

Numerical Solution of Stochastic Differential Equations with Jumps in Finance

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Kloeden, P.E. & Pl, E.: Numerical Solutions of Stochastic Differential Equations

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Springer Finance

E. Platen · D. Heath

The benchmark approach provides a general framework for financial market modeling, which extends beyond the standard risk neutral pricing theory. It allows for a unified treatment of portfolio optimization, derivative pricing, integrated risk management and insurance risk modeling. The existence of an equivalent risk neutral pricing measure is not required. Instead, it leads to pricing formulae with respect to the real world probability measure. This yields important modeling freedom which turns out to be necessary for the derivation of realistic, parsimonious market models.

The first part of the book describes the necessary tools from probability theory, statistics, stochastic calculus and the theory of stochastic differential equations with jumps. The second part is devoted to financial modeling under the benchmark approach. Various quantitative methods for the fair pricing and hedging of derivatives are explained. The general framework is used to provide an understanding of the nature of stochastic volatility.

The book is intended for a wide audience that includes quantitative analysts, postgraduate students and practitioners in finance, economics and insurance. It aims to be a self-contained, accessible but mathematically rigorous introduction to quantitative finance for readers that have a reasonable mathematical or quantitative background. Finally, the book should stimulate interest in the benchmark approach by describing some of its power and wide applicability.

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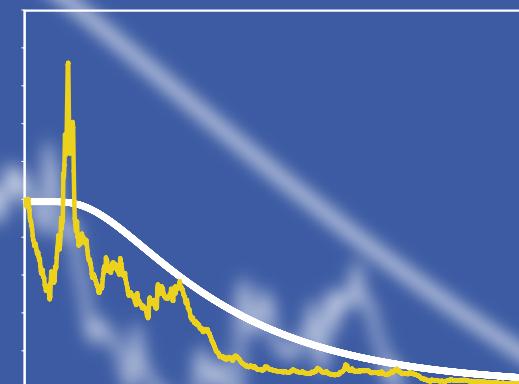


A Benchmark Approach to Quantitative Finance

Springer Finance

**Eckhard Platen
David Heath**

A Benchmark Approach to Quantitative Finance



Springer

Jump-Diffusion Multi-Factor Models

Björk, Kabanov & Runggaldier (1997)

- continuous time
- Markovian
- explicit transition densities in special cases
- benchmark framework
- discrete time approximations
- suitable for simulation
- Markov chain approximations

Pathwise Approximations:

- scenario simulation of entire markets
 - testing statistical techniques on simulated trajectories
 - filtering hidden state variables
Pl. & Runggaldier (2005, 2007)
 - hedge simulation
 - dynamic financial analysis
 - extreme value simulation
 - stress testing
- ⇒ higher order strong schemes
predictor-corrector methods

Probability Approximations:

- derivative prices
- sensitivities
- expected utilities
- portfolio selection
- risk measures
- long term risk management

⇒ Monte Carlo simulation, higher order weak schemes,
predictor-corrector variance reduction, Quasi Monte Carlo,
or Markov chain approximations, lattice methods

Essential Requirements:

- parsimonious models
- respect no-arbitrage in discrete time approximation
- numerically stable methods
- efficient methods for high-dimensional models
- higher order schemes, predictor-corrector

Continuous and Event Driven Risk

- Wiener processes $W^k, k \in \{1, 2, \dots, m\}$

- counting processes p^k

intensity h^k

jump martingale q^k

$$dW_t^{m+k} = dq_t^k = (dp_t^k - h_t^k dt) (h_t^k)^{-\frac{1}{2}}$$

$$k \in \{1, 2, \dots, d-m\}$$

$$\mathbf{W}_t = (W_t^1, \dots, W_t^m, q_t^1, \dots, q_t^{d-m})^\top$$

Primary Security Accounts

$$dS_t^j = S_{t-}^j \left(a_t^j dt + \sum_{k=1}^d b_t^{j,k} dW_t^k \right)$$

Assumption 1

$$b_t^{j,k} \geq -\sqrt{h_t^{k-m}}$$

$$k \in \{m+1, \dots, d\}.$$

Assumption 2

Generalized volatility matrix $b_t = [b_t^{j,k}]_{j,k=1}^d$ invertible.

- market price of risk

$$\theta_t = (\theta_t^1, \dots, \theta_t^d)^\top = b_t^{-1} [a_t - r_t \mathbf{1}]$$

- primary security account

$$dS_t^j = S_{t-}^j \left(r_t dt + \sum_{k=1}^d b_t^{j,k} (\theta_t^k dt + dW_t^k) \right)$$

- portfolio

$$dS_t^\delta = \sum_{j=0}^d \delta_t^j dS_t^j$$

- fraction

$$\pi_{\delta,t}^j = \delta_t^j \frac{S_t^j}{S_t^\delta}$$

- portfolio

$$dS_t^\delta = S_{t-}^\delta \left\{ r_t dt + \pi_{\delta,t-}^\top b_t (\theta_t dt + dW_t) \right\}$$

Assumption 3

$$\sqrt{h_t^{k-m}} > \theta_t^k$$

- generalized GOP volatility

$$c_t^k = \begin{cases} \theta_t^k & \text{for } k \in \{1, 2, \dots, m\} \\ \frac{\theta_t^k}{1 - \theta_t^k (h_t^{k-m})^{-\frac{1}{2}}} & \text{for } k \in \{m+1, \dots, d\} \end{cases}$$

- GOP fractions

$$\pi_{\delta_*, t} = (\pi_{\delta_*, t}^1, \dots, \pi_{\delta_*, t}^d)^\top = (c_t^\top b_t^{-1})^\top$$

- Growth Optimal Portfolio

$$dS_t^{\delta_*} = S_{t-}^{\delta_*} \left(r_t dt + c_t^\top (\theta_t dt + dW_t) \right)$$

- optimal growth rate

$$g_t^{\delta_*} = r_t + \frac{1}{2} \sum_{k=1}^m (\theta_t^k)^2$$

$$- \sum_{k=m+1}^d h_t^{k-m} \left(\ln \left(1 + \frac{\theta_t^k}{\sqrt{h_t^{k-m}} - \theta_t^k} \right) + \frac{\theta_t^k}{\sqrt{h_t^{k-m}}} \right)$$

- **benchmarked portfolio**

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$$

Theorem 4 *Any nonnegative benchmarked portfolio \hat{S}^δ is an $(\underline{\mathcal{A}}, P)$ -supermartingale.*

⇒ no strong arbitrage

but there may exist:

free lunch with vanishing risk (Delbaen & Schachermayer (2006))

free snacks or cheap thrills (Loewenstein & Willard (2000))

Multi-Factor Model

model mainly:

