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# Numerical solution of partial differential equations 

## FINITE DIFFERENCE METHODS

THIRD EDITION

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