve the conservation form of the equation (1.7).

Both simple and rotated schemes can be devised [11]. The term $(\partial/\partial x)(\rho u)$ can be expanded in a smooth region as $\rho(1 - u^2/a^2)\phi_{xx} - \rho(uv/a^2)\phi_{xy}$. In a simple scheme $Q = 0$, and $\partial P/\partial x$ is constructed as an upwind approximation to $-\Delta x(\partial/\partial x)\mu\phi_{xx}$, where $\mu = \min\{0, \rho(1 - u^2/a^2)\}$. Thus at supersonic points the term $\rho(1 - u^2/a^2)\phi_{xx}$ is canceled and replaced by its value at the adjacent upwind point.

The rotated scheme is designed to introduce viscosity terms similar to those introduced by the rotated scheme for the quasilinear form. Let the switching function μ be defined as

$$\mu = \max\left\{0, \left(1 - \frac{a^2}{q^2}\right)\right\}. \tag{5.3}$$

Then P and Q are constructed as approximations to

$$-\mu\{(1-\epsilon)|u|\Delta x \rho_x + \epsilon \Delta x^2 u \rho_{xx}\}$$

and

$$-\mu\{(1-\epsilon)|v|\Delta y\rho_y + \epsilon\Delta y^2 v\rho_{yy}\}$$

where ϵ is a parameter controlling the accuracy. If $\epsilon = 1 - \lambda\Delta x$ and λ is a constant the scheme is second order accurate. If $\epsilon = 0$ it is first order accurate, and at supersonic points where $u > 0$, $v > 0$, P then approximates

$$\Delta x \frac{\rho}{a^2}\left(1 - \frac{a^2}{q^2}\right)(u^2 u_x + uvv_x).$$

In these expressions the derivatives of ρ are represented by upwind difference formulas. Thus the formula for the viscosity becomes

$$T_{ij} = -\frac{P_{i+1/2,j} - P_{i-1/2,j}}{\Delta x} - \frac{Q_{i,j+1/2} - Q_{i,j-1/2}}{\Delta y} \tag{5.4}$$

where if $u_{i+1/2,j} > 0$

$$P_{i+1/2,j} = u_{i+1/2,j}\,\mu_{ij}\{\rho_{i+1/2,j} - \rho_{i-1/2,j} - \epsilon(\rho_{i-1/2,j} - \rho_{i-3/2,j})\}$$

and if $u_{i+1/2,j} < 0$

$$P_{i+1/2,j} = u_{i+1/2,j}\,\mu_{i+1,j}\{\rho_{i+1/2,j} - \rho_{i+3/2,j} - \epsilon(\rho_{i+3/2,j} - \rho_{i+5/2,j})\} \tag{5.5}$$

while $Q_{i,j+1/2}$ is defined by a similar formula.

6 Analysis of Relaxation Schemes by the Time Dependent Analogy

As in the treatment of the small disturbance equation, relaxation methods can be used to solve the nonlinear difference equations generated by the various approximations to the potential flow equation. In the simplest case, when the equation is expressed in quasilinear form and the upwind differencing is restricted to the x coordinate, this presents no particular difficulty. In each cycle the coefficients $a^2 - u^2$, $2uv$, and $a^2 - v^2$ are first calculated using the result $\phi_{ij}^{(n)}$ of the previous cycle. Then new values $\phi_{ij}^{(n+1)}$ are determined by solving a set of linear equations on each successive vertical line $x = i\Delta x$, with the latest available values $\phi_{i-2,j}^{(n+1)}$, $\phi_{i-1,j}^{(n+1)}$ and $\phi_{i+1,j}^{(n)}$ on the adjacent upstream and downstream lines substituted in the difference formulas for ϕ_{xx}, and ϕ_{xy} and ϕ_{yy}. This gives a block triangular form to the matrix N in equation (2.1). If the coefficients of the second derivatives were frozen the method would reduce to a marching scheme in the supersonic zone, since the values on each vertical line would then depend only on the updated values just determined on the upstream lines.

When the rotated difference scheme is used, the difference equation at a supersonic point includes contributions to ϕ_{nn} from adjacent downward points. The time dependent analogy suggested in Section 2 then provides a useful insight into the nature of the relaxation process, which no longer resembles a marching scheme in the supersonic zone [10]. The typical form of a central difference appximation to ϕ_{xx} at a point on the line $x = i\Delta x$ is

$$\frac{\phi_{i-1,j}^{(n+1)} - (1 + r\Delta x)\phi_{ij}^{(n+1)} - (1 - r\Delta x)\phi_{ij}^{(n)} + \phi_{i+1,j}^{(n)}}{\Delta x^2}$$

when an updated value is used at $x = (i-1)\Delta x$, an old value is used at $x = (i+1)\Delta x$ because the new value is not yet available, and a linear combination depending on a parameter r is used at $x = i\Delta x$. Introducing the correction $C_{ij} = \phi_{ij}^{(n+1)} - \phi_{ij}^{(n)}$, this is equivalent to

$$\frac{\phi_{i-1,j}^{(n)} - 2\phi_{ij}^{(n)} + \phi_{i+1,j}^{(n)}}{\Delta x^2} - \frac{C_{ij} - C_{i-1,j}}{\Delta x^2} - \frac{rC_{ij}}{\Delta x}.$$

The structure of the operator N in equation (2.1) is thus determined by the particular combination of new and old values used in the various difference formulas contributing to ϕ_{ss} and ϕ_{nn}.

If we write the equivalent time dependent equation (2.2) in a locally aligned $s - n$ coordinate system, we now find that its principal part can be expressed as

$$(M^2 - 1)\phi_{ss} - \phi_{nn} + 2\alpha\phi_{st} + 2\beta\phi_{nt} = 0 \qquad (6.1)$$

where M is the local Mach number, and the coefficients α and β depend on the split between new and old values of ϕ in the difference formulas. The substitution $T = t - \alpha s/(M^2-1) + \beta n$ reduces this equation to the diagonal form

$$(M^2 - 1)\phi_{ss} - \phi_{nn} - \left(\frac{\alpha^2}{M^2 - 1} - \beta^2\right)\phi_{TT} = 0.$$

If $M > 1$ either s or n is timelike, depending on the sign of the coefficient of ϕ_{TT}, while T is spacelike. Since s is the timelike direction of the steady state equation, it ought also to be the timelike direction when this equation is embedded in a time dependent process. Thus when $M > 1$ the coefficients α and β should satisfy the compatibility condition

$$\alpha > \beta\sqrt{M^2 - 1} \tag{6.2}$$

The characteristic cone of equation (6.1) touches the $s-n$ plane. As long as condition (6.2) holds it slants upstream in the reverse time direction, as illustrated in Figure 2. The iterative scheme will then have a proper region of dependence as long as we sweep the flow field in a direction such that the updated region always includes the upwind line of tangency between the characteristic cone and the $s-n$ plane. It can be seen from Figure 2 that the region of dependence of a subsonic point contains the t axis, with the result that it is important to include a damping term $\gamma\phi_t$ to attenuate the influence of the initial guess. The coefficient γ is controlled by the choice of an overrelaxation factor [12]. The situation is different at a supersonic point. If the coefficients of equation (6.1) were constant with $M > 1$, the region of dependence would cease to intersect the initial data after a sufficient time interval. Instead it would intersect a surface containing the Cauchy data of the steady state problem. Thus no damping due to ϕ_t is required in the supersonic zone for the process to reach a steady state.

These considerations lead to the following method for deriving the operator N. We substitute new values $\phi_{ij}^{(n+1)}$ whenever they are available in the central difference formulas at subsonic points, and also in the central difference formulas contributing to ϕ_{nn} at supersonic points. In order to satisfy the compatibility condition (6.2) at supersonic points, however, we do not use new values in the upwind difference formulas contributing to ϕ_{ss}. Instead, if $u > 0$, ϕ_{xx} is represented by

$$\frac{2\phi_{ij}^{(n+1)} - \phi_{ij}^{(n)} - 2\phi_{i-1,j}^{(n+1)} + \phi_{i-2,j}^{(n)}}{\Delta x^2}.$$

This can be regarded as an approximation to $\phi_{xx} - 2(\Delta t/\Delta x)\phi_{xt}$. Similar formulas are used for ϕ_{xy} and ϕ_{yy} with the result that the approximation to ϕ_{ss} introduces a term $2(M^2 - 1)((u/q)(\Delta t/\Delta x) + (v/q)(\Delta t/\Delta y))\phi_{st}$ in the equivalent time dependent equation. Finally to make sure that (6.2) is satisfied when M is close to unity we add a term to augment further the coefficient of ϕ_{st}. If $u > 0$ and $v > 0$ this term is

$$\frac{\omega_S}{\Delta x}\left\{\frac{u}{\Delta x}(C_{ij} - C_{i-1,j}) + \frac{v}{\Delta y}(C_{ij} - C_{i,j-1})\right\}$$

where ω_S is a relaxation factor with a value ≥ 0. The best rate of convergence is obtained by using the smallest possible value of ω_S. Often it is sufficient to take $\omega_S = 0$.

Similar schemes are easily constructed for the difference approximations to the conservation form (1.7). Since the conservation form is equivalent to the quasilinear form multiplied by ρ/a^2, we have only to multiply the operator N by ρ/a^2 to produce a time dependent process for the conservation form which converges at about the same rate as the process for the quasilinear form [11]. An advantage of this procedure is that the iterative scheme does not have to be modified to reflect every variation in the difference equations. We can use the same operator N, for example, for all values of the viscosity parameter ϵ.

7 Accelerated Iterative Method

If we consider iterative schemes of the class defined by equation (2.1), we can expect to improve the rate of convergence by choosing N as the closest possible approximation to the operator used to evaluate the residual R. In this we are constrained by the need to limit the number of operations required for each cycle. In recent years fast direct methods have been developed for solving finite difference approximations to Poisson's equation on a rectangle [13, 14]. On an $N \times N$ square these require a number of operations proportional to $N^2 \log N$. The coefficient A in the small disturbance equation (1.10) is an approximation to $1 - M^2$, where M is the local Mach number, so if M is small the Laplacian is a fair approximation to the operator on the left-hand side of the equation. This suggests the use of the discrete Laplacian for N in equation (2.1), with the scaling function σ replaced by a fixed relaxation factor ω. An iteration of this kind was proposed by Martin and Lomax [15].