- **benchmarked primary security accounts**

$$\hat{S}_t^j = \frac{S_t^j}{S_t^{\delta_*}}$$

$$j \in \{0, 1, \dots, d\}$$

supermartingales, often SDE driftless,
local martingales, sometimes martingales

savings account

$$S_t^0 = \exp \left\{ \int_0^t r_s \, ds \right\}$$

\implies GOP

$$S_t^{\delta_*} = \frac{S_t^0}{\hat{S}_t^0}$$

\implies stock

$$S_t^j = \hat{S}_t^j S_t^{\delta_*}$$

additionally dividend rates

foreign interest rates

Example

Black-Scholes Type Market

$$d\hat{S}_t^j = -\hat{S}_{t-}^j \sum_{k=1}^d \sigma_t^{j,k} dW_t^k$$

$$h_t^j, \sigma_t^{j,k}, r_t$$

Examples

- Merton jump-diffusion model

$$dX_t = X_{t-} (\mu dt + \sigma dW_t + dp_t),$$

↓

$$X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{i=1}^{N_t} \xi_i$$

- Bates model

$$dS_t = S_{t-} \left(\alpha dt + \sqrt{V_t} dW_t^S + dp_t \right)$$

$$dV_t = \xi(\eta - V_t) dt + \theta \sqrt{V_t} dW_t^V$$

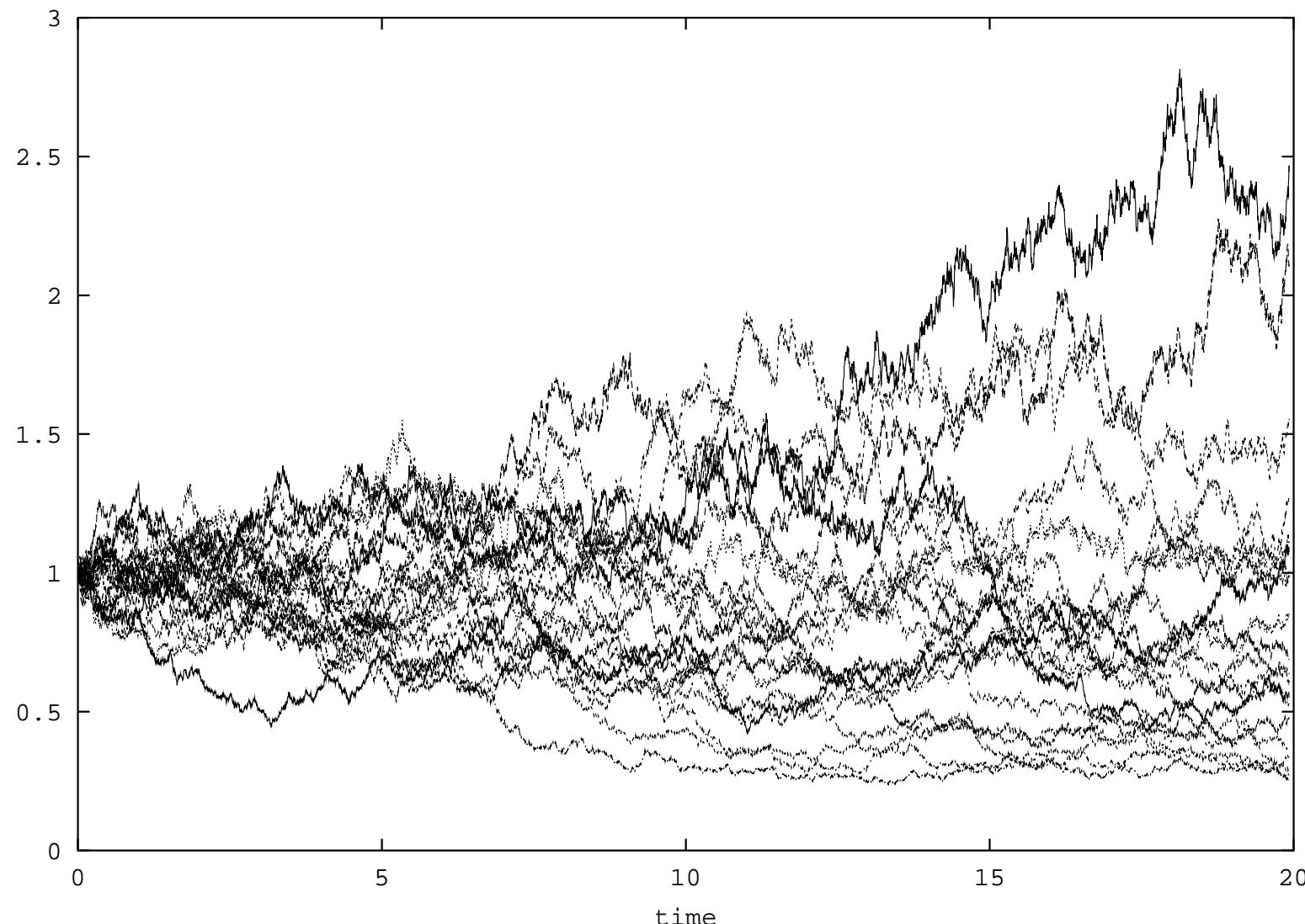


Figure 1: Simulated benchmarked primary security accounts.

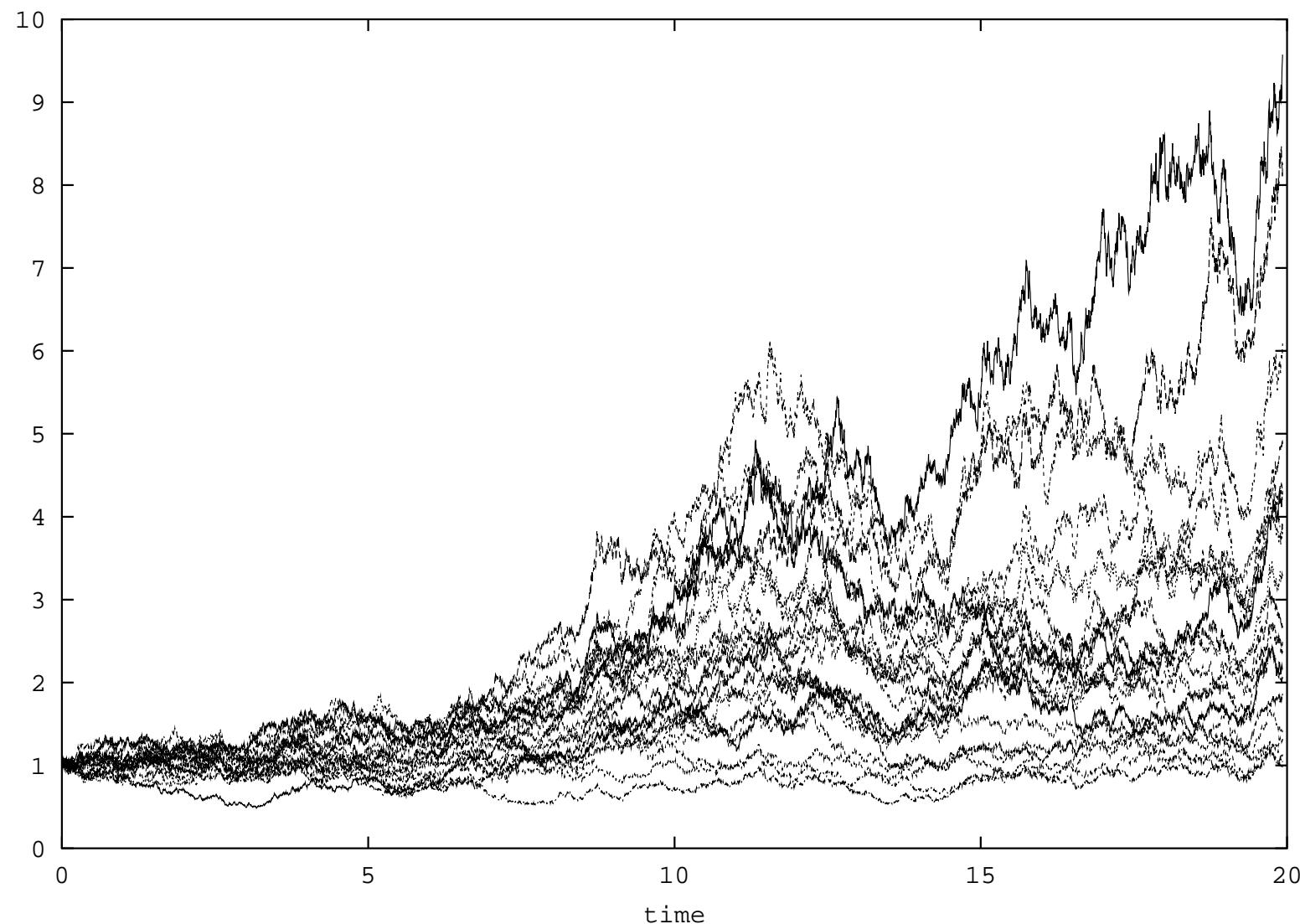


Figure 2: Simulated primary security accounts.

