Indian Journal of Engineering & Materials Sciences
Vol. 11, February 2004, pp. 49-54

Numerical solution of the flow of a second-order fluid under an enclosed rotating disc

H G Sharma & K S Biradar*

Department of Mathematics, Indian Institute of Technology, Roorkee 247 667, India

Received 23 April 2003; accepted 2 December 2003

The solution of a non-linear boundary value problem arising due to the steady flow of an incompressible second-order fluid (flowing with a small mass rate of symmetrical radial outflow *m*, taken negative for a net radial inflow) under finite rotating disc (enclosed within a co-axial cylindrical casing) has been obtained numerically using finite difference method. The resulting equations are converted into a set of difference equations. Starting from the known values of flow functions for small values of the Reynolds number, the solution is extended for larger Reynolds number by making use of Newton-Raphson iterative method and Gauss elimination method. Effects of second order forces in the flow on the velocity field have been investigated in detail in the regions of recirculation and no-recirculation for the cases of radial outflow and inflow and illustrated graphically. Such flows are useful in mechanical and chemical industries.

The problem arising out of the flow of a liquid over an enclosed rotating disc (enclosed in a cylindrical casing) or a shrouded disc has important engineering applications as its generalization could be of help in studies concerning air cooling of turbine discs and pedestal bearing with central feeding of lubricant, windage losses and leakage flow in centrifugal pump or compressor. The problem of the flow over an enclosed rotating disc was first studied by Soo[1] for a Newtonian fluid. Sharma[2] has given an improved formulation for the velocity profile assumed by Soo. Sharma and Gupta[3] and Sharma and Sharma[4] extended the study for elastico-viscous and second-order fluids respectively. Sharma and Gupta[5] and Sharma and Singh[6] have considered the flows of second-order fluid under an enclosed rotating disc without and with heat transfer respectively. In all these investigations approximate methods of solution have been used. Sharma and Biradar[7,8] have studied numerically the flow of a second-order fluid over an enclosed rotating disc without and with uniform suction and injection respectively.

The purpose of the present paper is to investigate numerically, the steady flow of a second-order fluid under a rotating disc enclosed within a co-axial cylindrical casing considered by Sharma and Gupta[5]. The symmetrical radial steady outflow has a small mass rate 'm' of radial outflow ('$-m$' for radial inflow). The inlet condition is taken as a simple radial source flow of strength 'm' along the axis of rotation. The presence of the shroud induces circulation about the axis of rotation starting from radius r_0. The base of the casing could be considered as a stationary disc (called stator) placed parallel to and at a distance equal to gap-length from the rotating disc (called rotator). The equations of motion have been solved by approximate method as well as finite difference technique for small as well as larger values of Reynolds numbers. The second-order effects on the velocity components have been investigated in detail in the regions of recirculation[†] and no-recirculation for the cases of radial outflow and inflow.

The results obtained by approximate and numerical methods have been compared to show that the numerical method provides values nearer to those obtained by approximate methods. Moreover results obtained by numerical methods are valid not only for small Reynolds numbers but for higher Reynolds numbers also.

Theoretical

Mathematics Modelling of the Problem

In a three-dimensional cylindrical set of co-ordinates (r, θ, z), the system consists of a disc of radius r_s (coinciding with plane $z = z_0$), rotating at a constant angular velocity Ω about the axis of rotation $(r = 0)$ and situated at a distance $z = z_0 \, (<< r_s)$ from

E-mail: hgsfdfma@iitr.ernet.in
*Present address : Department of Mathematics, P.D.A. College of Engineering, Gulbarga 585 102, India

[†]Recirculation is the phenomena arising out of a radial outflow near the rotating disc and radial inflow near the stationary disc.