In order to estimate the rate of convergence which might be expected, consider the Prandtl Glauert equation, which is obtained by replacing A by $1 - M_\infty^2$ in equation (1.10). Let H and V be positive definite operators representing $-\partial^2/\partial x^2$ and $-\partial^2/\partial y^2$ with the appropriate boundary conditions. Also let ϕ be the solution, and let $e^{(n)} = \phi^{(n)} - \phi$ be the error after n cycles. With $\omega = 1$ the iteration then gives

$$(H + V)e^{(n+1)} = M_\infty^2 H e^{(n)}.$$

Thus

$$\left\| H^{1/2} e^{(n+1)} \right\| \leq M_\infty^2 \left\| K \right\| \left\| H^{1/2} e^{(n)} \right\|$$

where K is the symmetric operator $H^{1/2}(H + V)^{-1} H^{1/2}$. If we use a Euclidean norm then

$$\|K\| = \max \frac{(x, Kx)}{(x, x)} = \max \frac{(y, Hy)}{(y, Hy) + (y, Vy)}$$

where $H^{1/2}y = K^{1/2}x$. Thus $\|K\| < 1$. This estimate serves to indicate that for subsonic flows the scheme should converge at a rate independent of the mesh size.

If we consider the case of linearized supersonic flow, with A replaced by $1 - M_\infty^2$ and $M_\infty > 1$, on the other hand, it can be shown that the iteration would diverge. In this case, if E is a shifting operator which replaces ϕ_{ij} by $\phi_{i-1,j}$, the iteration gives

$$(H + V)e^{(n+1)} = \{I + (M_\infty^2 - 1)E\}He^{(n)}$$

and $H + V$ does not dominate $\{I + (M_\infty^2 - 1)E\}H$.

Thus we cannot expect the Poisson iteration to converge when it is used to calculate a transonic flow with a supersonic zone of appreciable size. If, however, it could be supplemented with another method which gives fast convergence in the supersonic zone, the two in combination might produce an effective iterative scheme. In fact the usual line relaxation scheme for the small disturbance equation is just such a method, since it acts like a marching scheme in the supersonic zone. Thus it would give the solution in the supersonic zone in a single sweep, if it were not for the change from cycle to cycle in the nonlinear coefficients A_{ij} and the data at the sonic line. This leads to the idea of using a two stage iteration [16], in which the first stage is a Poisson step, and the second stage consists of p relaxation steps to sweep the errors from the supersonic zone. The best value of p is most easily determined by numerical experiments. These have confirmed that in calculations using the small disturbance equation, a single relaxation step after each Poisson step is sufficient to give fast convergence. An alternative approach is to use a desymmetrized operator N of a form which still permits the use of a fast direct method. Good results have been reported by Martin, using a scheme of this type formulated for an equivalent system of first order equations [17].

The two stage iteration has the advantage that it is easily extended to treat the potential flow equation. This can be scaled so that the Laplacian represents its linear part by dividing the quasilinear form (1.1) by a^2 or the conservation form (1.7) by ρ. Thus we define the Poisson iteration by letting N be the Laplacian in equation (2.1), and setting $\sigma = \omega/a^2$ for the quasilinear form, or ω/ρ for the conservation form, where ω is a relaxation factor with a value in the range $1 \leq \omega \leq 2$. This gives rapid convergence for subsonic flows. For transonic flows we again use p relaxation steps after each Poisson step. The method has proved particularly effective when it is used with the simple difference scheme in quasilinear form, with the upwind differencing restricted to one coordinate. The best rate of convergence is then usually obtained with one or two relaxation steps after each Poisson step. The rotated difference schemes require a larger number of relaxation steps because the relaxation method no longer acts like a marching scheme in the supersonic zone. Typically the best rate of convergence is then obtained with $p \sim 5 - 8$.

8 Three Dimensional Calculations

A similar approach can be used for three dimensional transonic flow calculations [18, 19]. The three dimensional small disturbance equation

$$\frac{\partial}{\partial x}\left(K\phi_x - \frac{\gamma+1}{2}\phi_x^2\right) + \phi_{yy} + \phi_{zz} = 0 \tag{8.1}$$

can be approximated by the obvious generalization of the scheme proposed in Section 3. Let p_{ijk} be a central difference approximation to the first term and q_{ijk} a central difference approximation to $\phi_{yy} + \phi_{zz}$. Then we evaluate the residual as

$$R_{ijk} = p_{ijk} + q_{ijk} - \mu_{ijk}p_{ijk} + \mu_{i-1,j,k}p_{i-1,j,k} \tag{8.2}$$

where the switching function μ_{ijk} is unity at supersonic points and zero at subsonic points.

The time dependent analogy is again helpful in devising a relaxation method for solving the difference equations. If we solve on vertical lines, for example, the use of mixed new and old values in the approximation to ϕ_{zz} introduces a term containing ϕ_{zt} in the equivalent time dependent equation. In order to make sure that we obtain a wave equation in which x is timelike at supersonic points, as it is in the steady state equation, a compensating term $\alpha\phi_{xt}$ should then be added, with $\alpha > 0$.

An alternative method is to relax the equations on successive longitudinal lines with the following scheme. Let the residual R_{ijk} and the coefficient

$$A_{ijk} = K - (\gamma - 1)\frac{\phi_{i+1,j,k} - \phi_{i-1,j,k}}{2\Delta x} \tag{8.3}$$

be evaluated using the result $\phi_{ijk}^{(n)}$ of the n^{th} cycle. Then the correction $C_{ijk} = \phi_{ijk}^{(n+1)} - \phi_{ijk}^{(n)}$ is determined by solving the equation

$$\begin{aligned}(1-\mu_{ijk})A_{ijk}\frac{C_{i+1,j,k} - 2C_{ijk} + C_{i-1,j,k}}{\Delta x^2} - (\alpha - 2\mu_{i-1,j,k}A_{i-1,j,k})\frac{C_{ijk} - C_{i-1,j,k}}{\Delta x^2} \\ - \frac{C_{ijk} - C_{i,j-1,k}}{\Delta y^2} - \frac{C_{ijk} - C_{i,j,k-1}}{\Delta z^2} - (1-\mu_{ijk})\left(\frac{2}{\omega} - 1\right)\left(\frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}\right)C_{ijk} + R_{ijk} = 0\end{aligned} \tag{8.4}$$

where ω is the subsonic over-relaxation factor, and α is a parameter controlling the coefficient of ϕ_{xt} in the equivalent time dependent equation.

A rotated difference scheme for the three dimensional transonic potential flow equation can be devised by writing it locally as

$$(a^2 - q^2)\phi_{ss} + a^2(\Delta\phi - \phi_{ss}) = 0.$$

In this equation Δ is the Laplacian and

$$\phi_{ss} = \frac{1}{q^2}\{u^2\phi_{xx} + v^2\phi_{yy} + w^2\phi_{zz} + 2uv\phi_{xy} + 2vw\phi_{yz} + 2uw\phi_{xz}\}$$

where u, v and w are the velocity components ϕ_x, ϕ_y and ϕ_z and q is the speed $\sqrt{u^2 + v^2 + w^2}$. Now upwind differencing is used for ϕ_{ss} in the supersonic zone as before. Horizontal or vertical line relaxation schemes can be devised in the same manner as in the two dimensional case.

9 Typical Results

Methods contructed along these lines have been quite widely used in the last few years. They have proved particularly effective for calculating two dimensional and axially symmetric flows [5−10, 20−21]. Some typical results are presented here.

As a check on the accuracy attainable with these methods some results are first given for the Tricomi equation

$$y\phi_{xx} + \phi_{yy} = 0$$

for which exact polynomial solutions can easily be constructed. The basic scheme described in Section 3 was applied to this equation in the rectangle $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, with Dirichlet boundary conditions on the sides $y = \pm 1$ and $x = \pm 1$ with $y \geq 0$, Cauchy data on the side $x = -1$ with $y < 0$, and no data on the side $x = 1$ with $y < 0$. Central difference formulas were used for ϕ_{yy} everywhere and for ϕ_{xx} when $y > 0$. When $y < 0$, ϕ_{xx} was approximated by the upwind formula

$$\frac{1}{\Delta x^2}\left\{\phi_{ij} - 2\phi_{i-1,j} + \phi_{i-2,j} + \epsilon\left(\phi_{ij} - 3\phi_{i-1,j} + 3\phi_{i-2,j} - \phi_{i-3,j}\right)\right\}$$

which is first order accurate if $\epsilon = 0$ and second order accurate if $\epsilon = 1$. An equal mesh spacing $\Delta x = \Delta y = h$ was used in each coordinate direction, and exact values of ϕ were provided on the boundaries carrying Dirichlet data, and also at $x = (-1+h)$, $x = -(1+2h)$

on the boundary carrying Cauchy data. The solution $\bar{\phi}(h)$ of the difference equations was obtained on three grids with $h = 1/16$, $1/32$ and $1/64$ by the iterative method described in Section 3. On each grid the iterations were continued until the residual, normalized by multiplying by h^2, was $< 10^{-12}$ at every interior point. This generally required about 300 cycles on the fine mesh. The results were then compared with the exact solution. A typical polynomial solution is

$$\phi = x^4 y - x^2 y^4 + \frac{y^7}{21}.$$

Table 1 shows the error $||\phi - \bar{\phi}(h)||$ and $||\phi_x - \bar{\phi}_x(h)||$ for this case, where $\bar{\phi}_x(h)$ was estimated by a central difference formula. The norm was defined as $||a|| = \left(\frac{1}{N}\sum_i \sum_j a_{ij}^2\right)^{1/2}$, where the sum is over the interior mesh points, and N is the number of these points. For the error in ϕ_x points on the line $x = 1 - h$ were excluded from the sum to avoid estimating ϕ_x by differencing between interior points and points on the boundary $x = 1$.