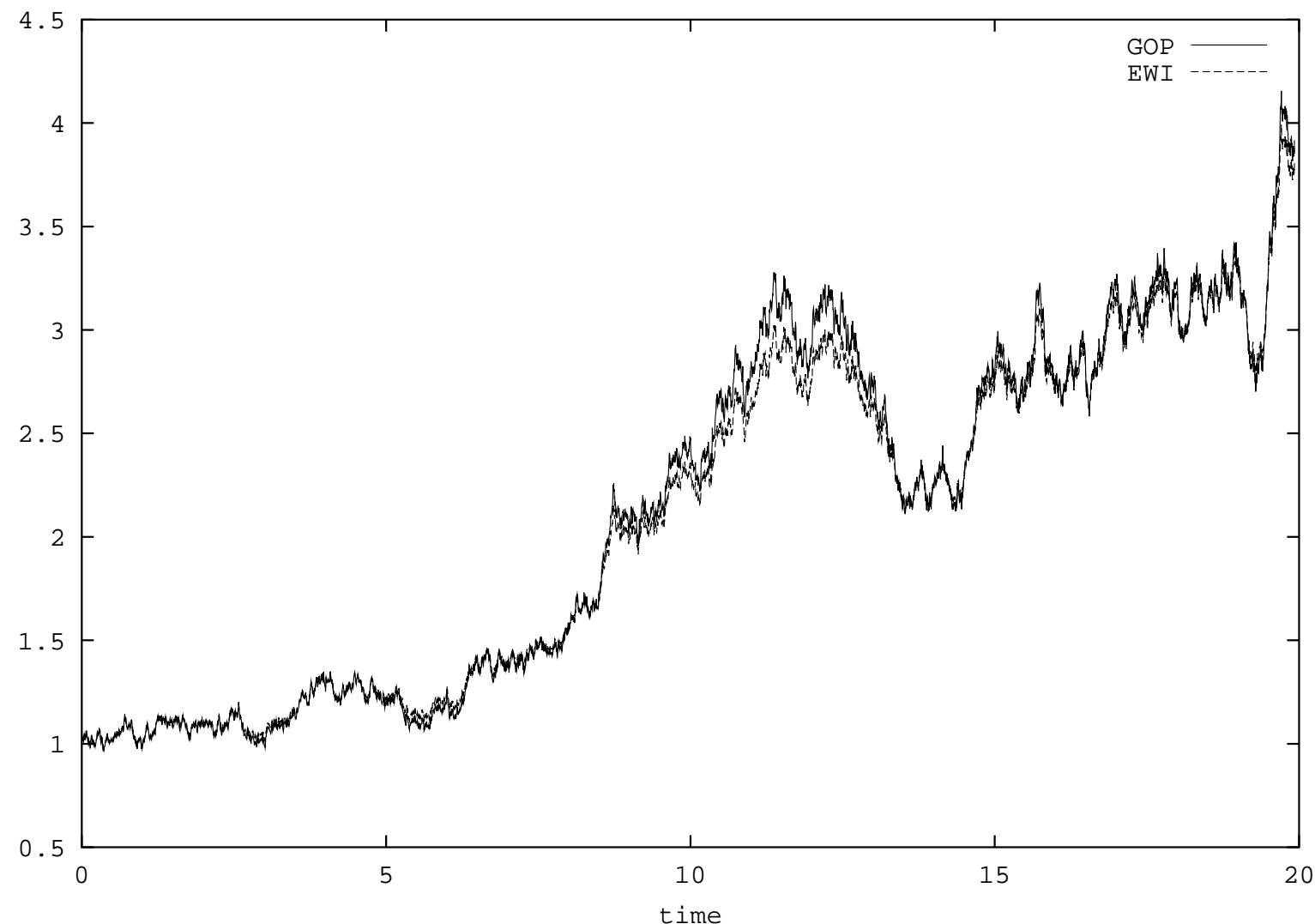


Figure 3: Simulated GOP and EWI for $d = 50$.

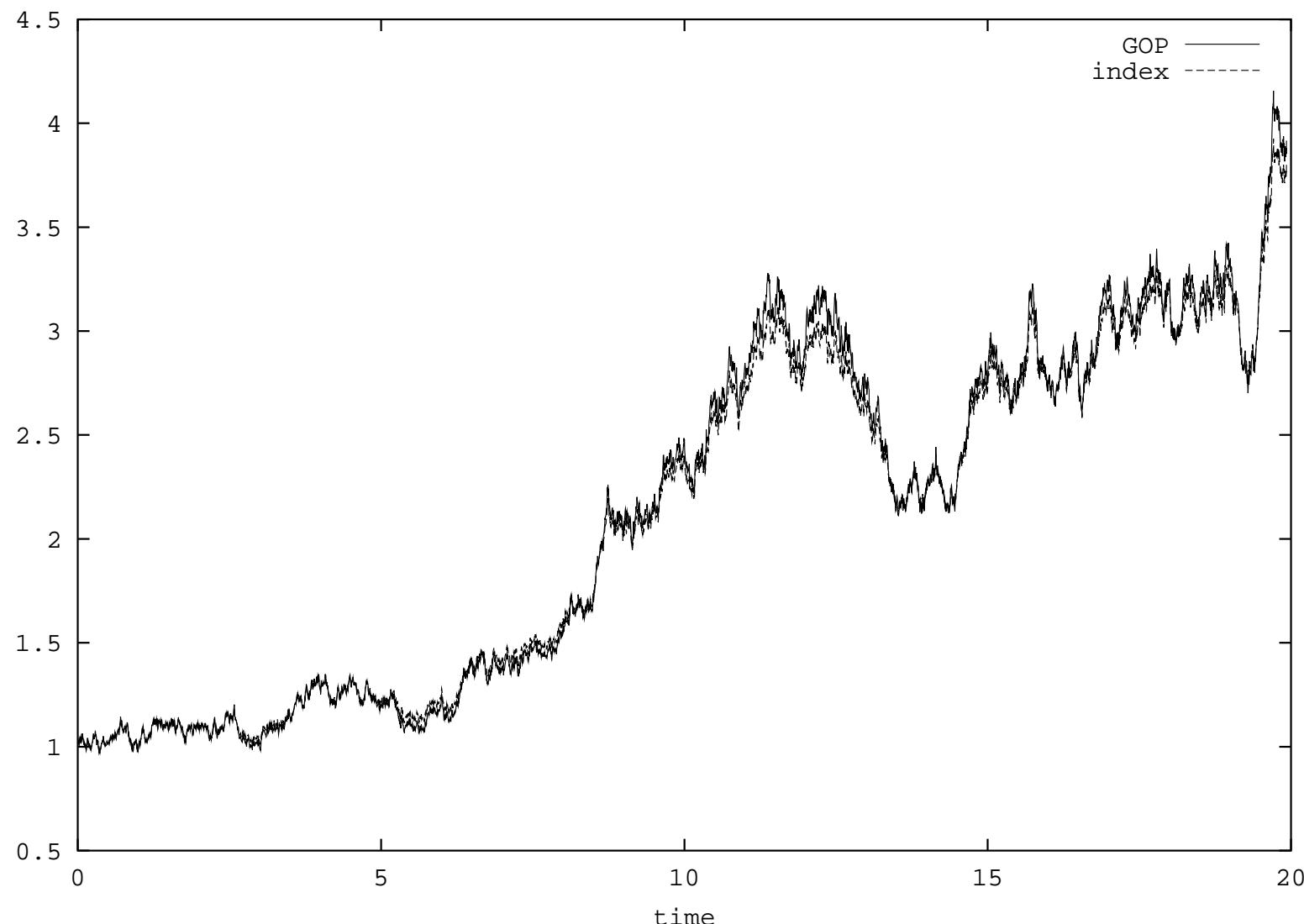


Figure 4: Simulated accumulation index and GOP.

Diversification

- diversified portfolios

$$\left| \pi_{\delta,t}^j \right| \leq \frac{K_2}{d^{\frac{1}{2} + K_1}}$$

Theorem 5

*In a regular market any **diversified portfolio** is an approximate **GOP**.*

Pl. (2005)