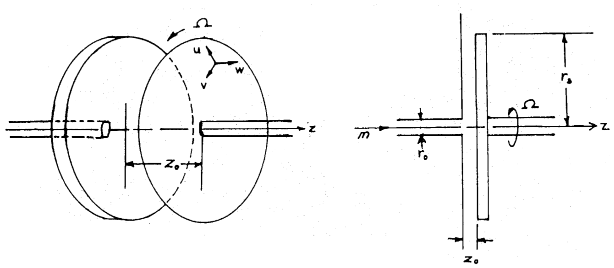


Fig.1—Co-ordinates system of an enclosed rotating disc showing velocity profile at radius r at the condition of no net radial flow

a stationary disc (coinciding with the plane $z=0$) forming part of a co-axial cylindrical casing (Fig. 1). We call the rotating disc as 'rotor' and stationary disc as 'stator'. An incompressible second-order fluid fills the space between the discs. The symmetrical radial steady outflow has a small mass rate m ($m<0$ for radial inflow). The inlet condition is taken as a simple radial source flow along the z-axis starting from radius r_0. The radius of the disc r_s is sufficiently large as compared to the gap-length so that edge effects may be negligible.

Following Sharma and Sharma[4], the governing equations, boundary conditions and velocity field are as follows:

The constitutive equation[9]:

$$T_{ij} = -p\delta_{ij} + 2\mu_1 d_{ij} + 2\mu_2 e_{ij} + 4\mu_3 C_{ij} , \quad \ldots (1)$$

where $d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$,

$$e_{ij} = \frac{1}{2}(a_{i,j} + a_{j,i} + 2u^{m}{}_{,i}\, u_{m,j} .$$

and $C_{ij} = d^{m}{}_{i}\, d_{mj}$.

Momentum equation for no extraneous forces:

$$\rho\left[\frac{\partial u_i}{\partial t} + u^{m} u_{i,m}\right] = t^{m}{}_{i,m} . \quad \ldots (2)$$

Equation of continuity for steady flow:

$$u^{i}{}_{,i} = 0 . \quad \ldots (3)$$

Velocity field for the axi-symmetric flow compatible with continuity criterion and the pressure:

$$u = -r\Omega H'(\zeta) + \frac{mM'(\zeta)}{2\pi r\rho z_0} ,$$

$$v = r\Omega G(\zeta) + \frac{\ell N(\zeta)}{2\pi r\rho z_0} ,$$

$$w = 2\Omega z_0 H(\zeta) ,$$

$$P = P_0(\zeta) + \xi^2 P_1(\zeta) + P_2(\zeta)\log\xi . \quad \ldots (4)$$

Boundary conditions:

$$z = 0 : \; u = 0 , \; v = 0 , \; w = 0 ,$$

$$z = z_0 : \; u = 0, \; v = r\Omega , \; w = 0 , \quad \ldots (5)$$

where T_{ij}, u_i, a_i are the stress tensor, velocity and acceleration vectors, P is the hydrostatic pressure, ρ is the density of the fluid and μ_1, μ_2, μ_3 are the material constants known as coefficients of Newtonian-viscosity, elastico-viscosity and cross-viscosity respectively and comma (,) represents the covariant differentiation. (u, v, w) are the velocity components and $H(\zeta)$, $M(\zeta)$, $G(\zeta)$ and $N(\zeta)$ are non-dimensional flow functions of the dimensionless variable $\zeta (= z/z_0)$ and 'm' the small mass rate of radial outflow, is represented by

$$m = 2\pi\rho\int_0^{z_0} ru.dz, \quad \ldots (6)$$

m being positive for net radial outflow and negative for net radial inflow. ℓ is a constant associated with induced circulatory flow, assumed to be of order m.

The unknown perameters involved in the velocity field are to be determined from the following set of equations:

$$H^{IV} = 2R_z(HH''' + GG') - 2T_1R_z(2H''H''' + 4G'G'' + H'H^{IV} + HH^{V}) - 2T_2R_z(2H''H''' + H'H^{IV} + 3G'G''), \quad \ldots (7)$$

$$M^{IV} = 2R_z(H'M'' + HM''' - G'L - GL') - 2T_1R_z(2H''M''' + 2H'''M'' + 2H'M^{IV} + H^{IV}M' + HM^{V} - 4G''L' - 2G'L'' - 2G'''L) - 2T_2R_z(2H''M''' + 2H'''M'' + H'M^{IV} + H^{IV}M' - 3G''L' - 2G'L'' - G'''L), \quad \ldots (8)$$

$$G'' = 2R_z(HG' - H'G) + 2T_1R_z(H''G' - HG''') + 2T_2R_z(H''G' - H'G''), \quad \ldots (9)$$