The errors can be seen to decrease in the expected manner as the mesh width is decreased. For either $\epsilon = 0$ or 1 they are roughly consistent with the estimate

$$\phi = \bar{\phi}(h) + Ah^{\epsilon+1} + \mathcal{O}\left(h^{\epsilon+2}\right) \tag{9.1}$$

where the function $A(x, y)$ determines the distribution of the dominant error. Assuming this to be the case Richardson extrapolation was used to give the estimate $\phi = \tilde{\phi}(h) + \mathcal{O}\left(h^{\epsilon+2}\right)$ where

$$\tilde{\phi}(h) = \bar{\phi}(y) + \frac{1}{2^{\epsilon+1} - 1}\left(\bar{\phi}(h) - \bar{\phi}(2h)\right).$$

The derivative was also extrapolated by a similar formula. The results are shown in Table 2. The success of the extrapolation provides further confirmation of the error estimate (9.1).

Figures 4−6 show typical solutions of the transonic potential flow equation for flows past airfoils. Curvilinear coordinates were used for these calculations. These were generated

Mesh Width	Errors for $\epsilon = 0$		Errors for $\epsilon = 1$	
	$\|\|\phi - \bar{\phi}(h)\|\|$	$\|\|\phi_x - \bar{\phi}_x(h)\|\|$	$\|\|\phi - \bar{\phi}(h)\|\|$	$\|\|\phi_x - \bar{\phi}_x(h)\|\|$
$\frac{1}{16}$	$.428 \times 10^{-1}$	$.111$	$.429 \times 10^{-2}$	$.937 \times 10^{-2}$
$\frac{1}{32}$	$.242 \times 10^{-1}$	$.631 \times 10^{-1}$	$.107 \times 10^{-2}$	$.259 \times 10^{-2}$
$\frac{1}{64}$	$.130 \times 10^{-1}$	$.350 \times 10^{-1}$	$.218 \times 10^{-3}$	$.675 \times 10^{-3}$

Table 1: Errors in solution of the Tricomi equation

Mesh Width of Finer Mesh	Errors for $\epsilon = 0$		Errors for $\epsilon = 1$	
	$\lvert\lvert\phi - \tilde{\phi}(h)\rvert\rvert$	$\lvert\lvert\phi_x - \tilde{\phi}_x(h)\rvert\rvert$	$\lvert\lvert\phi - \tilde{\phi}(h)\rvert\rvert$	$\lvert\lvert\phi_x - \tilde{\phi}_x(h)\rvert\rvert$
$\frac{1}{32}$	$.673 \times 10^{-2}$	$.210 \times 10^{-1}$	$.117 \times 10^{-3}$	$.544 \times 10^{-3}$
$\frac{1}{64}$	$.211 \times 10^{-2}$	$.719 \times 10^{-2}$	$.123 \times 10^{-4}$	$.804 \times 10^{-4}$

Table 2: Result of Richardson Extrapolation

by mapping the exterior of the profile conformally onto the interior of a unit circle, and introducing polar coordinates r and θ in the circle [8−11]. This simplifies the representation of the Neumann boundary condition. It also provides a regular and finite mesh suitable for the application of a fast Poisson solver to accelerate the iterative scheme. To check the influence of the mesh width on the result each calculation was performed on three grids, first with 64 cells in the θ direction and 16 cells in the r direction, then with 128×32 cells and finally with 256×64 cells. On the second two grids the interpolated solution of the previous grid was used to provide the initial guess. A convenient measure of the local flow condition is the pressure coefficient $C_p = (p - p_\infty) / \frac{1}{2}\rho_\infty q_\infty^2$, where the subscript ∞ denotes free stream values. Each figure shows the calculated pressure coefficient over the surface of the profile. The pressure critical pressure coefficient at which the flow has sonic velocity is marked by a horizontal line on the axis. Lift and drag coefficients CL and CD are also shown. These were obtained by integrating the surface pressure.

Figure 4 shows an airfoil designed by Garabedian to produce shock free flow [22, page 44]. The quasilinear form (1.1) was treated using the simple difference scheme described in Section 4, with upwind differencing restricted to the θ direction. The accelerated iterative method described in Section 7 was used, with 2 relaxation sweeps after each Poisson step. The Poisson solution was calculated using the Buneman algorithm [13] in the θ direction. The largest residual R_{ij} at any point of the field, normalized by multiplying by $\Delta\theta^2$, was used as a measure of convergence. It requires 24 cycles to reduce the largest residual from $.89 \times 10^{-2}$ to $.91 \times 10^{-9}$ on the 64×16 grid, 21 cycles to reduce it from $.16 \times 10^{-2}$ to $.84 \times 10^{-9}$ on the 128×32 grid, and 25 cycles to reduce it from $.38 \times 10^{-3}$ to $.81 \times 10^{-9}$ on the 256×64 grid. The entire calculation took 262 seconds on a CDC 6600. A Poisson step takes about the same amount of time as 2 relaxation sweeps, so in this case the calculation on the 128×32 grid was equivalent to about 84 relaxation sweeps. Typically it takes about 4000 sweeps to reduce the largest residual to 10^{-9} on a 128×32 grid by relaxation alone.

It has been found that the jump at a normal shock wave is consistently underestimated by calculations which do not use conservation form [7, 11]. The Mach number behind the shock wave is generally too close to unity. This can be corrected by using conservation form. The schemes described in Section 5 have proved less accurate than the nonconservation schemes of Section 4, however, in the treatment of shock free flows, particularly on coarse grids. They also have the disadvantage of requiring more computer time. Two examples of results obtained with the rotated difference scheme in conservation form, equations (5.1)−(5.4), are shown in Figures 5 and 6. The NACA 64A410, shown in Figure 5, is a typical example of an airfoil which produces a shock wave at quite a low Mach number. The accelerated iterative scheme was used in this calculation, with 8 relaxation steps after each Poisson step. 49 cycles were required to reduce the largest residual to 10^{-9} on the 256×64 grid. Figure 6 shows a symmetric airfoil (suitable for a vertical tail) designed by Hicks and Murman to give a low wave drag at Mach .80 [23]. The degeneration to a flow containing two shock waves is typical of an airfoil designed to operate efficiently at transonic speeds when either the Mach number or (in the case of a lifting airfoil) the lift coefficient is slightly reduced. The forward shock can be seen to be completely eliminated on the 64×16 grid. This calculation was performed by relaxation alone. 2000 cycles were used to reduce the largest residual to $.22 \times 10^{-6}$ on the 256×64 grid. In both calculations the viscosity parameter ϵ was zero, giving the first order accuracy. The accuracy can generally be improved by using values of ϵ between 0 and 1. In the easier cases it is possible to set $\epsilon = 1$. This may, however, lead to divergence for the more sensitive flows, such as the flow past a shock free airfoil near its design point.

Figure 7 shows an example of a three dimensional calculation for a wing with a roughly elliptic plan form. In this case a curvilinear coordinate system was generated in two stages [19]. First parabolic coordinates were introduced in planes containing the wing section by the square root transformation

$$X_1 + \mathrm{i}\, Y_1 = \left(x - x_0(z) + \mathrm{i}\,(y - y_0(z))\right)^{1/2}, \qquad Z_1 = z.$$

The singular line $x_0(z) + \mathrm{i}\, y_0(z)$ was located just behind the leading edge in order to unwrap the wing to form a shallow bump $Y_1 = S\,(X_1, Z_1)$. Then a shearing transformation

$$X = X_1, \qquad Y = Y_1 - S\,(X_1, Z_1)\,, \qquad Z = Z_1$$

was used to map the wing surface to a coordinate surface. The calculation was performed on a mesh with $192 \times 16 \times 32$ cells in the X, Y and Z directions. After a preliminary calculation on a mesh with $96 \times 8 \times 16$ cells, 100 relaxation cycles were used on the fine mesh to reduce the largest residual (multiplied by ΔX^2) to $\sim 10^{-5}$. The figure shows the wing configuration, and the upper and lower surface pressure distributions in separate plots. On the upper surface there is a single shock wave near the center of the wing, but two shock waves near each tip where less lift is produced.

When solutions of the transonic potential flow equation are compared with experimental data it is important to allow for viscous effects, which are dominant in the boundary layer adjacent to the surface. It has been found that remarkably good agreement can frequently be obtained simply by correcting the profile to allow for the displacement effect of the boundary layer [22]. This procedure, which is effective in the regime where the shock waves are not strong enough to cause a separated flow, often proves more successful with the quasilinear form than with the conservation form. The shock jumps observed in practice in the presence of a boundary layer are weaker than the jumps predicted by the Rankine Hugoniot theory for normal shock waves. The attenuation of the shock jumps which results from the use of the quasilinear form provides a partial simulation of this effect even when no correction is made for the boundary layer.

10 Conclusion

Finite difference methods with an upwind bias in the hyperbolic region are now quite well established as practical tools for transonic flow calculations. Three dimensional calculations are presently restricted by limitations of computer memory capacity and time.

Much work remains to be done to improve these methods. In particular no estimates of global error bounds have been obtained. As long as the difference scheme is in conservation form, the solution of the difference equations should satisfy the proper jump conditions in the limit as the mesh width approaches zero. With the mesh widths realizable in practice, however, there is not enough resolution to provide a good representation of an oblique shock wave. If a reliable shock fitting scheme could be devised the results should be improved. If such a technique could be combined with the use of higher order accurate difference formulas it should be possible to use relatively coarse grids. This would open the way to more extensive three dimensional applications. Relaxation has proved a reliable but slow method for solving the nonlinear difference equations. Faster iterative methods are now in hand for treating two dimensional flows. Methods of comparable efficiency are needed for three dimensional calculations.