- robust characterization
- similar to Central Limit Theorem
- model independent

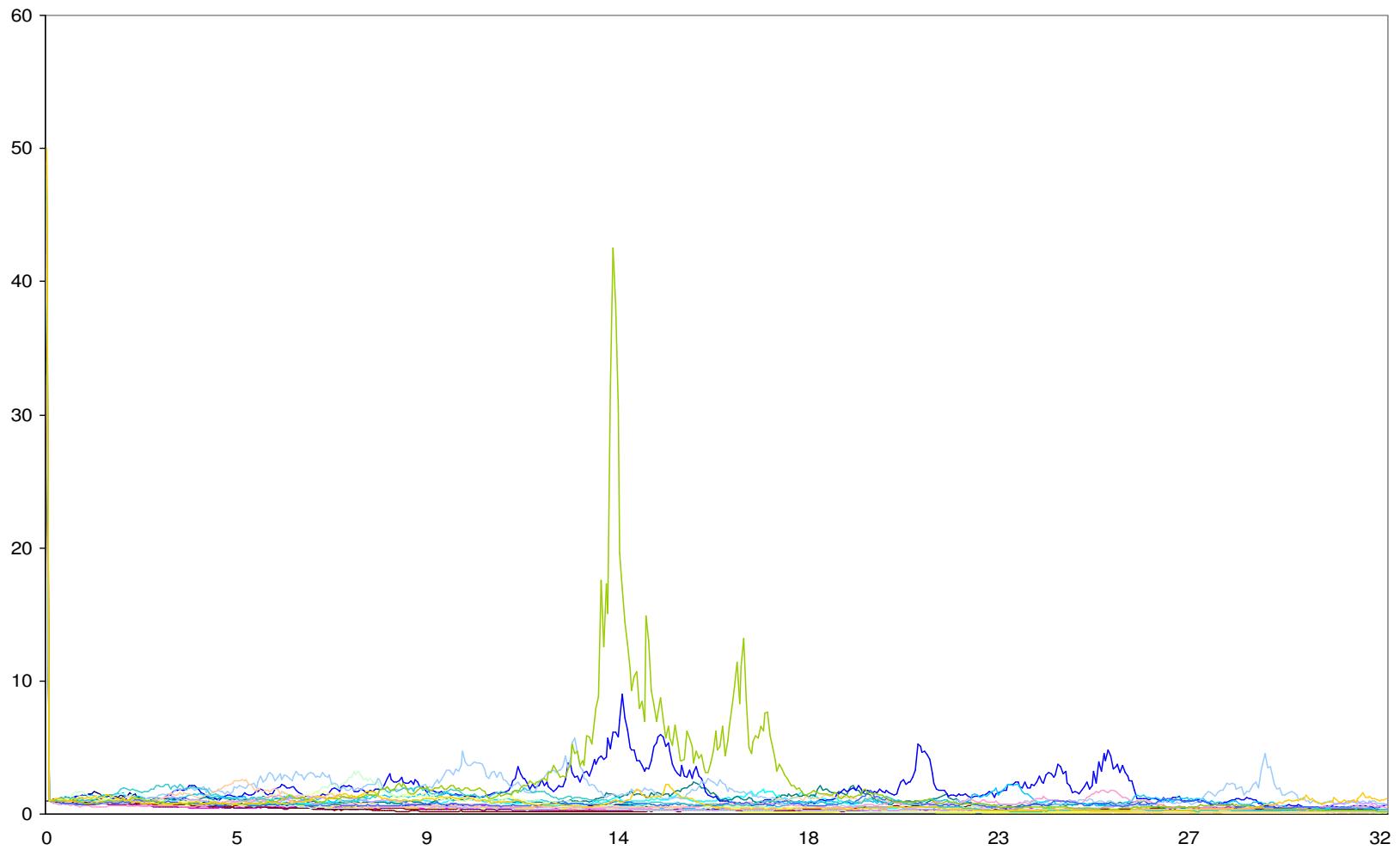


Figure 5: Benchmarked primary security accounts.

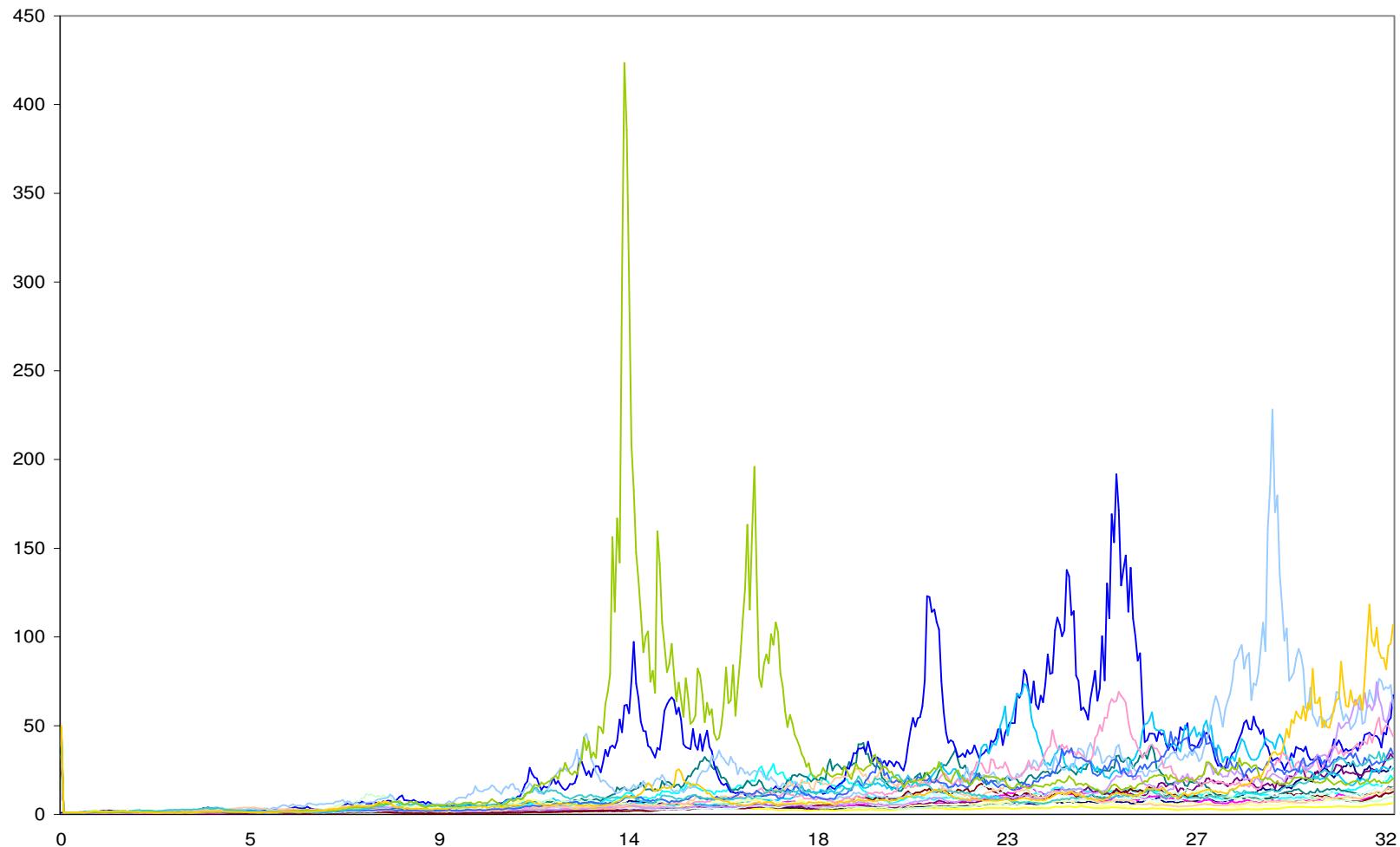


Figure 6: Primary security accounts under the MMM.