$$L'' = 2R_z(M'G + HL') - 2T_1 R_z(2M''G' + 2M'G'' + H'''L + HL''' + H''L' + H'L'') - 2T_2 R_z(2M''G' + M'G'' + H''L' + H'L'' + H'''L + HL'''), \quad \ldots (10)$$

where primes denote differentiation with respect to $\zeta(= z/z_0)$: $T_1(= \upsilon_2/z_0^2)$, $T_2(= \upsilon_3/z_0^2)$ are the dimensionless parameters representing ratio of the second-order and the inertial effects and denoting elastico-viscous and cross-viscous effects and $R_z(= \Omega z_0^2/\upsilon_1)$ is the Reynolds number based on the gap-length and $L(\zeta) = (R_\ell / R_m) N(\zeta)$; $R_\ell(= \ell/\pi\rho z_0 \upsilon_1)$ and $R_m(= m/\pi\rho z_0 \upsilon_1)$ being Reynolds numbers based on induced circulatory flow and radial outflow respectively.

Solution of the Problem

Approximate solution

Approximate solution of the Eqs (7)-(10) can be obtained for small values of Reynolds number R_z and correct to the squares of R_z and first power of (R_m/R_z) as obtained by Sharma and Gupta[5] by expanding the flow functions in powers of R_z. Using expression (4), the dimensionless velocity components $\overline{U}, \overline{V}$ and $\overline{W}$ correct to $O(R_z^2)$ are:

$$\overline{U}\left[= \frac{u}{\Omega z_0}\right] = -\frac{R_z}{60}\xi(5\zeta^4 - 9\zeta^2 + 4\zeta) - \frac{3R_m}{R_z \xi}(6\zeta^2 - \zeta) + \frac{R_m R_z}{8400\xi}\{60\zeta^8 - 180\zeta^7 + 168\zeta^6 - 294\zeta^5 + 420\zeta^4 - 263\zeta^2 + 89\zeta\} - \frac{R_m R_z T}{1050\xi}\{105\zeta^6 - 630\zeta^5 + 980\zeta^4 - 910\zeta^3 + 636\zeta^2 - 181\zeta\} + \frac{4R_m R_z T^2}{5\xi}\{5\zeta^4 - 10\zeta^3 + 6\zeta^2 - \zeta\}, \quad \ldots (11)$$

$$\overline{V}\left(= \frac{v}{\Omega z_0}\right) = \xi\zeta - \frac{R_z^2}{6300}\xi(20\zeta^7 - 63\zeta^5 + 35\zeta^4 + 8\zeta) + \frac{R_z^2}{30}T\xi(\zeta^5 - 3\zeta^3 + 2\zeta^2) - \frac{R_m}{10\xi}(3\zeta^5 - 5\zeta^4 + 2\zeta) + \frac{2R_m T}{\xi}(2\zeta^3 - 3\zeta^2 + \zeta), \quad \ldots (12)$$

and

$$\overline{W}\left[= \frac{w}{\Omega z_0}\right] = \frac{R_z}{30}(\zeta^5 - 3\zeta^3 + 2\zeta^2), \quad \ldots (13)$$

where T (= T_1 + T_2) represents total second-order effects and $\xi = \frac{r}{z_0}$.

The above expressions for velocity components show that for the present case and with the approximation introduced, the cross-viscous and elastico-viscous effects are additive. For $T_1 = T_2 = 0$, we get corresponding expressions for Newtonian fluid.

Dimensionless form of the radii at which there is no-recirculation for the cases of net radial outflow (m>0) and net radial inflow (m<0) respectively are found to satisfy the following conditions:

$$\text{(i) } R_m > 0; \left[\frac{\partial \overline{U}}{\partial \zeta}\right]_{\zeta=0} \geq 0, \left[\frac{\partial \overline{U}}{\partial \zeta}\right]_{\zeta=1} \leq 0, \quad \ldots (14)$$

$$\text{(ii) } R_m < 0 \ (R_m = -R_n); \left[\frac{\partial U}{\partial \zeta}\right]_{\zeta=0} \leq 0, \left[\frac{\partial \overline{U}}{\partial \zeta}\right]_{\zeta=1} \geq 0. \quad \ldots (15)$$