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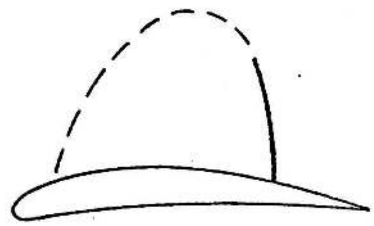
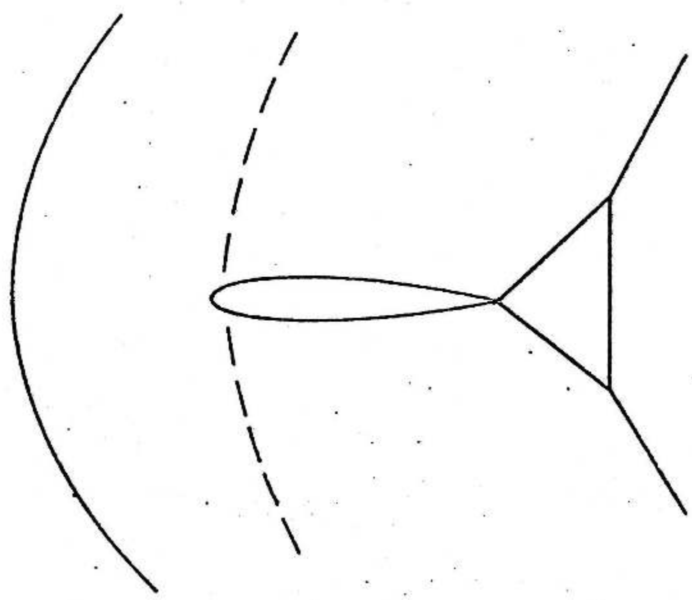


(a) SUBSONIC -- ONE SHOCK WAVE

(b) SUBSONIC -- TWO SHOCK WAVES



(c) SUPERSONIC -- BOW WAVE AND FISH TAIL SHOCK

FIGURE 1

TRANSONIC FLOW PATTERNS

—— SHOCK WAVE   --- SONIC LINE

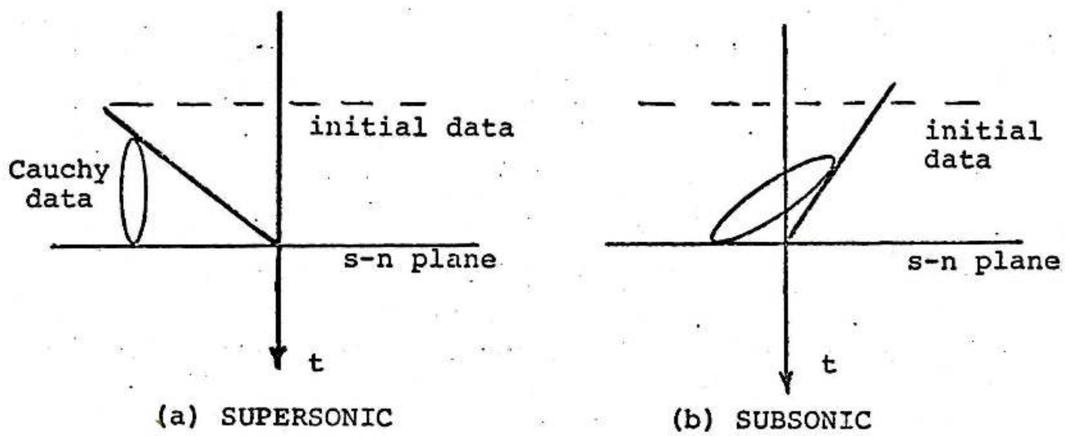


FIGURE 2

CHARACTERISTIC CONE OF EQUIVALENT TIME DEPENDENT EQUATION

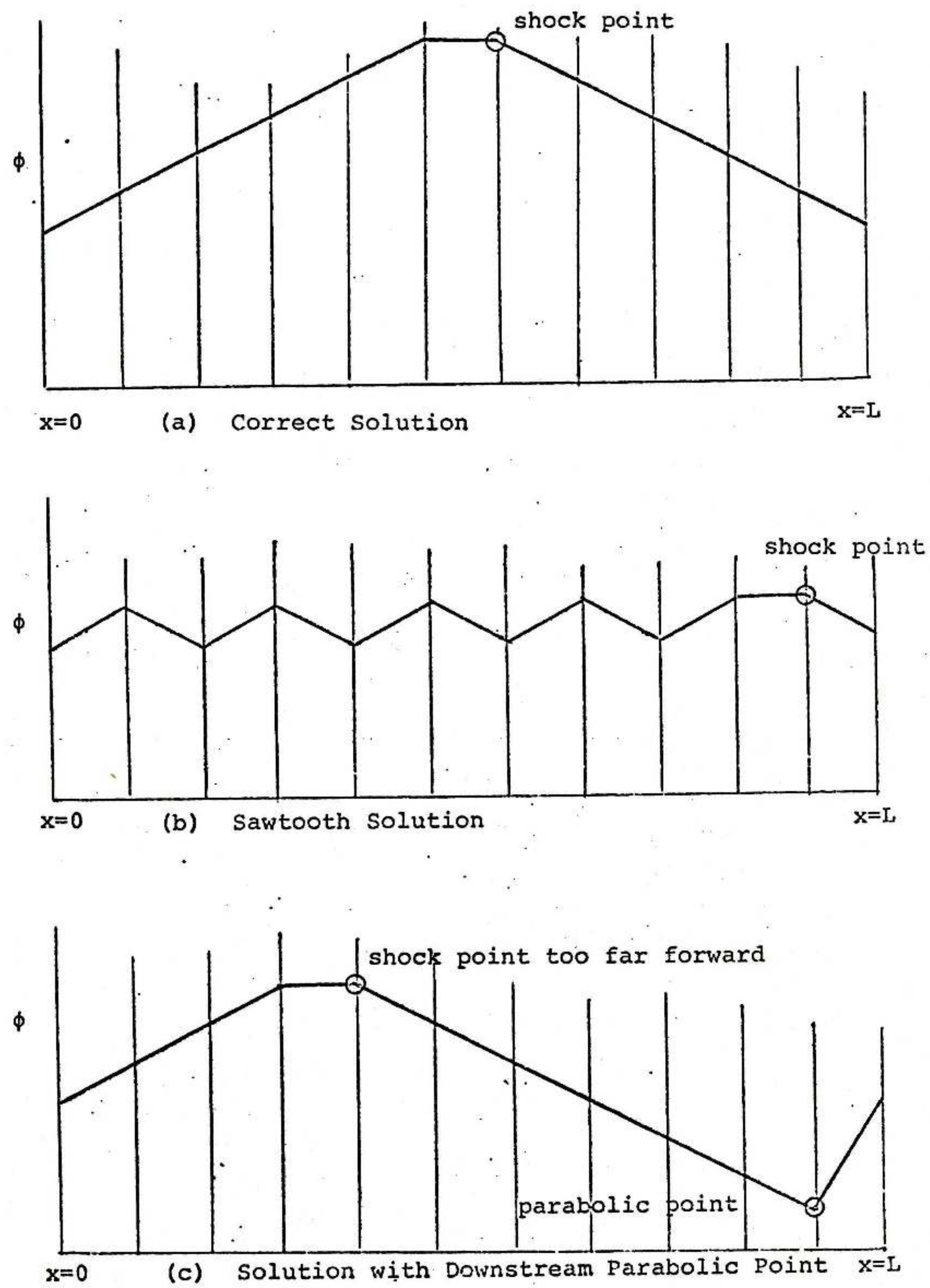
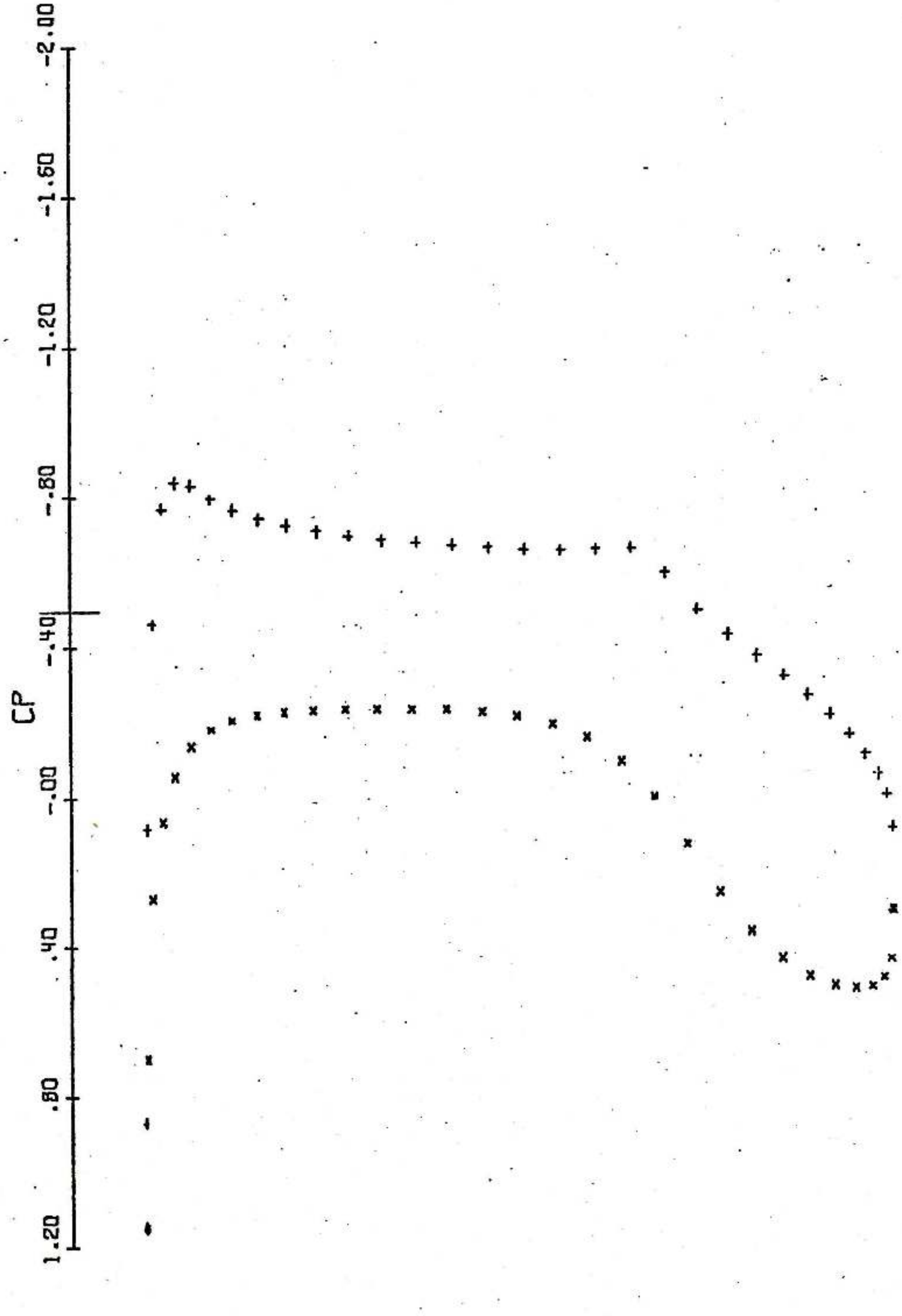