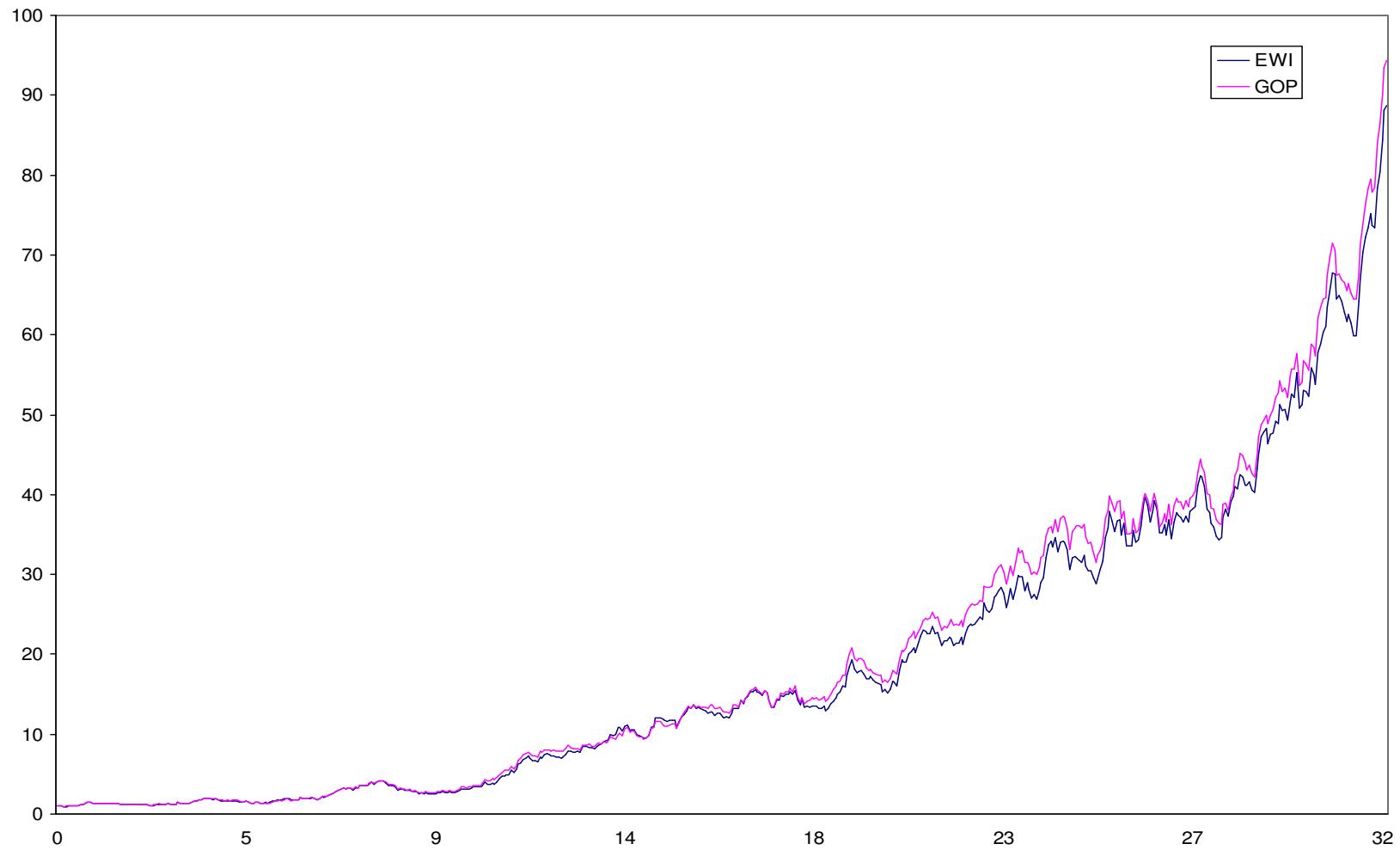


Figure 7: GOP and EWI.

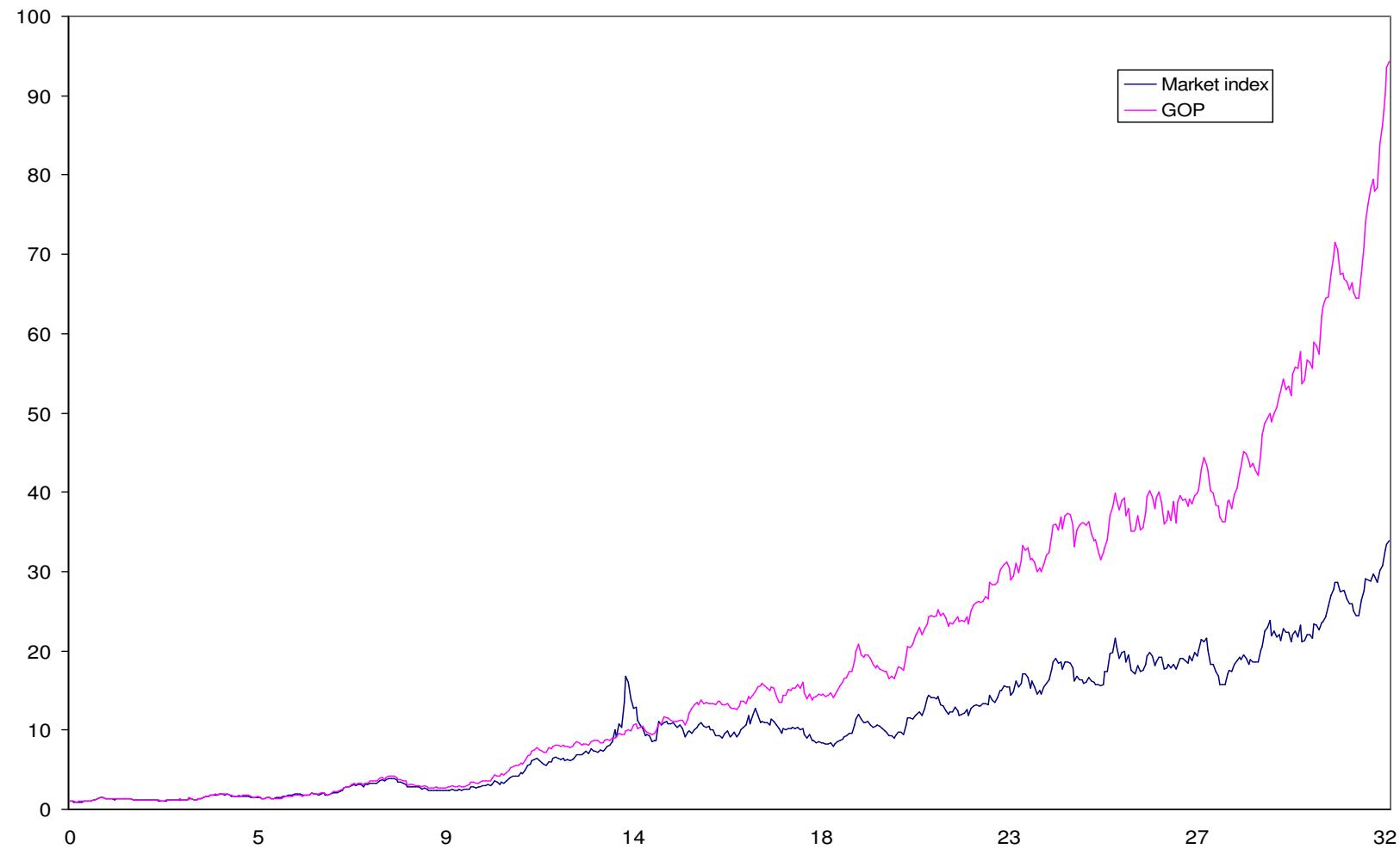


Figure 8: GOP and market index.

- **fair security**

benchmarked security (\mathcal{A}, P) -martingale \iff fair

- **minimal replicating portfolio**

fair nonnegative portfolio S^δ with $S_\tau^\delta = H_\tau$

\implies minimal nonnegative replicating portfolio

- **fair pricing formula**

$$V_{H_\tau}(t) = S_t^{\delta_*} E \left(\frac{H_\tau}{S_\tau^{\delta_*}} \mid \mathcal{A}_t \right)$$

No need for equivalent risk neutral probability measure!

Fair Hedging

- fair portfolio S_t^δ
- benchmarked fair portfolio

$$\hat{S}_t^\delta = E \left(\frac{H_\tau}{S_\tau^{\delta_*}} \mid \mathcal{A}_t \right)$$

- **martingale representation**

$$\frac{H_\tau}{S_\tau^{\delta_*}} = E \left(\frac{H_\tau}{S_\tau^{\delta_*}} \mid \mathcal{A}_t \right) + \sum_{k=1}^d \int_t^\tau x_{H_\tau}^k(s) dW_s^k + M_{H_\tau}(t)$$

M_{H_τ} - $(\underline{\mathcal{A}}, P)$ -martingale (pooled)

$$E ([M_{H_\tau}, W^k]_t) = 0$$

Föllmer & Schweizer (1991)

No need for equivalent risk neutral probability measure!

Simulation of SDEs with Jumps

- **strong schemes** (paths)

Taylor

explicit

derivative-free

implicit

balanced implicit

predictor-corrector

- **weak schemes** (probabilities)

Taylor

simplified

explicit

derivative-free

implicit, predictor-corrector

- **intensity of jump process**
 - regular schemes \implies high intensity
 - jump-adapted schemes \implies low intensity

SDE with Jumps

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t + c(t-, X_{t-}) dp_t$$

$$X_0 \in \Re^d$$

- $p_t = N_t$: Poisson process, intensity $\lambda < \infty$
- $p_t = \sum_{i=1}^{N_t} (\xi_i - 1)$: compound Poisson, ξ_i i.i.d r.v.
- Poisson random measure

$$\int_{\mathcal{E}} c(t-, X_{t-}, v) p_\phi(dv \times dt)$$

- $\{(\tau_i, \xi_i), i = 1, 2, \dots, N_T\}$

Numerical Schemes

- time discretization

$$t_n = n\Delta$$

- discrete time approximation

$$Y_{n+1}^\Delta = Y_n^\Delta + a(Y_n^\Delta)\Delta + b(Y_n^\Delta)\Delta W_n + c(Y_n^\Delta)\Delta p_n$$

Strong Convergence

- **Applications:** scenario analysis, filtering and hedge simulation
- **strong order γ** if

$$\varepsilon_s(\Delta) = \sqrt{E \left(|X_T - Y_N^\Delta|^2 \right)} \leq K \Delta^\gamma$$