The maximum values $\xi_1\left[= \frac{r_1}{z_0}\right]$ and $\xi_2\left[= \frac{r_2}{z_0}\right]$ of these non-dimensional radii are easily found to be:

$$\xi_1^2 = \frac{R_m}{560 R_z^2}\left[25200 - R_z^2(6720T^2 - 1448T - 89)\right], \quad \ldots (16)$$

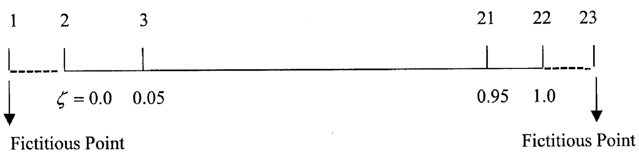
$$\xi_2^2 = \frac{R_m}{840 R_z^2}\left[25200 - R_z^2(6720T^2 + 1912T + 1)\right]. \quad \ldots (17)$$

For the case $R_m > 0$ there is recirculation except in the region $0 \leq \xi \leq \xi_1$ and for $R_m < 0$ there will be no-recirculation in the region $0 \leq \xi \leq \xi_2$.

Numerical solution

Eqs (7)-(10) are non-linear in H, M, G and L, but Eqs (7) and (9) are independent of M and L.

We replace the derivatives therein by finite difference approximations. All H_j's and G_j's in the finite difference equations are functions of the variable ζ. We divide the range of $\zeta(0,1)$ in twenty parts each of length $h = 0.05$ with mesh points 2 to 22. Points 1 and 23 are fictitious points.



The boundary conditions can be rewritten as:

$$
\begin{aligned}
&H_2 = H_{22} = 0, \\
&H_{23} = H_{21}, H_1 = H_3, \\
&M_2 = 0, M_{22} = 1, \\
&M_1 = M_3, M_{23} = M_{21}, \\
&L_2 = L_{22} = 0, \\
&G_2 = 0, \ G_{22} = 1
\end{aligned}
\qquad \ldots (18)
$$

The finite difference equations obtained represent 38 non-linear equations. Functions $F_{1,\mathrm{j}}, F_{3,\mathrm{j}}$ are functions of 38 variables, $G_3, G_4 \ldots G_{20}, G_{21}$, $H_3\, H_4 \ldots H_{20}\, H_{21}$.

Using the approximate solution for velocity field obtained, we obtain first approximation at different mesh points in (0, 1). The differences between the exact and the approximate values are denoted by $\Delta H_{\mathrm{j}}, \Delta G_{\mathrm{j}}$ and can be calculated by $H_{\mathrm{j}} = \overline{H}_{\mathrm{j}} + \Delta H_{\mathrm{j}}$, $G = \overline{G}_{\mathrm{j}} + \Delta G_{\mathrm{j}}$. Expanding $F_{1,\mathrm{j}}$ and $F_{3,\mathrm{j}}$ by means of Taylor's series and neglecting the second and higher powers of the small quantities, Newton-Raphson's iterative method is applied to solve this system, under the boundary conditions (18). A better approximation to *H* and *G* is thus found. The procedure is repeated till the desired accuracy is achieved. Substituting the values of G_{j} and H_{j} into corresponding difference Eqs (8) and (10) at different mesh points, Gauss elimination method is used to solve the system of linear equations. The values of M' and L are also calculated and finally $\overline{U}$, $\overline{V}$ and $\overline{W}$ are obtained.

Results and Discussion

The numerical computations have been made for R_z=0.5, 5.0 and 10.0 and R_m=0.02, -0.02. The maximum radii for no recirculation in case of net radial outflow and inflow for R_z = 0.5 for varying T are calculated by making use of the expressions (16) and (17) and for R_z = 5.0 and 10.0 for varying T by using forward difference and backward difference formulae. It is found that these radii decrease with an increase in T and also with an increase in R_z at fixed T at R_z = 0.5. Reverse is the case for R_z = 5.0 and 10.0 with an increase in T.

