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Numerical Solution of the Korteweg De Vries Equation by Finite Difference and Adomian Decomposition Method.

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Abstract

The Korteweg de Vries (KDV) equation which is a non-linear PDE plays an important role in studying the propagation of low amplitude water waves in shallow water bodies, the solution to this equation leads to solitary waves or solitons. In this paper, we present the analytic solution and use the explicit and implicit finite difference schemes and the Adomian decomposition method to obtain approximate solutions to the KDV equation. As the behavior of the solitons generated from the KDV depends on the nature of the initial wave, this work aims to study two possible scenarios (hyperbolic tangent initial condition and a sinusoidal initial condition) and obtained solution analytically, numerically with the aforementioned methods. Comparison between the four different solutions is done with the aid of tables and diagrams. We observed that valid analytical solutions for the KDV equation are restricted to time values close to the initial time and that the Adomian decomposition method is a wonderful tool for solving the KDV equation and other non-linear PDEs.

Keywords: *Korteweg de Vries equation, Adomian decomposition method, Solitons, Finite difference, Numerical analysis.*

1 Introduction

The Korteweg-de-Vries equation (KDV) which is a non-linear PDE of third order has been of interest since 150 years ago. The KDV equation is used to study the unusual water waves that occur in shallow, narrow channels such as canals.
In 1844, John Scott Russell while conducting experiments to determine the most efficient design for canal boats observed a phenomenon on the Edinburgh-Glasgow canal. He observed that water in the channel put in motion by a boat drawn by a pair of horses accumulated in a state of

violent agitation and then rolled forward with great velocity assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water which continued its course along the channel without change of form or diminution of speed. Its height gradually diminished after one or two miles. He called this singular and beautiful phenomenon the Wave of Translation (1). Russell deduced empirically that the speed c of the wave is related to the depth h of the water in the canal and to the amplitude A of the wave by

$$c^2 = g(h + A) \tag{1}$$

where g is the acceleration due to gravity. The Korteweg-de-Vries equation (KDV) was originally developed by (2) in order to describe the behavior of one-dimensional shallow water waves with small but finite amplitudes. More recently, this equation also has been found to describe wave phenomena in Plasma physics (3), (4), anharmonic crystals (5), (6), bubble liquid mixture (7), (8) etc. The solutions to the KDV PDE are called solitons or solitary waves.

2 Theoretical Background.

The dynamics of solitary waves is modeled by the KDV equation. The KDV is a non-linear, dispersive, non dissipative equation which has soliton solutions. The General Korteweg de Vries equation (GKDV) is of the form

$$\frac{\partial u(x,t)}{\partial t} + \epsilon u(x,t)^p \frac{\partial u(x,t)}{\partial x} + \mu \frac{\partial^3 u(x,t)}{\partial x^3} = 0 \tag{2}$$

Where $p = 1, 2, 3, \ldots$ is a positive integer and ϵ, μ are positive parameters. The Korteweg de Vries equation developed by (2) is similar to the GKDV with $p = 1$ and is of the form

$$\frac{\partial u(x,t)}{\partial t} + \epsilon u(x,t) \frac{\partial u(x,t)}{\partial x} + \mu \frac{\partial^3 u(x,t)}{\partial x^3} = 0 \tag{3}$$

$u(x,t)$ describes the elongation of the wave at place x and at time t. KDV is non-linear because of the product shown in the second summand and is of third order because of the third derivative in the third summand. The non-linear term, $\epsilon u \frac{\partial u}{\partial x}$ is similar to the usual wave equation $c \frac{\partial u}{\partial x}$ term. This implies that as long as u does not change too much, the wave propagates with a speed proportional to ϵu. The non-linear term introduces the possibility of shock waves into the solution. The $\mu \frac{\partial^3 u}{\partial x^3}$ term produces dispersive broadening that can exactly compensate the narrowing caused by the non-linear term under proper conditions (9).

KDV has been studied analytically by (10), (11), (12), (13) and (14). KDV has motivated considerable research into numerical solution by several methods. Recently the study of solitons has been the focus of many research groups (15), (16), (17), (18) and (19).