Weak Convergence

- **Applications:** derivative pricing, utilities, risk measures

- **weak order** β if

$$\varepsilon_w(\Delta) = |E(g(X_T)) - E(g(Y_N^\Delta))| \leq K\Delta^\beta$$

Literature on Strong Schemes with Jumps

- Pl (1982), Mikulevicius & Pl (1988)
 $\implies \gamma \in \{0.5, 1, \dots\}$ Taylor schemes and jump-adapted
- Maghsoodi (1996, 1998) \implies strong schemes $\gamma \leq 1.5$
- Jacod & Protter (1998) \implies Euler scheme for semimartingales
- Gardoñ (2004) $\implies \gamma \in \{0.5, 1, \dots\}$ strong schemes
- Higham & Kloeden (2005) \implies implicit Euler scheme
- Bruti-Liberati & Pl (2005) $\implies \gamma \in \{0.5, 1, \dots\}$
explicit, implicit, derivative-free, predictor-corrector

Euler Scheme

- Euler scheme

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n + c(Y_n)\Delta p_n$$

where

$$\Delta W_n \sim \mathcal{N}(0, \Delta) \quad \text{and} \quad \Delta p_n = N_{t_{n+1}} - N_{t_n} \sim Poiss(\lambda \Delta)$$

- $\gamma = 0.5$

Strong Taylor Scheme

Wagner-Platen expansion \implies

$$\begin{aligned} Y_{n+1} = & \ Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n + c(Y_n)\Delta p_n + b(Y_n)b'(Y_n) I_{(1,1)} \\ & + b(Y_n)c'(Y_n) I_{(1,-1)} + \{b(Y_n + c(Y_n)) - b(Y_n)\} I_{(-1,1)} \\ & + \{c(Y_n + c(Y_n)) - c(Y_n)\} I_{(-1,-1)} \end{aligned}$$

with

$$\begin{aligned} I_{(1,1)} &= \frac{1}{2}\{(\Delta W_n)^2 - \Delta\}, & I_{(-1,-1)} &= \frac{1}{2}\{(\Delta p_n)^2 - \Delta p_n\} \\ I_{(1,-1)} &= \sum_{i=N(t_n)+1}^{N(t_{n+1})} W_{\tau_i} - \Delta p_n W_{t_n}, & I_{(-1,1)} &= \Delta p_n \Delta W_n - I_{(1,-1)} \end{aligned}$$

- simulation jump times τ_i : $W_{\tau_i} \implies I_{(1,-1)}$ and $I_{(-1,1)}$
- Computational effort heavily dependent on intensity λ

Derivative-Free Strong Schemes

avoid computation of derivatives



order 1.0 derivative-free strong scheme

Implicit Strong Schemes

wide stability regions



implicit Euler scheme

order 1.0 implicit strong Taylor scheme

Predictor-Corrector Euler Scheme

- corrector

$$Y_{n+1} = Y_n + \left(\theta \bar{a}_\eta(\bar{Y}_{n+1}) + (1 - \theta) \bar{a}_\eta(Y_n) \right) \Delta_n$$

$$+ \left(\eta b(\bar{Y}_{n+1}) + (1 - \eta) b(Y_n) \right) \Delta W_n + \sum_{i=p(t_n)+1}^{p(t_{n+1})} c(\xi_i)$$

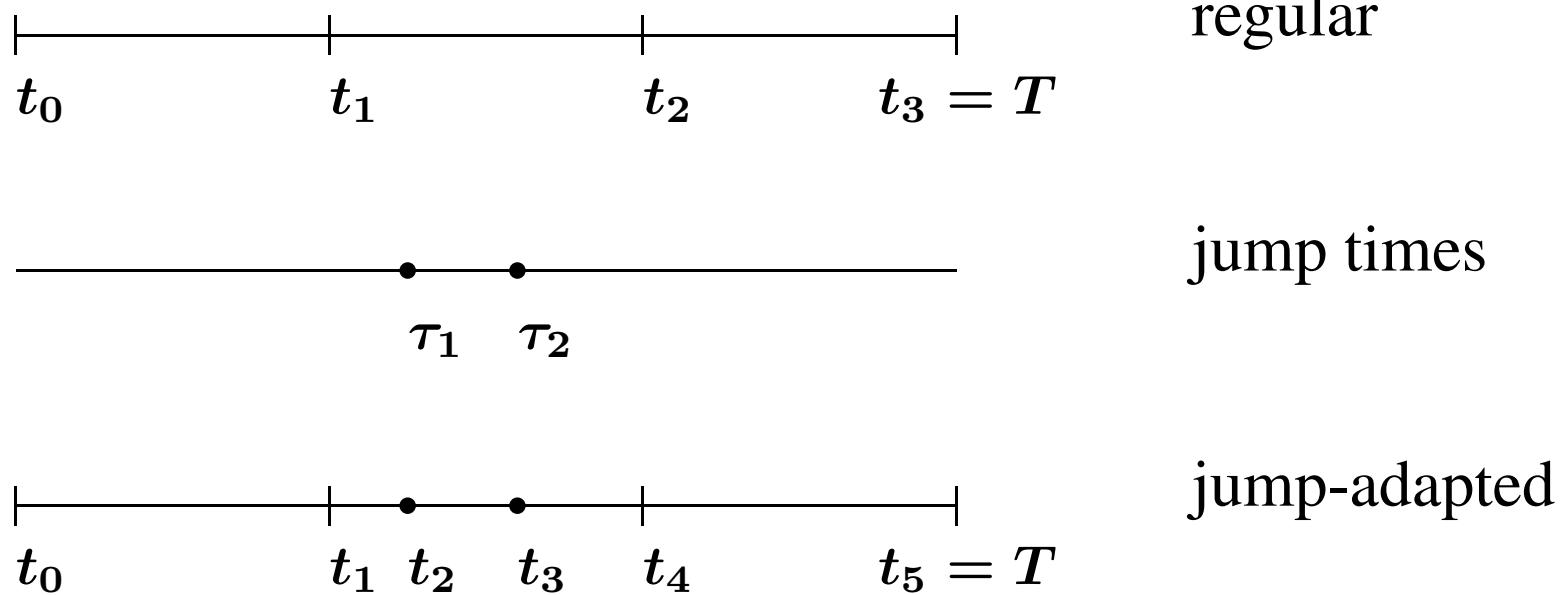
$$\bar{a}_\eta = a - \eta b b'$$

- predictor

$$\bar{Y}_{n+1} = Y_n + a(Y_n) \Delta_n + b(Y_n) \Delta W_n + \sum_{i=p(t_n)+1}^{p(t_{n+1})} c(\xi_i)$$

$\theta, \eta \in [0, 1]$ degree of implicitness

Jump-Adapted Time Discretization



Jump-Adapted Strong Approximations

jump-adapted time discretisation



jump times included in time discretisation

- jump-adapted Euler scheme

$$Y_{t_{n+1}-} = Y_{t_n} + a(Y_{t_n})\Delta_{t_n} + b(Y_{t_n})\Delta W_{t_n}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1}-} + c(Y_{t_{n+1}-}) \Delta p_n$$

- $\gamma = 0.5$

Merton SDE : $\mu = 0.05$, $\sigma = 0.2$, $\psi = -0.2$, $\lambda = 10$, $X_0 = 1$, $T = 1$