The values of dimensionless radial and transverse components of velocity for the cases

$R_{\mathrm{m}} > 0$ and $R_{\mathrm{m}} < 0 \ (R_{\mathrm{m}} = -R_{\mathrm{n}})$,

$$U^{(+)}_{\xi_1(\mathrm{T})} = \left[\overline{U}\sqrt{R_z / R_{\mathrm{m}}}\right]_{\xi_1(\mathrm{T})}, \quad U^{(-)}_{\xi_2(\mathrm{T})} = \left[\overline{U}\sqrt{R_z / R_{\mathrm{n}}}\right]_{\xi_2(\mathrm{T})}$$

$$V^{(+)}_{\xi_1(\mathrm{T})} = \left[\overline{V}\sqrt{R_z / R_{\mathrm{m}}}\right]_{\xi_1(\mathrm{T})}, \quad V^{(-)}_{\xi_2(\mathrm{T})} = \left[\overline{V}\sqrt{R_z / R_{\mathrm{n}}}\right]_{\xi_2(\mathrm{T})}$$

for different values of T and for maximum and other values of ξ_1 and ξ_2 have been calculated and shown in Figs 2-5. It is seen that if T increases, the radial component of velocity for maximum radii for no-recirculation decreases near the rotor and increases near the stator both for radial outflow and inflow for R_z = 0.5 and increases near the rotor and decreases near the stator for R_z = 5.0 and 10.0. Figs 2 and 3 representing the behaviour of the radial velocity at maximum radii for T = 0, 2 in the cases $R_{\mathrm{m}} > 0$ and $R_{\mathrm{m}} < 0$, for R_z = 0.5, 5.0 and 10.0, exhibit the associated phenomena of no-recirculation. To discuss the recirculation behaviour the radial velocity

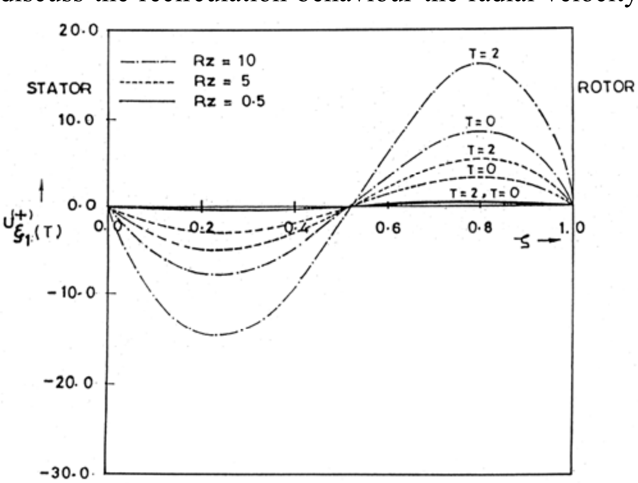


Fig. 2—Variation of radical velocity at maximum radii (for the case R_{m}>0)

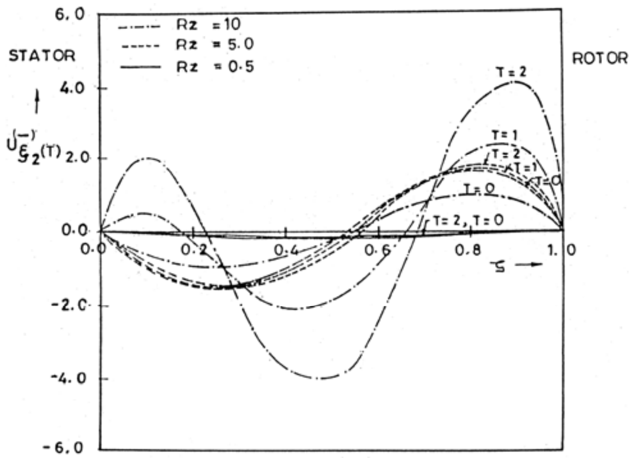


Fig. 3—Variation of radical velocity at maximum radii (for the case R_m<0)