FIGURE 3

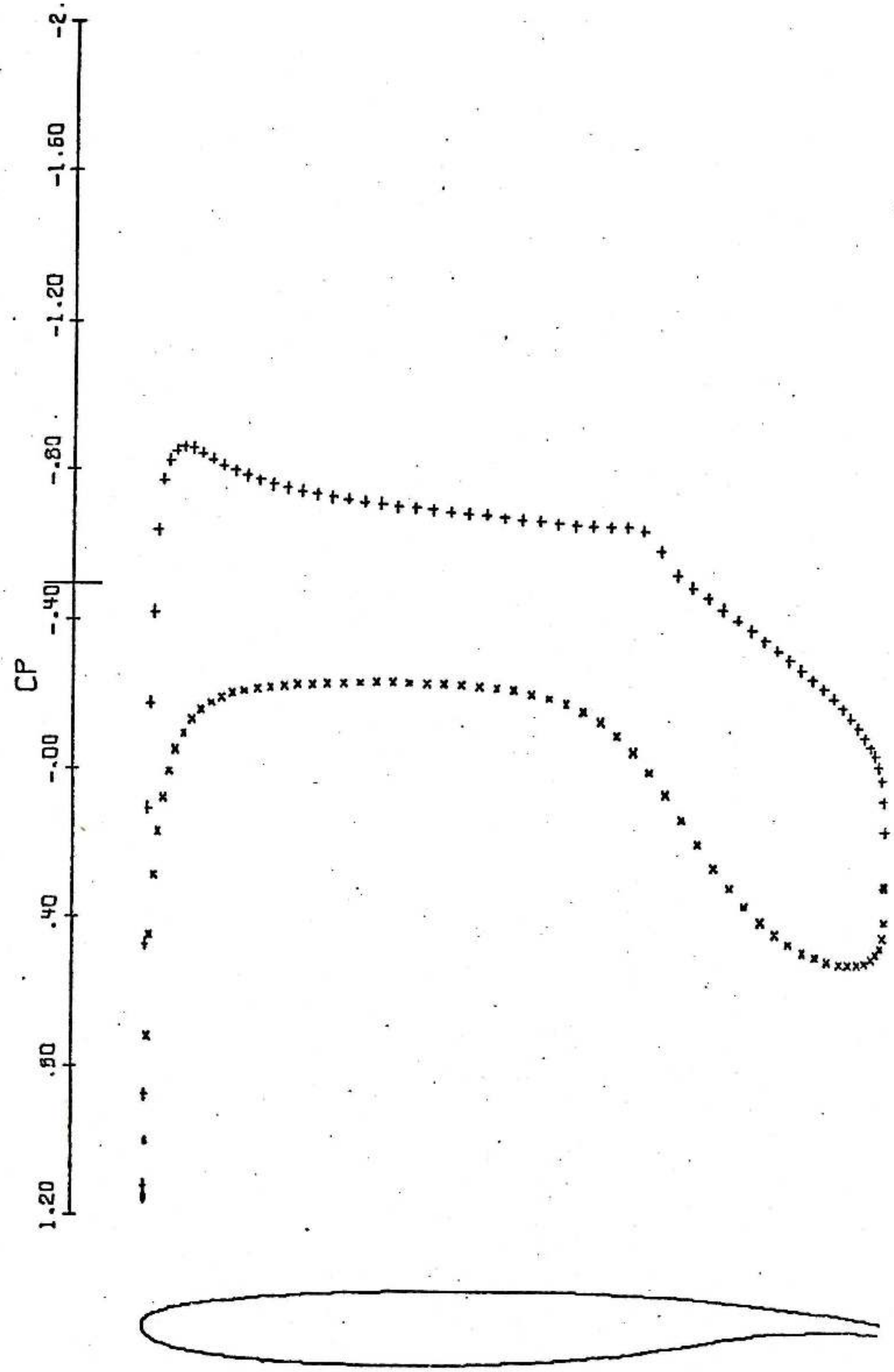
ONE DIMENSIONAL FLOW IN A CHANNEL



78-06-10 AIRFOIL

MACH .780 ANGLE OF ATTACK 0°

CL .5689 CD .0002 GRID 64x16

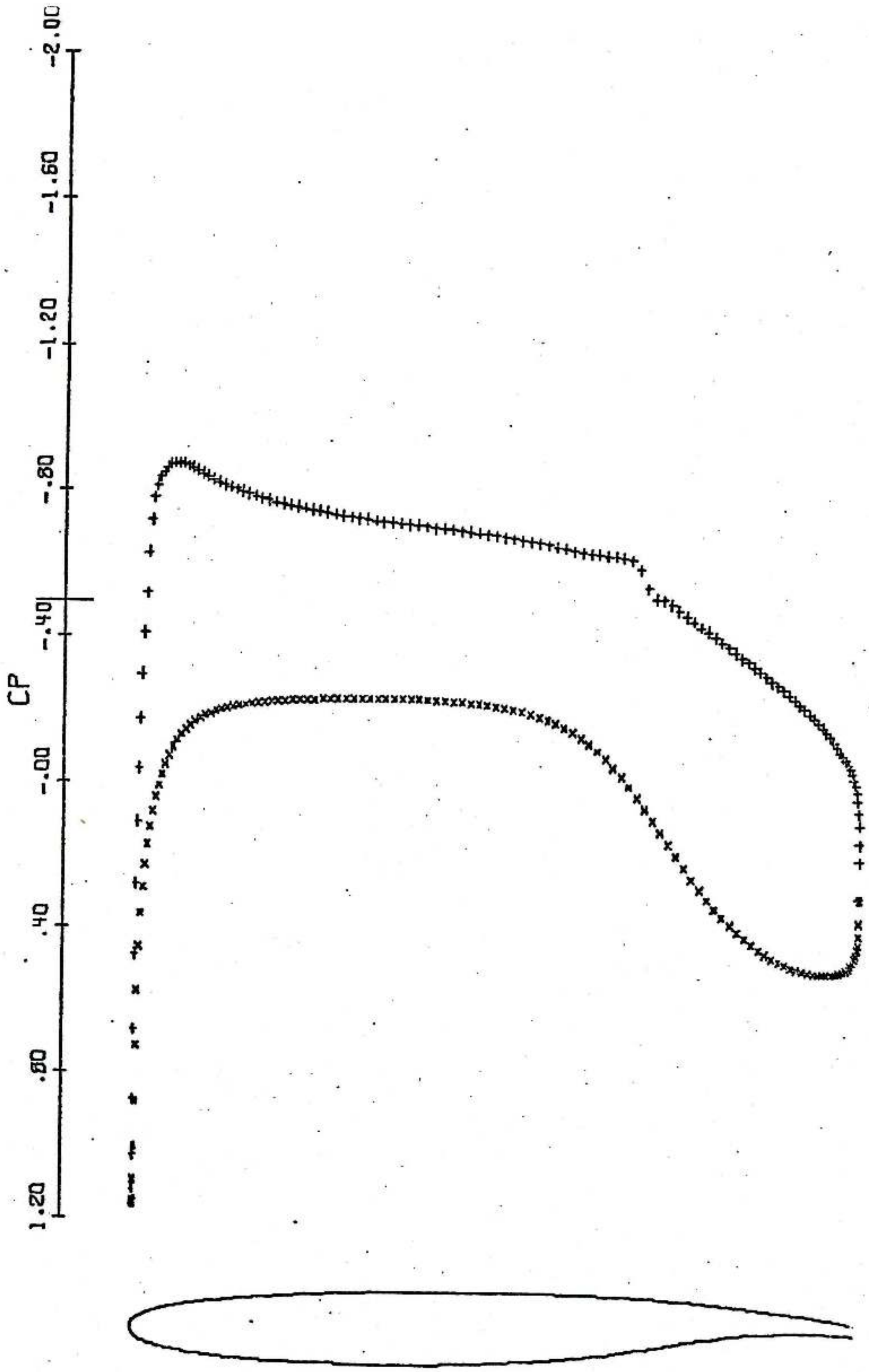


78-06-10 AIRFOIL

MACH .780 ANGLE OF ATTACK 0°

CL .5844 CD .0006 GRID 128x32

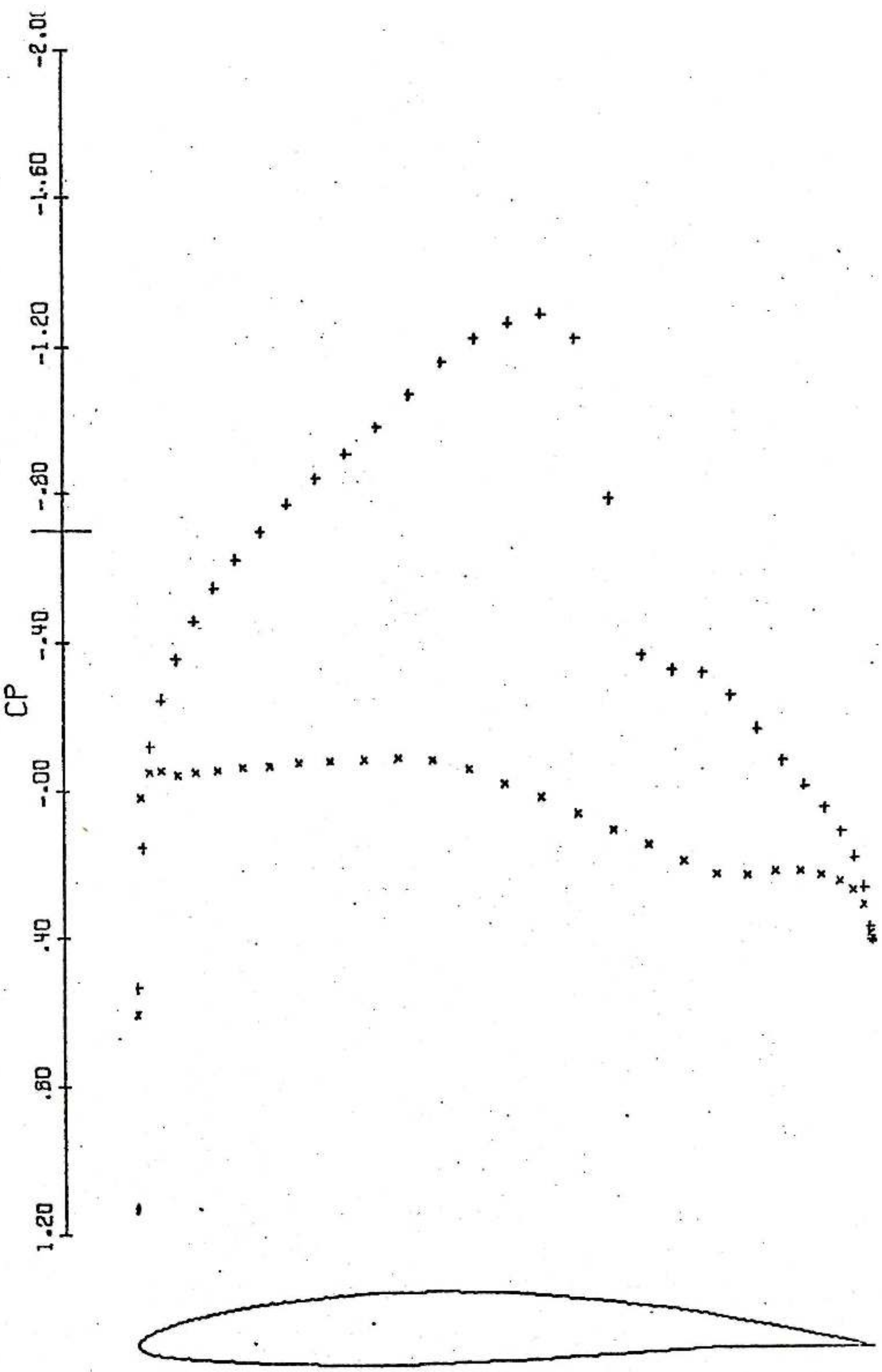