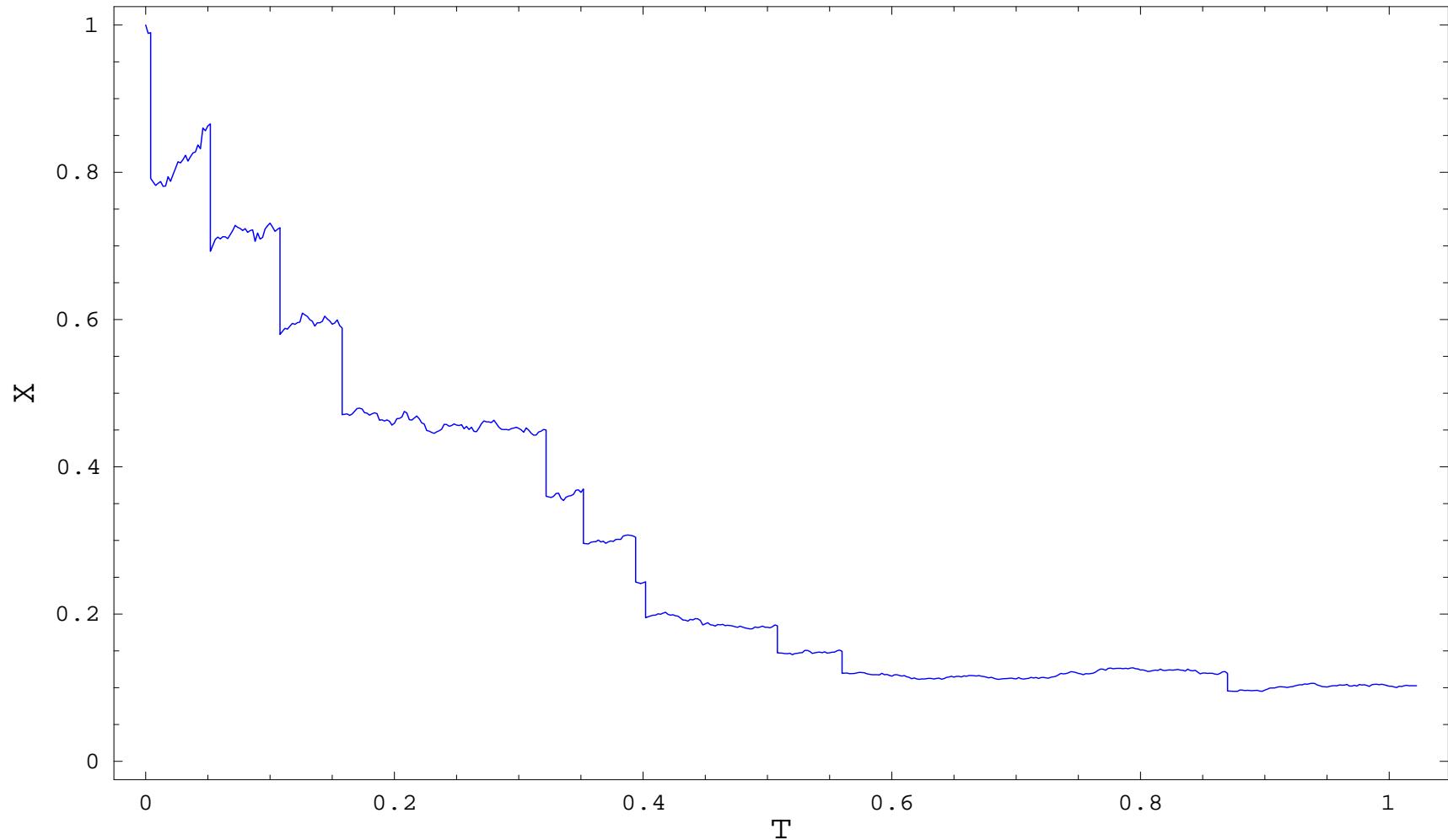


Figure 9: Plot of a jump-diffusion path.

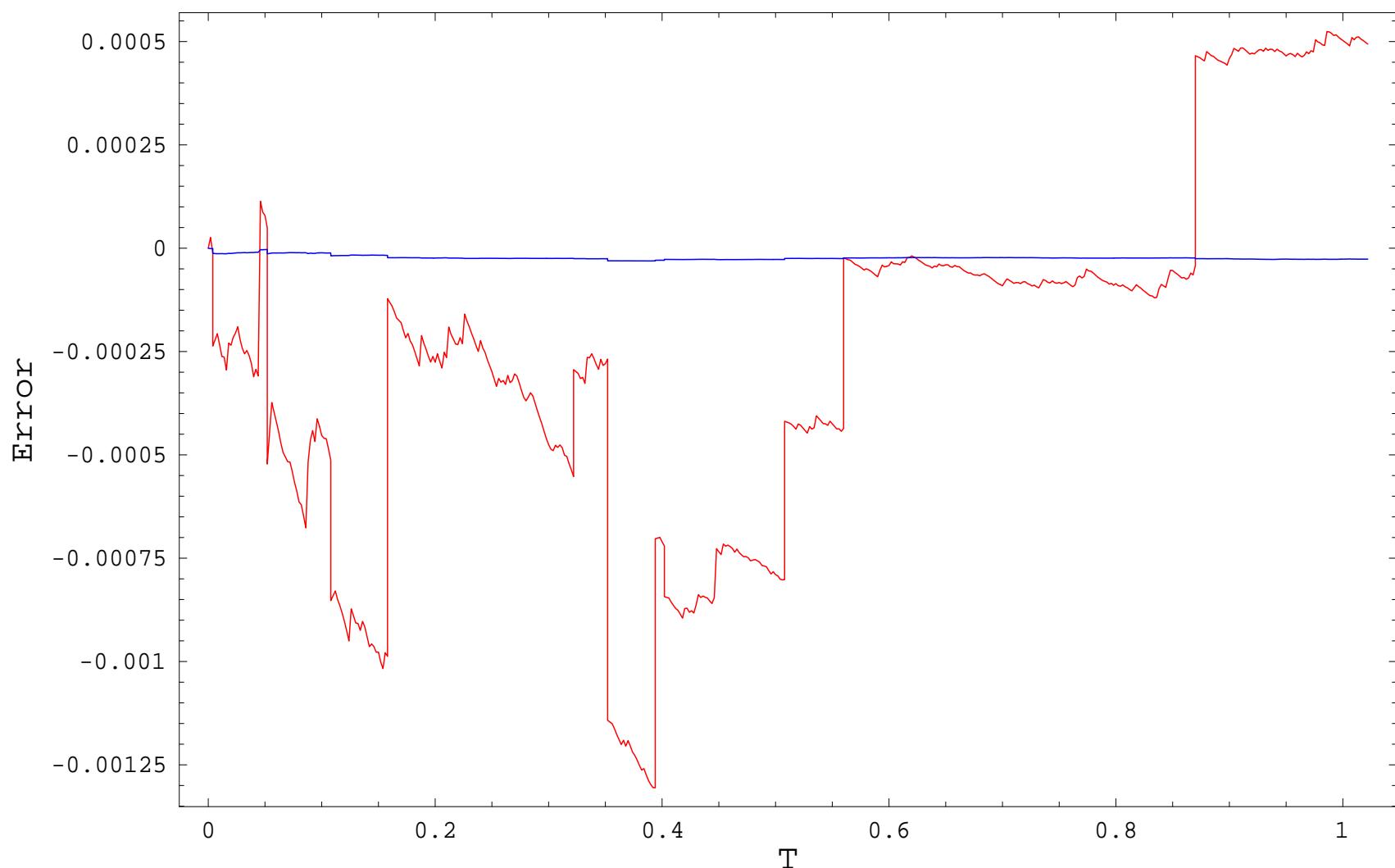


Figure 10: Plot of the strong error for Euler(red) and 1.0 Taylor(blue) scheme.