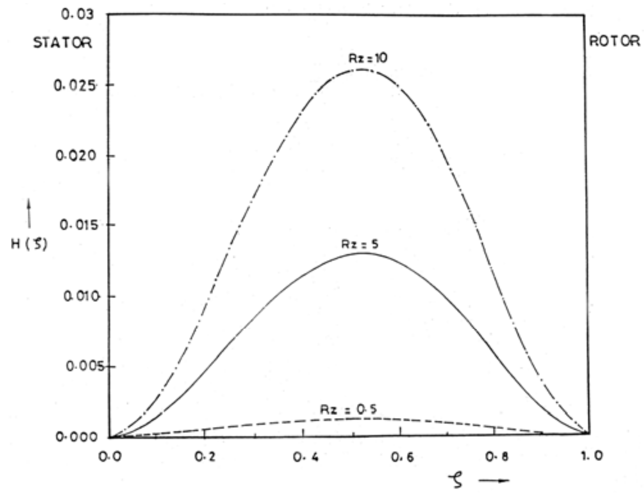


Fig. 6—Variation of H' (ζ) with ζ

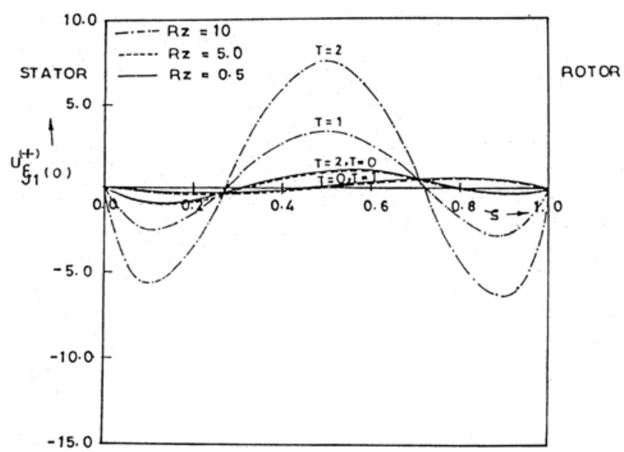


Fig. 4—Variation of radical velocity at fixed radius (for the case R_m>0)

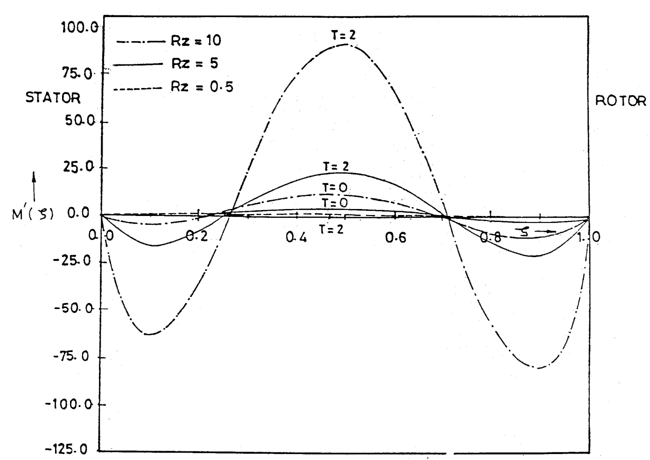


Fig. 7—Variation of M' (ζ) with T

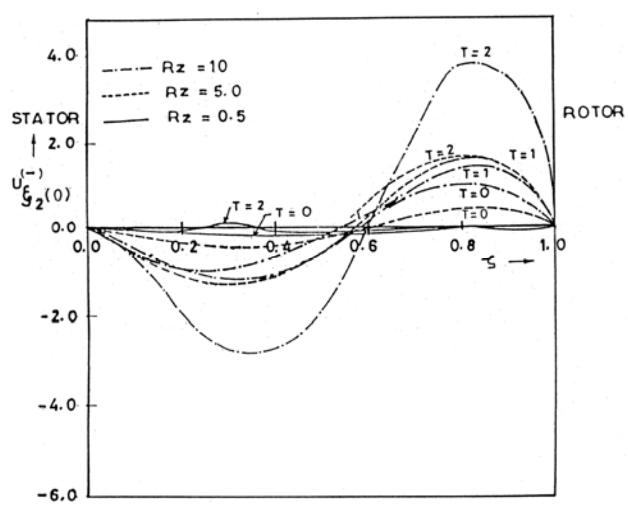


Fig. 5—Variation of radical velocity at fixed radius (for the case R_m<0)