The aim of the research is to discover whether non-linear and dispersive systems can support waves with particle-like properties. Starting with two different form of the initial condition, we determine the propagation of the wave profile over time for a wave of length $l = 52metres$. The initial conditions are

$$Initial\ condition\ 1: U(x, t = 0) = \frac{1}{2}\left[1 - \tanh\left(\frac{x - 25}{5}\right)\right]\ \ with\ \ U(x = 0, t) = 1, U(x_{max}, t) = 0 \qquad (4)$$

$$Initial\ condition\ 2: U(x, t = 0) = \sin\left(\frac{\pi x}{n}\right)\ \ with\ U(x = 0, t) = U(x_{max}, t) = 0 \qquad (5)$$

With the parameters $\epsilon = 0.2$, $\mu = 0.1$, $|u| = 1$ and $c = \epsilon|u| = 0.2$. Also, we choose $\Delta x = 0.4$ and $\Delta t = 0.1$. It should be noted that there are specific values of ϵ and μ that can produce solitary waves.

3 Methodology

3.1 Analytical solution of the KDV equation

Finding analytical solutions to linear PDE is simplified by the principle of linear superposition, which tells us that the sum of two solutions is also a solution. When the description of a physical system is made more realistic by including higher-order effects, there results non-linear PDEs which are more difficult to solve analytically in contrast to trying to solve linear PDEs analytically (9).

Recall that the simplest mathematical wave is a function of the form $u(x,t) = f(x - ct)$ which is a solution of the simple PDE $u_t + cu_x = 0$ where c denotes the speed of the wave. The well known wave equation $u_{tt} - c^2 u_{xx} = 0$ leads to two wave fronts represented by the terms $f(x - ct)$ and $f(x + ct)$. We start here by assuming a trial solution of the form

$$u(x,t) = f(x - ct) = f(\xi) \qquad (6)$$

Equation [3] becomes

$$-c\frac{df}{d\xi} + \epsilon f\frac{df}{d\xi} + \mu\frac{d^3 f}{d\xi^3} = 0 \qquad (7)$$

$$\frac{df}{f\sqrt{\frac{c}{\mu} - \frac{\epsilon f}{3\mu}}} = d\xi$$

According to (20), these solutions can be represented in terms of elliptic integrals as

$$\int_0^f \frac{d\tau}{\tau\sqrt{\frac{c}{\mu} - \frac{\epsilon\tau}{3\mu}}} = \int_{\xi_0}^{\xi} d\eta \qquad (8)$$

The integral on the left hand side of equation [8] can be evaluated by using a transformation

$$\tau = \frac{3c}{\epsilon}\operatorname{sech}^2 w \qquad (9)$$

$$\frac{d\tau}{dw} = -\frac{3c}{\epsilon}\frac{\sinh w}{\cosh^3 w} \tag{10}$$

Using equation [6], [8], [9] and [10] we get [11]

$$f(\xi) = \frac{3c}{\epsilon}\operatorname{sech}^2\left((\xi - \xi_0)\sqrt{c/\mu}\right) \tag{11}$$

By substituting equation [11] into equation [6] we have

$$u(x,t) = \frac{3c}{\epsilon}\operatorname{sech}^2\left[\sqrt{c/\mu}\,(x - ct - \xi_0)\right] \tag{12}$$

The analytical result is implemented by using the initial condition to determine the value of ξ_0. For the first initial condition, we get the value of ξ_0

$$u(x,0) = \frac{3c}{\epsilon}\operatorname{sech}^2\left[\sqrt{c/\mu}\,(x - \xi_0)\right] = \frac{1}{2}\left[1 - \tanh\left(\frac{x-25}{5}\right)\right]$$

$$\xi_0 = x - \sqrt{\mu/c}\,\operatorname{sech}^{-1}\left[\sqrt{\frac{\epsilon}{6c}\left[1 - \tanh\left(\frac{x-25}{5}\right)\right]}\right] \tag{13}$$

For the second initial condition, we get the value of ξ_0 as

$$u(x,0) = \frac{3c}{\epsilon}\operatorname{sech}^2\left[\sqrt{c/\mu}\,(x - \xi_0)\right] = \sin\left(\frac{\pi x}{n}\right)$$

$$\xi_0 = x - \sqrt{\mu/c}\,\operatorname{sech}^{-1}\left[\sqrt{\frac{\epsilon}{3c}\sin\left(\frac{\pi x}{n}\right)}\right] \tag{14}$$

The complete computer program to obtain the analytical results is done with the help of the Computer Algebra System Mathematica 5.0 by Wolfram Research Inc.