FIGURE 4(B)



78-06-10 AIRFOIL

MACH .780 ANGLE OF ATTACK 0°

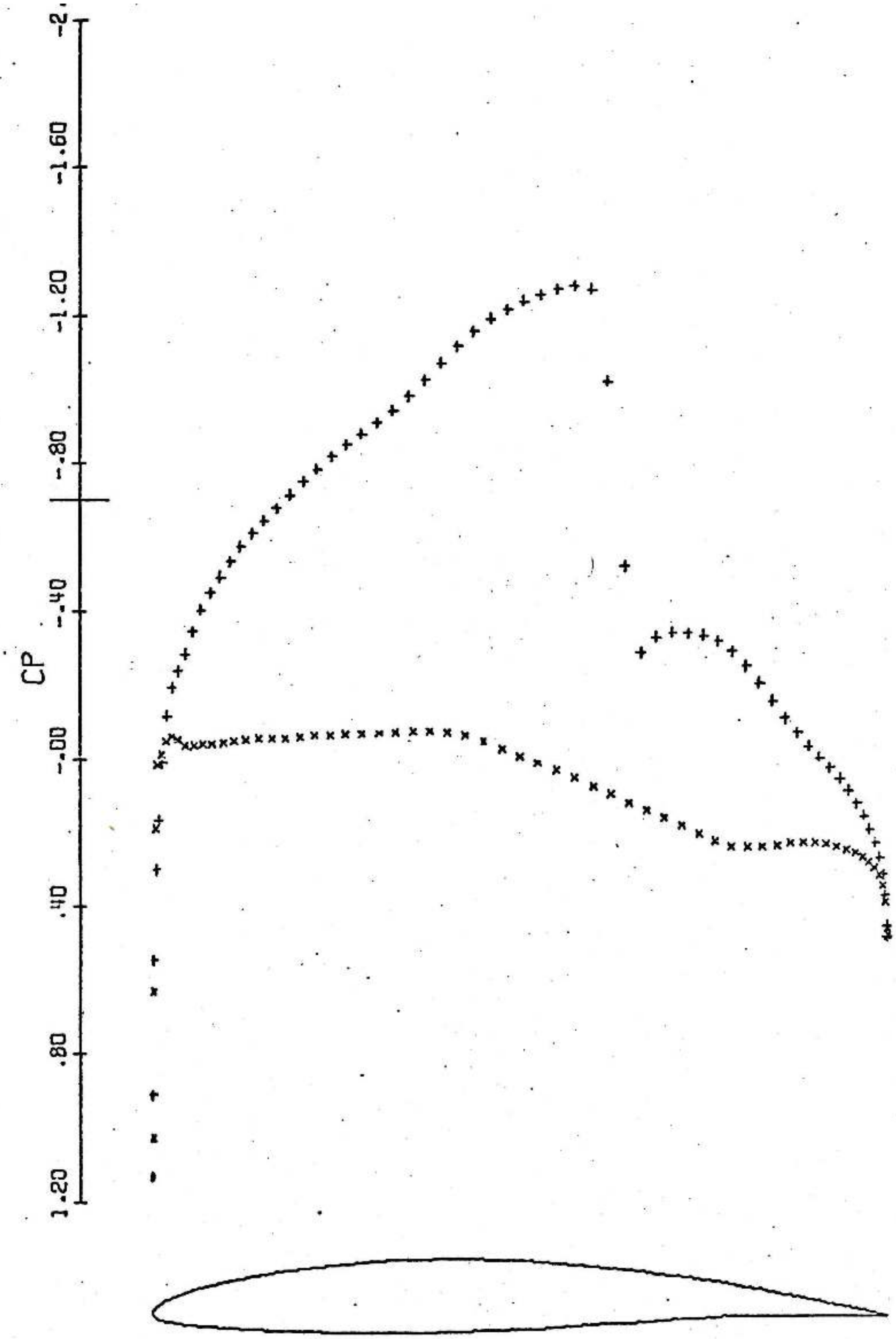
CL .5906 CD .0004 GRID 256x64



NACA 64A410

MACH .720 ANGLE OF ATTACK 0°

CL .6813 CD .0032 GRID 64x16

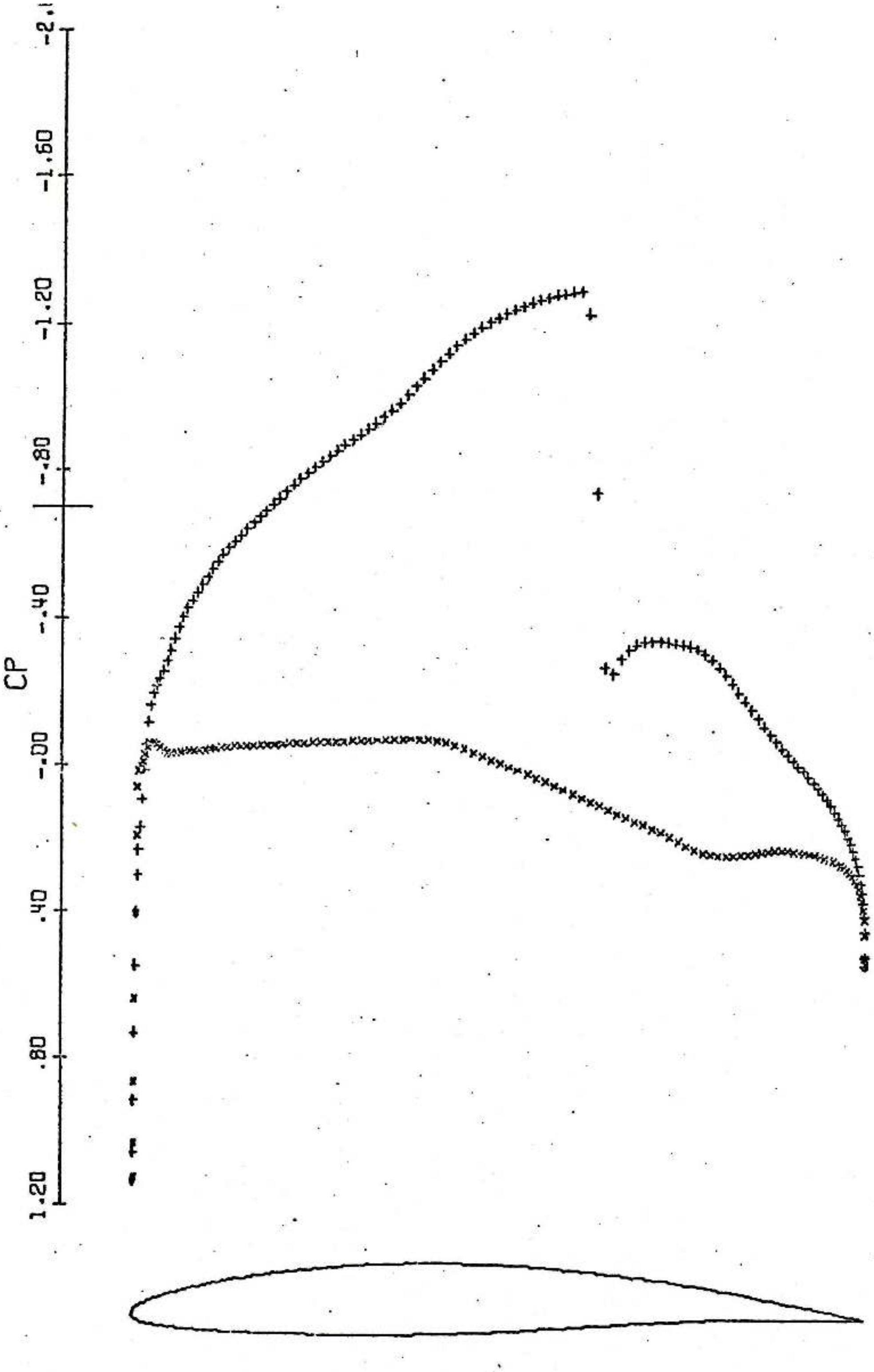


NACA 64A410

MACH .720 ANGLE OF ATTACK 0°

CL .6666 CD .0029 GRID 128x32

FIGURE 5(B)