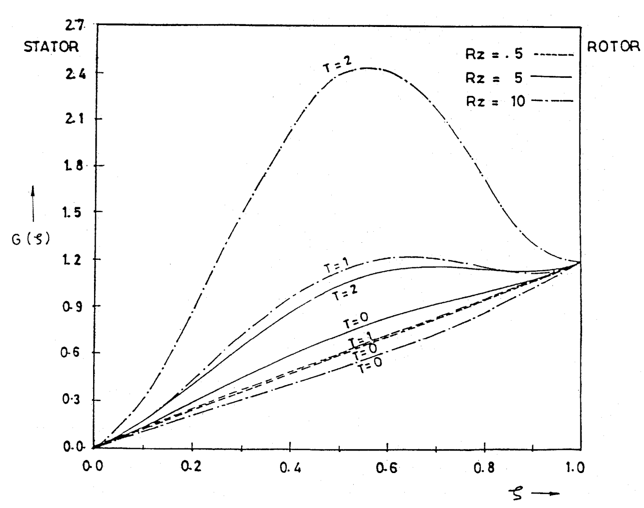


Fig. 8—Variation of G (ζ) with ζ

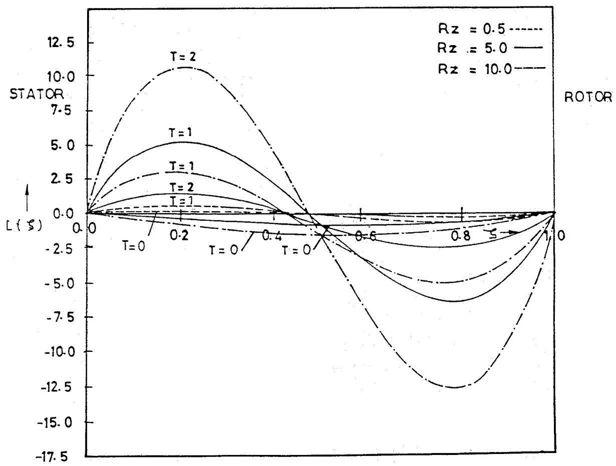


Fig. 9—Variation of L (ζ) with T

$U^{(+)}_{\xi_1(0)}$ and $U^{(-)}_{\xi_2(0)}$ are computed at fixed radii for $R_m > 0$ and $R_m < 0$ and shown through Figs 4 and 5. It is found that there is no-recirculation for $R_z = 0.5$ and recirculation for $R_z = 5.0$ and 10.0 at T = 2 for $R_m > 0$. This shows that a change in the direction of net radial flow changes the behaviour of radial velocity at fixed radii with T. The transverse velocity component for both the cases $R_{\hat{m}} > 0$ and $R_m < 0$ at maximum radii decreases near both the rotor $(\zeta = 1)$ and stator $(\zeta = 0)$ with an increase in T for $R_z = 0.5$, while it increases near both the rotor and stator for $R_{\hat{z}} = 5.0$ and 10.0. Thus it is clear that an increase in T produces more and more recirculation, however, there is no-recirculation for the viscous case. The variation of non-dimensional velocity functions H', M', G and L with Reynolds number R_z are obtained and represented graphically through Figs 6-9. It is seen that the values of H', which is independent of T decreases towards both the rotor and stator. The behaviour of M' for all values of $R_z = 0.5$, 5.0 and 10.0 is to increase with an increase in T near both the rotor and stator. The values of L are observed to increase for $R_z = 0.5$, 5.0 and 10.0 near the stator and decrease near the rotor with an increase in T.

Conclusions

This study concludes that the numerical method provides quite good results not only for small values of the Reynolds number but for larger Reynolds numbers also. The transverse shearing stress, moment, dimensionless moment coefficient on the stationary disc, the radial pressure variation on the stationary disc between the radii ξ and ξ_0, average normal force on the stationary disc up to a radius ξ_s can also be obtained[5].

Acknowledgement

Authors are thankful to the referee for useful comments. One of the authors (KSB) is also thankful to the University Grant Commission, Govt. of India for Quality Improvement Programme Fellowship during the investigation.

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