NOTE: There is a limitation to getting reliable solutions analytically as the boundary conditions were not used in obtaining the analytical solution in contrast with the separation of variable method used for linear PDEs where the analytical solutions are constrained to both the boundary and initial conditions. The analytical solution in [12] gives reliable values for time close to the initial condition $t = 0$. We therefore get the analytical solution only for the interval $0 \le t \le 1$ to compare with the numerical results to be obtained later.

3.2 Numerical solution of the KDV equation

3.2.1 Explicit scheme (Zabusky and Kruskal scheme)

The KDV equation can be solved numerically using a centered, finite difference scheme (10).

$$x = i\Delta x, \qquad t = j\Delta t$$

In terms of the discrete variables the derivatives in the KDV equation are given by

$$\frac{U_{i,j+1} - U_{i,j-1}}{2\Delta t} + \frac{\epsilon}{6\Delta x}\left[U_{i+1,j} + U_{i,j} + U_{i-1,j}\right]\left[U_{i+1,j} - U_{i-1,j}\right] + \frac{\mu}{2(\Delta x)^3}\left[U_{i+2,j} - 2U_{i+1,j} + 2U_{i-1,j} - U_{i-2,j}\right] = 0$$

$$U_{i,j+1} = U_{i,j-1} - \frac{\epsilon\Delta t}{3\Delta x}\left[U_{i+1,j} + U_{i,j} + U_{i-1,j}\right]\left[U_{i+1,j} - U_{i-1,j}\right] - \frac{\mu\Delta t}{(\Delta x)^3}\left[U_{i+2,j} - 2U_{i+1,j} + 2U_{i-1,j} - U_{i-2,j}\right] = 0 \tag{15}$$

For the initial time step ($j = 0$), we apply a forward difference scheme in the time derivative to avoid $U_{i,-1}$ in the discretized equation.

$$\frac{U_{i,1} - U_{i,0}}{\Delta t} + \frac{\epsilon}{6\Delta x}\left[U_{i+1,0} + U_{i,0} + U_{i-1,0}\right]\left[U_{i+1,0} - U_{i-1,0}\right] + \frac{\mu}{2(\Delta x)^3}\left[U_{i+2,0} - 2U_{i+1,0} + 2U_{i-1,0} - U_{i-2,0}\right] = 0$$

$$U_{i,1} = U_{i,0} - \frac{\epsilon\Delta t}{6\Delta x}\left[U_{i+1,0} + U_{i,0} + U_{i-1,0}\right]\left[U_{i+1,0} - U_{i-1,0}\right] - \frac{\mu\Delta t}{2(\Delta x)^3}\left[U_{i+2,0} - 2U_{i+1,0} + 2U_{i-1,0} - U_{i-2,0}\right] \quad (16)$$

Fig 1 and Fig 2 show 3D graphical representation of the explicit scheme solution of the KDV equation after 2000 time steps. The graphs were produced using MATLAB 7.8.0 from Mathworks, Inc.

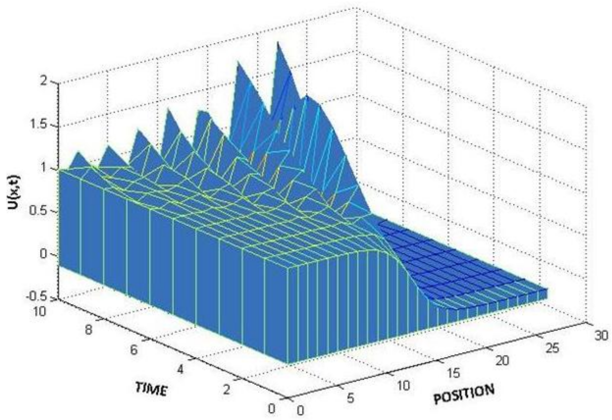

Figure 1**:** Explicit solution of the KDV equation (initial condition 1) after 2000 time steps (200 seconds).