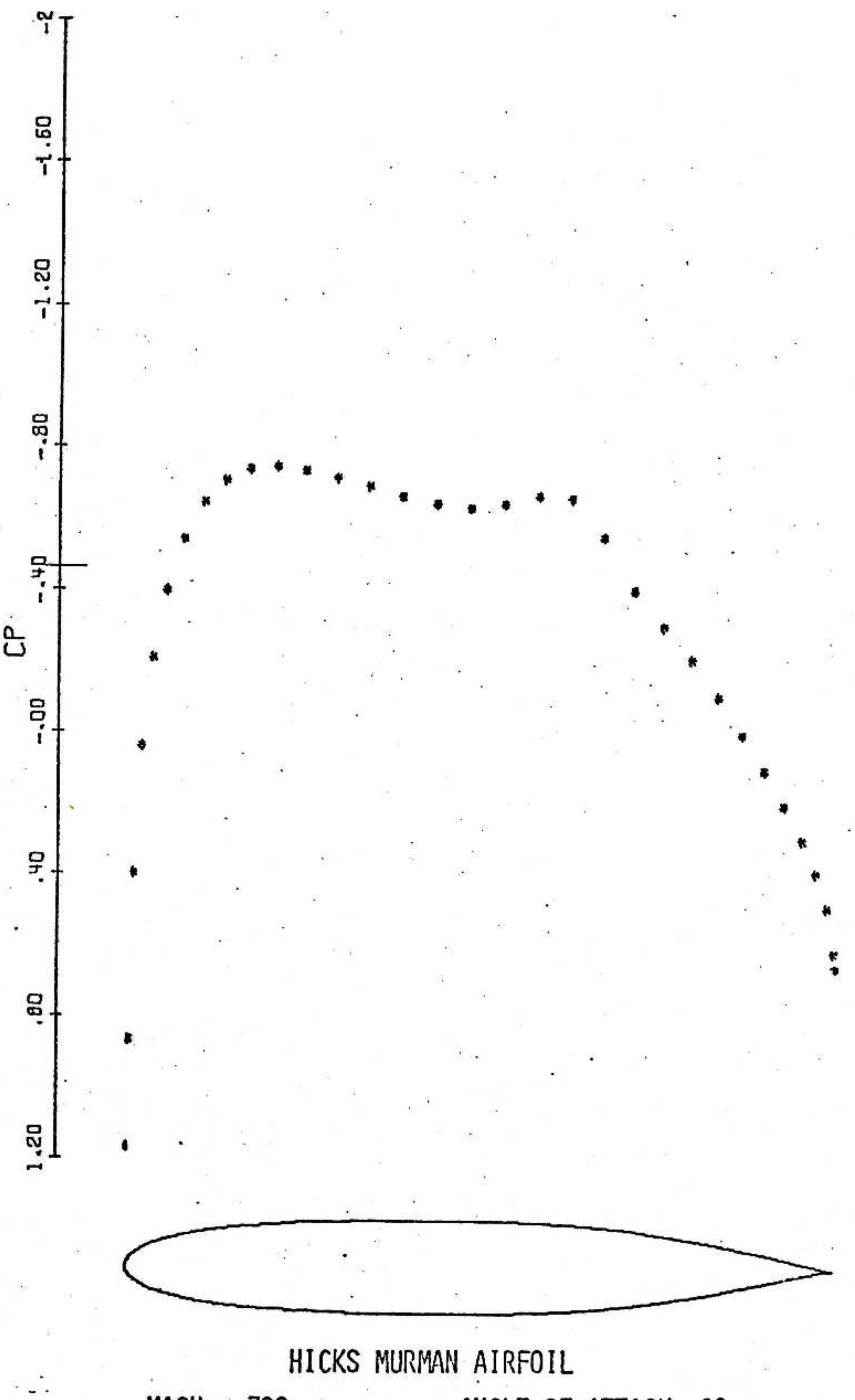


NACA 64A410

MACH .720 ANGLE OF ATTACK 0°

CL .6636 CD .0030 GRID 256x64

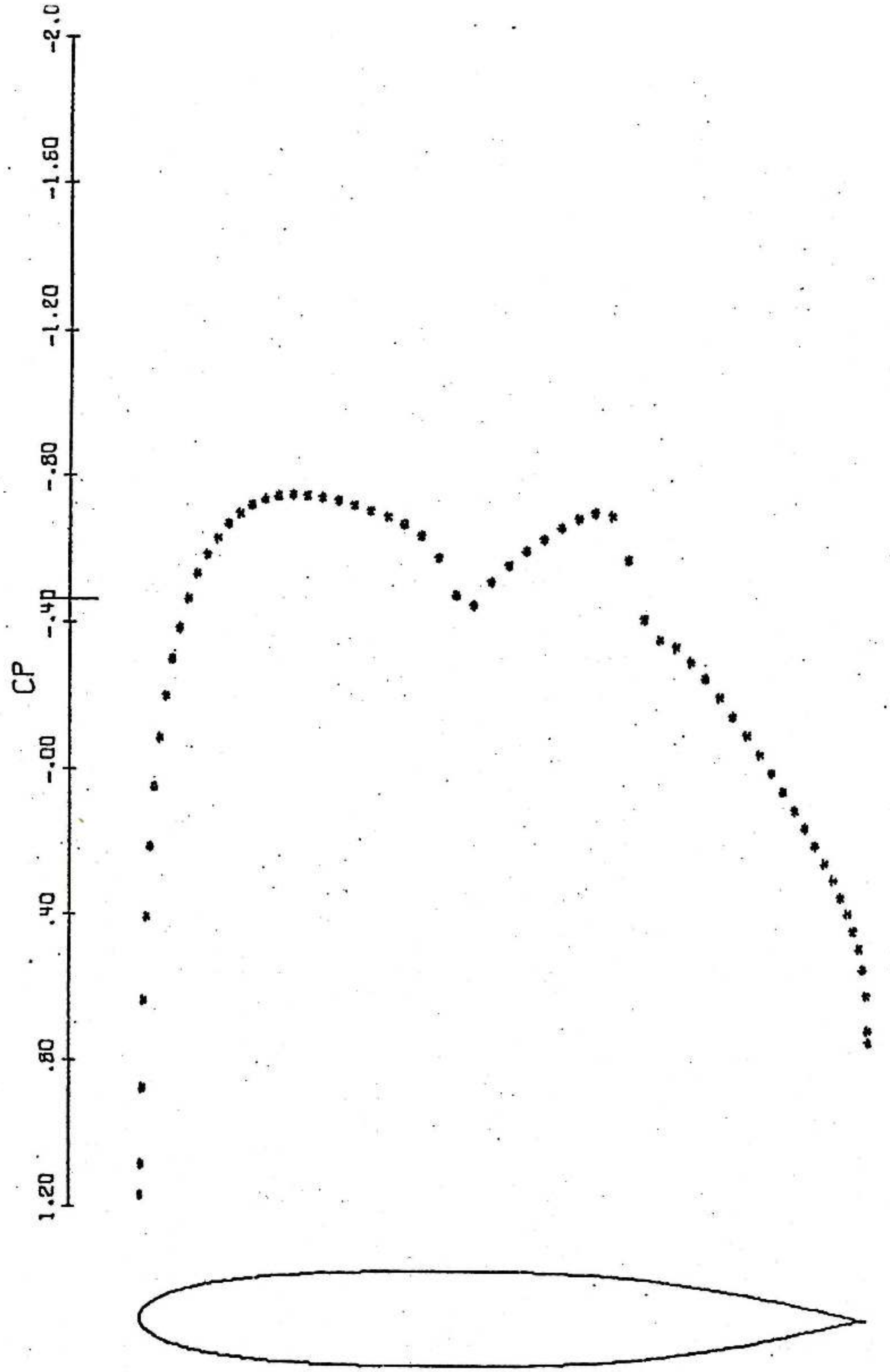
FIGURE 5(a)