Merton SDE : $\mu = -0.05$, $\sigma = 0.1$, $\lambda = 1$, $X_0 = 1$, $T = 0.5$

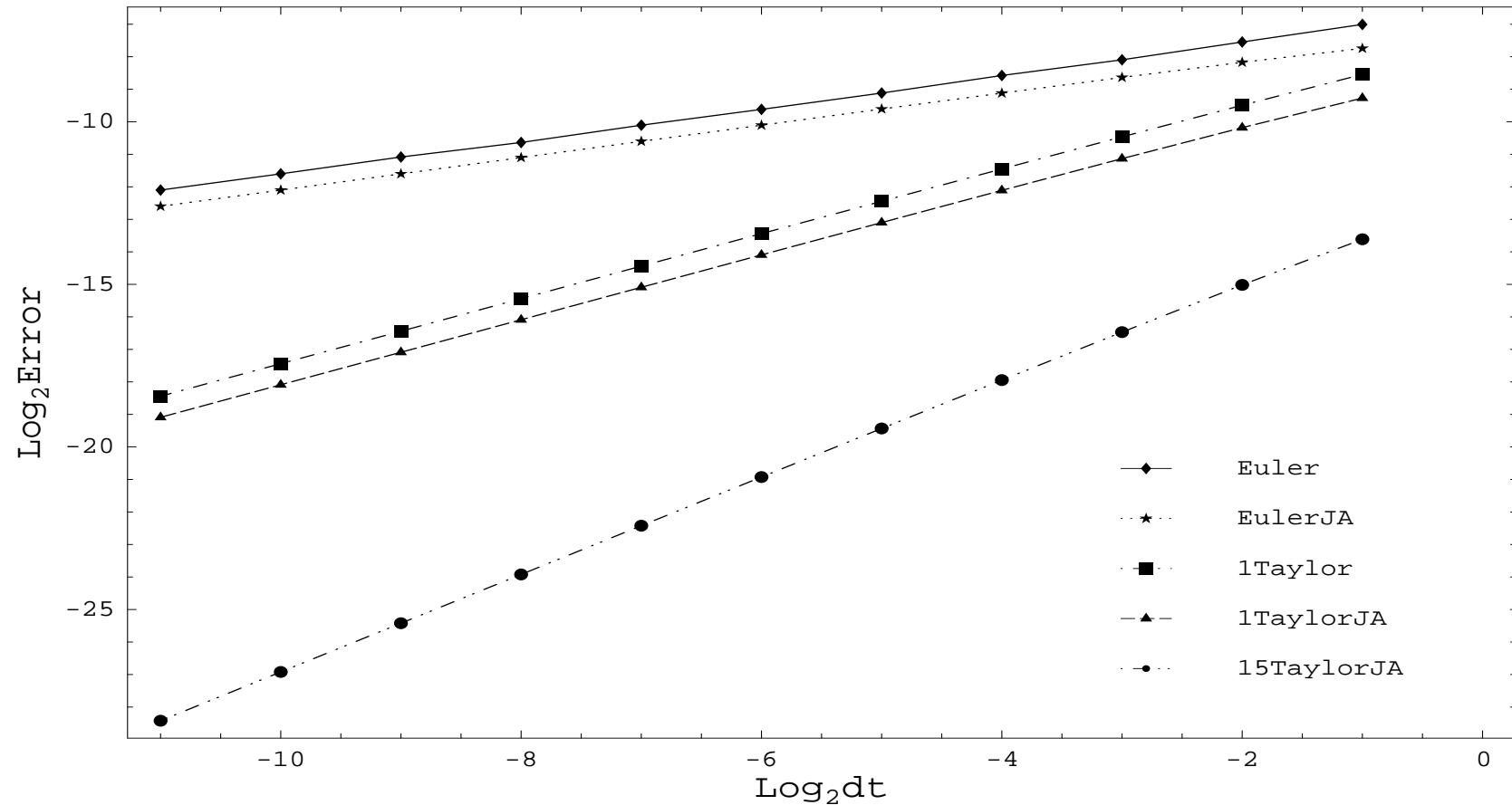


Figure 11: Log-log plot of strong error versus time step size.

Literature on Weak Schemes with Jumps

- Mikulevicius & Pl (1991)
 \implies jump-adapted order $\beta \in \{1, 2 \dots\}$ weak schemes
- Liu & Li (2000) \implies order $\beta \in \{1, 2 \dots\}$ weak Taylor, extrapolation and simplified schemes
- Kubilius & Pl (2002) and Glasserman & Merener (2003)
 \implies jump-adapted Euler with weaker assumptions on coefficients
- Bruti-Liberati & Pl (?) \implies jump-adapted order $\beta \in \{1, 2 \dots\}$
derivative-free, implicit and predictor-corrector schemes

Simplified Euler Scheme

- Euler scheme $\implies \beta = 1$

- simplified Euler scheme

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta \hat{W}_n + c(Y_n)(\hat{\xi}_n - 1)\Delta \hat{p}_n$$

- if $\Delta \hat{W}_n$ and $\Delta \hat{p}_n$ match the first 3 moments of ΔW_n and Δp_n up to an $O(\Delta^2)$ error $\implies \beta = 1$

-

$$P(\Delta \tilde{W}_n = \pm \sqrt{\Delta}) = \frac{1}{2}$$

Jump-Adapted Taylor Approximations

- jump-adapted Euler scheme $\implies \beta = 1$
- jump-adapted order 2 weak Taylor scheme

$$\begin{aligned} Y_{t_{n+1}-} &= Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n} + \frac{b b'}{2} \left((\Delta W_{t_n})^2 - \Delta_{t_n} \right) + a' b \Delta Z_{t_n} \\ &\quad + \frac{1}{2} \left(a a' + \frac{1}{2} a'' b^2 \right) \Delta_{t_n}^2 + \left(a b' + \frac{1}{2} b'' b^2 \right) \{ \Delta W_{t_n} \Delta_{t_n} - \Delta Z_{t_n} \} \end{aligned}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1}-} + c(Y_{t_{n+1}-}) \Delta p_n$$

- $\beta = 2$

Predictor-Corrector Schemes

- predictor-corrector \implies stability and efficiency

- **jump-adapted predictor-corrector Euler scheme**

$$Y_{t_{n+1}-} = Y_{t_n} + \frac{1}{2} \left\{ a(\bar{Y}_{t_{n+1}-}) + a \right\} \Delta_{t_n} + b \Delta W_{t_n}$$

with predictor

$$\bar{Y}_{t_{n+1}-} = Y_{t_n} + a \Delta_{t_n} + b \Delta W_{t_n}$$

- $\beta = 1$

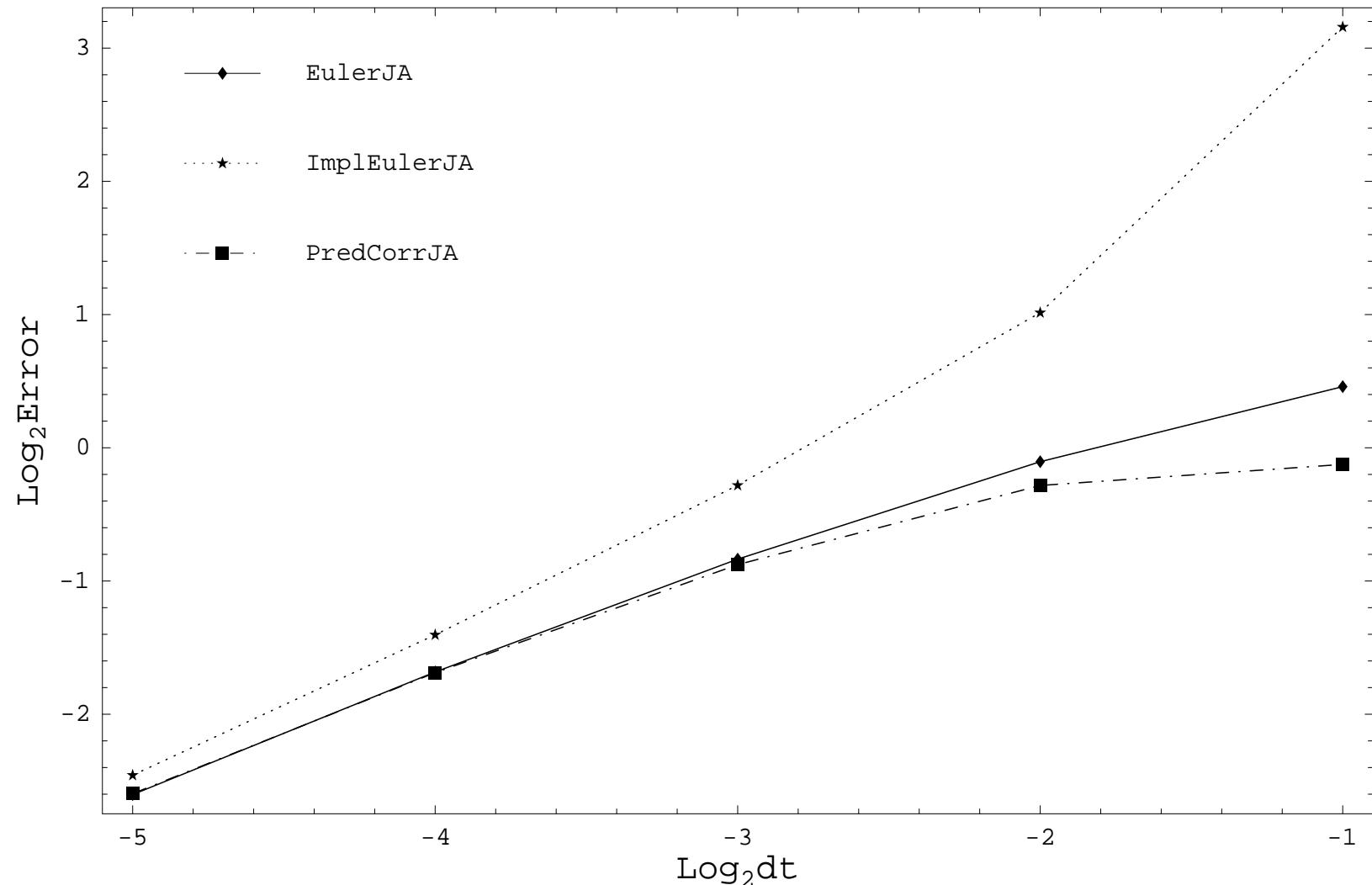


Figure 12: Log-log plot of weak error versus time step size.

Regular Approximations

- higher order schemes : time, Wiener and Poisson multiple integrals
- random jump size difficult to handle
- higher order schemes: computational effort dependent on intensity

Conclusions

- low intensity \implies jump-adapted higher order predictor-corrector
- high intensity \implies regular schemes
- distinction between strong and weak predictor-corrector schemes

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