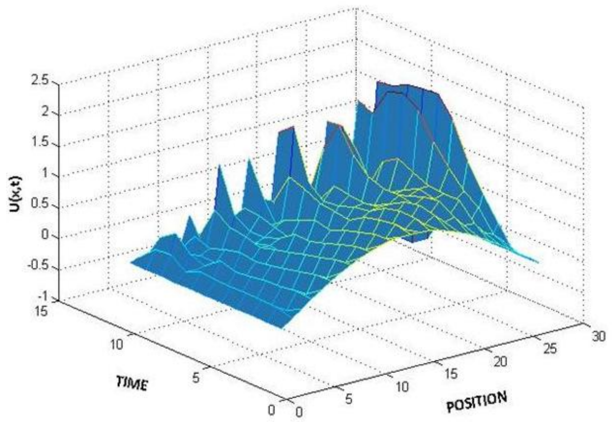

Figure 2**:** Explicit solution of the KDV equation (initial condition 1) after 2000 time steps (200 seconds).

3.2.2 Implicit scheme (Goda scheme)

An implicit scheme for approximating the KDV equation was proposed by (21) and is extended here to the KDV equation for all values of ϵ and μ.

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} + \frac{\epsilon}{6\Delta x}\{U_{i+1,j+1}(U_{i,j} + U_{i+1,j}) - U_{i-1,j+1}(U_{i,j} + U_{i-1,j})\} + \frac{\mu}{2(\Delta x)^3}\{U_{i+2,j+1} - 2U_{i+1,j+1} + 2U_{i-1,j+1} - U_{i-2,j+1}\} = 0 \quad (17)$$

Choosing $P = 3\mu\Delta t$, $Q = 6\mu\Delta t - \beta\epsilon(\Delta x)^2\Delta t$, $R = 6(\Delta x)^3$ and $S = -[6\mu\Delta t - \alpha\epsilon(\Delta x)^2\Delta t]$ we have a pentagonal system of linear equation to solve at each time step using LU decomposition scheme for determining the inverse of a matrix.

$$\begin{aligned}
&For\ i = 1: && RU_{1,j+1} + SU_{2,j+1} + PU_{3,j+1} = RU_{1,j} + PU_0 - QU_0\\
&For\ i = 2: && QU_{1,j+1} + RU_{2,j+1} + SU_{3,j+1} + PU_{4,j+1} = RU_{2,j} + PU_0\\
&For\ 3 < i < l-2: && -PU_{i-2,j+1} + QU_{i-1,j+1} + RU_{i,j+1} + SU_{i+1,j+1} + PU_{i+2,j+1} = RU_{i,j} \quad (18)\\
&For\ i = l-2: && -PU_{l-4,j+1} + QU_{l-3,j+1} + RU_{l-2,j+1} + SU_{l-1,j+1} = RU_{l-2,j} - PU_l\\
&For\ i = l-1: && -PU_{l-3,j+1} + QU_{l-2,j+1} + RU_{l-1,j+1} = RU_{l-1,j} - SU_l - PU_l
\end{aligned}$$

The pentagonal system of linear equation can be written in matrix form as

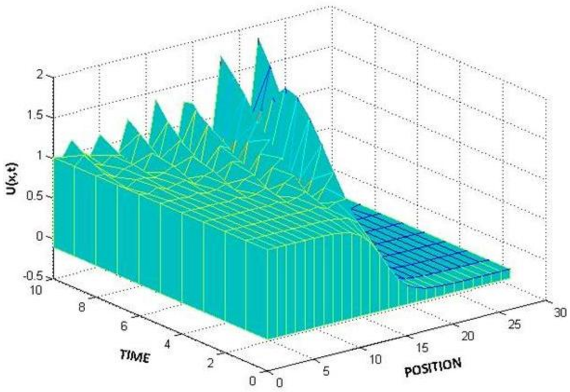

Figure 3**:** Implicit solution of the KDV equation (initial condition 1) after 2000 time steps (200 seconds).

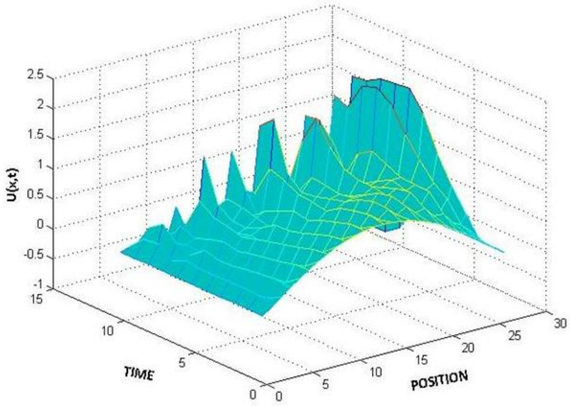

Figure 4: Implicit solution of the KDV equation (initial condition 2) after 2000 time steps (200 seconds).