HICKS MURMAN AIRFOIL

MACH .790 ANGLE OF ATTACK 0°

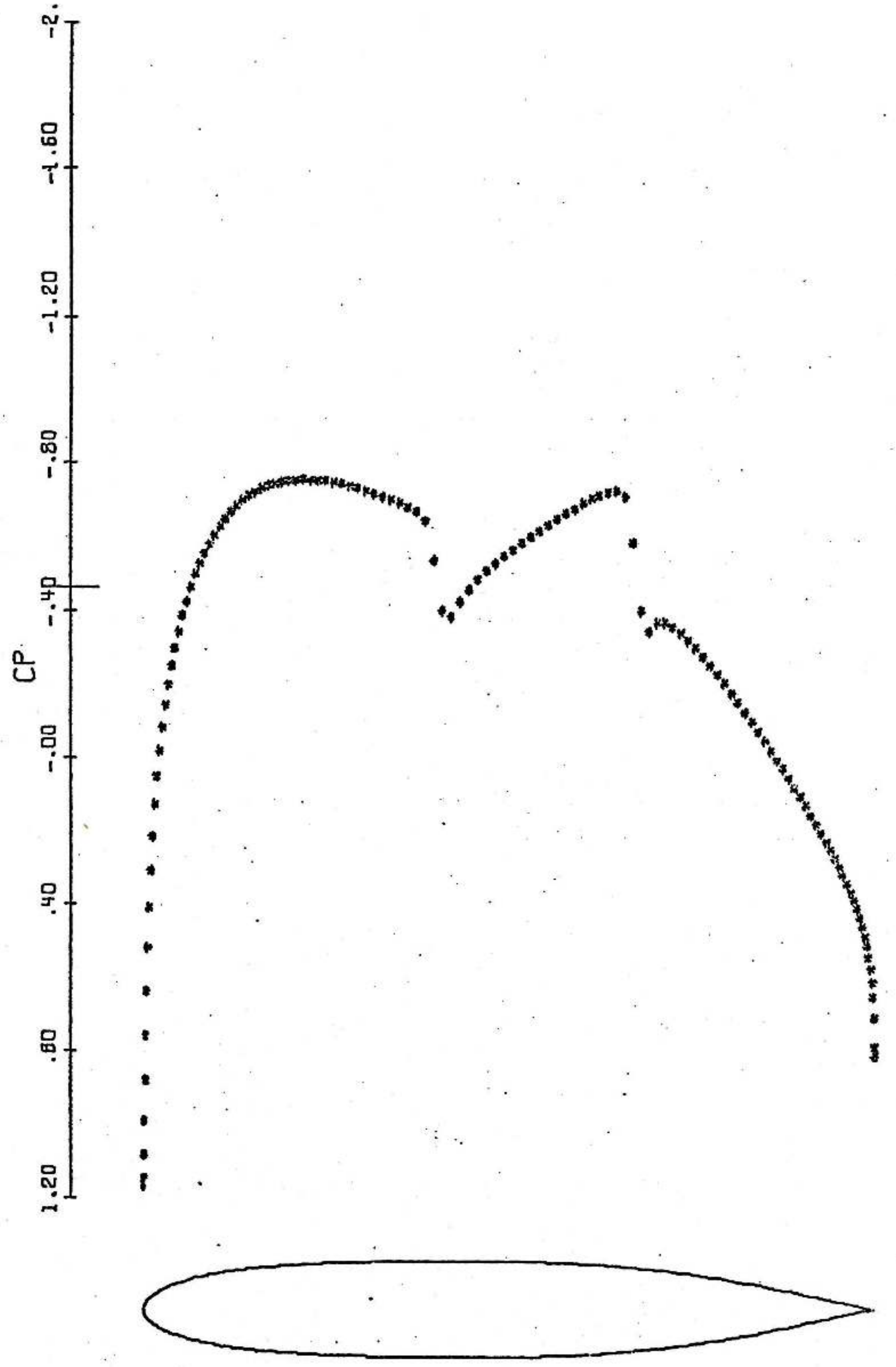
CD .0000 CD. .0002 GRID 64x16



HICKS MURMAN AIRFOIL

MACH .790 ANGLE OF ATTACK 0°

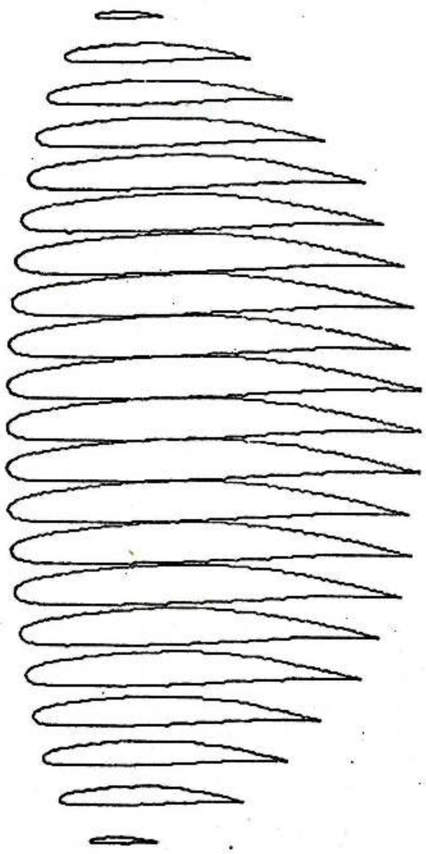
CL .0000 CD .0004 GRID 128x32

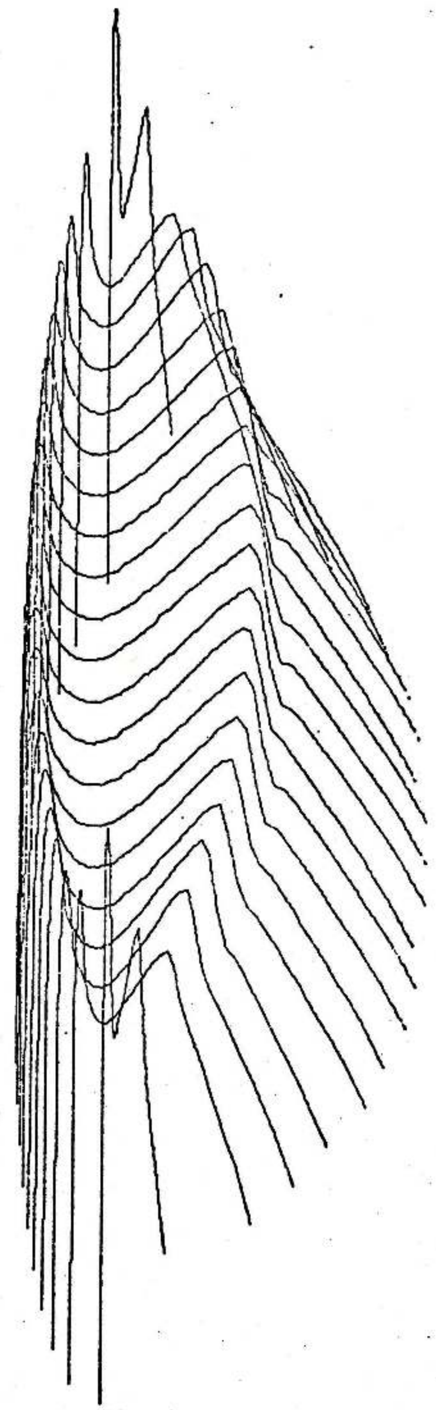


HICKS MURMAN AIRFOIL

MACH .790 ANGLE OF ATTACK 0°

CL .0000 CD .0005 GRID 256x64





VIEW OF WING

UPPER SURFACE PRESSURE

JONES SECTION

MACH .700

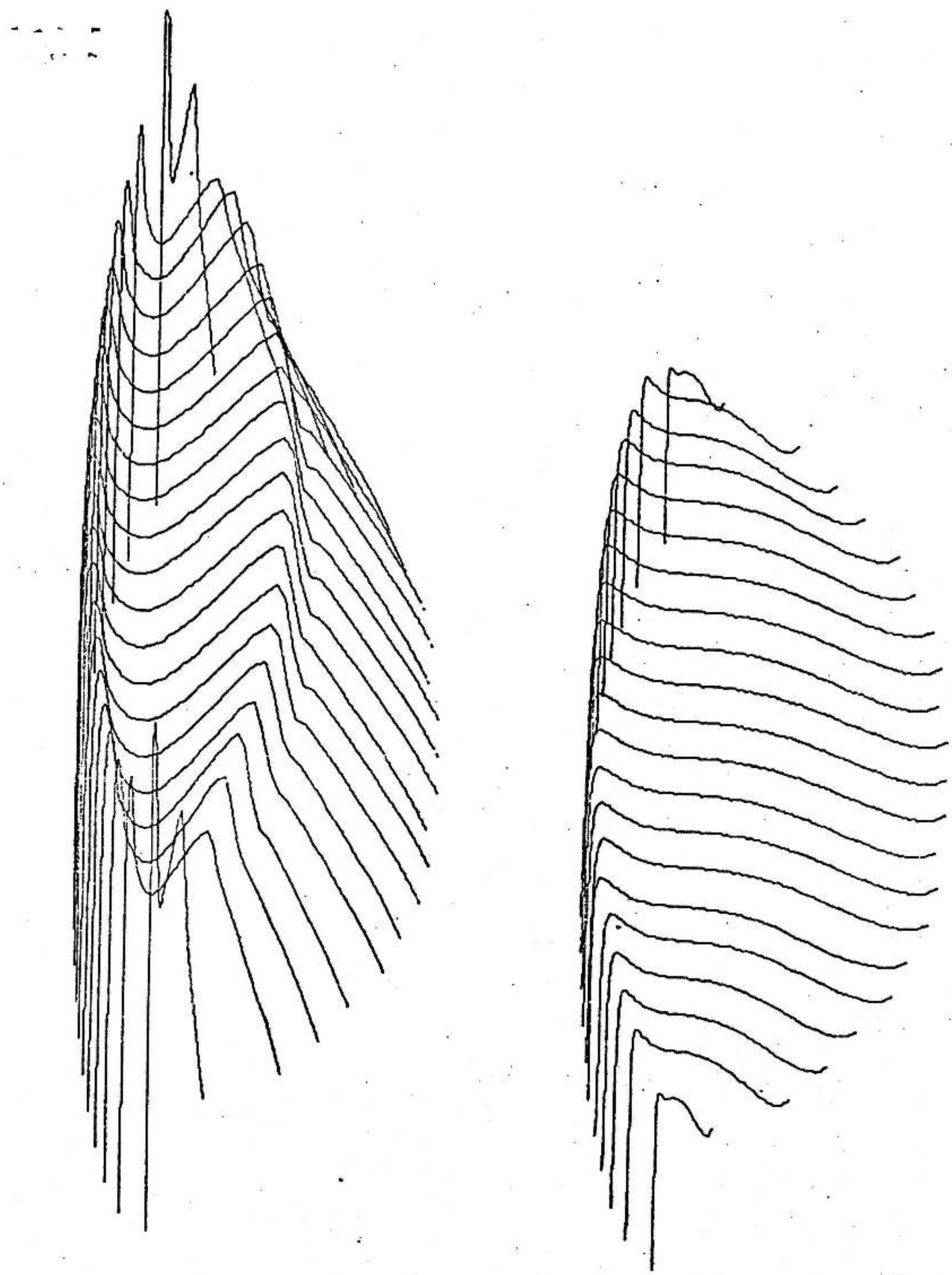
CL .8358

10 TO 1 ELLIPS

ANGLE OF ATTAC

CD .0147

FIGURE 7



UPPER SURFACE PRESSURE LOWER SURFACE PRESSURE

10 TO 1 ELLIPSE

ANGLE OF ATTACK 2.00°

CD .0147

FIGURE 7