$$\begin{bmatrix} R & S & P & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ Q & R & S & P & 0 & 0 & 0 & 0 & \cdots & 0 \\ -P & Q & R & S & P & 0 & 0 & 0 & 0 & 0 \\ 0 & -P & Q & R & S & P & 0 & 0 & \cdots & 0 \\ 0 & 0 & -P & Q & R & S & P & 0 & \cdots & 0 \\ 0 & 0 & 0 & -P & Q & R & S & P & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -P & Q & R & S \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -P & Q & R \end{bmatrix} \begin{bmatrix} U_1^{j+1} \\ U_2^{j+1} \\ U_3^{j+1} \\ U_4^{j+1} \\ \vdots \\ \vdots \\ \vdots \\ U_{l-2}^{j+1} \\ U_{l-1}^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} RU_{1,j} + (P-Q)U_0 \\ RU_{2,j} + PU_0 \\ RU_{3,j} \\ RU_{4,j} \\ \vdots \\ \vdots \\ \vdots \\ RU_{l-2,j} - PU_l \\ RU_{l-1,j} - (P+S)U_l \end{bmatrix}$$

The graphs showing the implicit scheme solution of the KDV equation are shown in Fig 3 and Fig 4.

3.3 Adomian Decomposition Scheme

The Adomian Decomposition scheme developed by George Adomian is a semi-numerical method which leads to approximated solutions of non-linear PDEs (22). The Adomian decomposition method has been used by researchers to obtain approximate solution to the KDV

equation for different ϵ and μ values. (23) applied the Adomian decomposition for $\epsilon = 1$ and $\mu = 1$ *and* 6 . (24) applied the method to the KDV equation with $\epsilon = -1$ and $\mu = -1$. In this work, we develop a formula for the KDV equation with any ϵ and μ values.
Recall the KDV equation in [3];

$$\frac{\partial u(x,t)}{\partial t} + \epsilon u(x,t)\frac{\partial u(x,t)}{\partial x} + \mu\frac{\partial^3 u(x,t)}{\partial x^3} = 0$$

We define the operators $L_t = \frac{\partial}{\partial t}$, $L_x = \frac{\partial^3}{\partial x^3}$ *and* $N = u\frac{\partial}{\partial x}$ and so equation [3] can be written in the form

$$L_t u + \epsilon N u + \mu L_x u = 0 \tag{19}$$

From [19] we can write

$$L_t u = -\epsilon N u - \mu L_x u \tag{20}$$

We also define the inverse operator to operator L_t *as* ${L_t}^{-1}$ given by

$$ {L_t}^{-1} = \int (\quad) dt \tag{21}$$

Applying the inverse operator in equation [21] on both sides of equation [20] gives

$$\begin{aligned} {L_t}^{-1} L_t u &= \eta_0(x) + {L_t}^{-1}(-\epsilon N u - \mu L_x u) \\ u &= \eta_0(x) - {L_t}^{-1}(\epsilon N u + \mu L_x u) \end{aligned} \qquad 22)$$

Where η_0 which is a constant of integration is the solution of the equation $\frac{\partial u}{\partial t} = 0$ and is just the initial condition which is purely a function of x.
The Adomian decomposition assumes that the solution $u(x,t)$ to the PDE can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} \lambda^n u_n(x,t) \tag{23}$$

and the decomposed form of the non-linear operator Nu into an infinite series of polynomials is given by

$$Nu = \sum_{n=0}^{\infty} \lambda^n A_n \tag{24}$$

A_n are the Adomian special polynomials (22) and λ is an arbitrary parameter which aids in the grouping of the terms. The parameterized form of equation [22] is written as

$$\begin{aligned} u &= \eta_0(x) - \lambda {L_t}^{-1}(\epsilon N u + \mu L_x u) \\ u_0 + \lambda^1 u_1 + \lambda^2 u_2 + \lambda^3 u_3 + \cdots & \\ &= \eta_0(x) - \lambda^1 {L_t}^{-1}(\epsilon A_0 + \mu L_x u_0) - \lambda^2 {L_t}^{-1}(\epsilon A_1 + \mu L_x u_1) \\ &\quad - \lambda^3 {L_t}^{-1}(\epsilon A_2 + \mu L_x u_2) + \dots\dots \end{aligned}$$

Comparing powers of λ gives

$$u_0 = \eta_0(x) \tag{25a}$$

$$u_1 = -L_t^{-1}(\epsilon A_0 + \mu L_x u_0) = -L_t^{-1}\left(\epsilon A_0 + \mu \frac{\partial^3 u_0}{\partial x^3}\right) \qquad (25b)$$

$$u_2 = -L_t^{-1}(\epsilon A_1 + \mu L_x u_1) = -L_t^{-1}\left(\epsilon A_1 + \mu \frac{\partial^3 u_1}{\partial x^3}\right) \qquad (25c)$$

$$u_3 = -L_t^{-1}(\epsilon A_2 + \mu L_x u_2) = -L_t^{-1}\left(\epsilon A_2 + \mu \frac{\partial^3 u_2}{\partial x^3}\right) \qquad (25d)$$

… ….

$$u_{n+1} = -L_t^{-1}(\epsilon A_n + \mu L_x u_n) = -L_t^{-1}\left(\epsilon A_n + \mu \frac{\partial^3 u_n}{\partial x^3}\right) \qquad (25e)$$

To obtain the Adomian polynomials we use equation [24]

$$Nu = u\frac{\partial u}{\partial x}$$

$$\sum_{n=0}^{\infty} \lambda^n A_n = \sum_{n=0}^{\infty} \lambda^n u_n \sum_{n=0}^{\infty} \lambda^n \frac{\partial u_n}{\partial x}$$

$$\begin{aligned} &A_0 + \lambda^1 A_1 + \lambda^2 A_2 + \lambda^3 A_3 + \dots \\ &= (u_0 + \lambda^1 u_1 + \lambda^2 u_2 + \lambda^3 u_3 + \dots.)\left(\frac{\partial u_0}{\partial x} + \lambda^1 \frac{\partial u_1}{\partial x} + \lambda^2 \frac{\partial u_2}{\partial x} + \lambda^3 \frac{\partial u_3}{\partial x} + \dots.\right) \\ &= u_0\frac{\partial u_0}{\partial x} + \lambda^1\left(u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}\right) + \lambda^2\left(u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x}\right) \\ &\quad + \lambda^3\left(u_0\frac{\partial u_3}{\partial x} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} + u_3\frac{\partial u_0}{\partial x}\right) + \dots.. \end{aligned} \qquad (26)$$

Comparing powers of λ gives

$$A_0 = u_0\frac{\partial u_0}{\partial x} \qquad (27a)$$

$$A_1 = u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} \qquad (27b)$$

$$A_2 = u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x} \qquad (27c)$$

$$A_3 = u_0\frac{\partial u_3}{\partial x} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} + u_3\frac{\partial u_0}{\partial x} \qquad (27d)$$

… … … … … . . … . … … … … … … … … … … … … … … … … ….

$$A_n = \sum_{k=0}^{n} u_k \frac{\partial u_{n-k}}{\partial x} \qquad (27e)$$

If we approximate the values of the solution to the KDV equation using three terms, then we have the solution

$$u = u_0 + u_1 + u_2 + u_3 + \dots.. \qquad (28)$$

The computer program to determine the values of $A_0, A_1, A_2, A_3, u_0, u_1, u_2, u_3$ and the solution is written in Mathematica 5.0 for initial condition 1 and 2 and the solutions are of the following form:

<u>For initial condition 1:</u>

$$\boldsymbol{2\ terms}: \quad u = \frac{1}{2} + \frac{1}{20} t\epsilon \operatorname{sech}\left[5 - \frac{x}{5}\right]^2 - \frac{1}{125} t\mu \operatorname{sech}\left[5 - \frac{x}{5}\right]^4 - \frac{1}{2}\tanh\left[\frac{1}{5}(-25 + x)\right] - \frac{1}{20} t\epsilon \operatorname{sech}\left[5 - \frac{x}{5}\right]^2 \tanh\left[\frac{1}{5}(-25 + x)\right] + \frac{2}{125} t\mu \operatorname{sech}\left[5 - \frac{x}{5}\right]^2 \tanh\left[\frac{1}{5}(-25 + x)\right]^2$$

The three terms solution has 36 terms while the four terms solution has 92 terms.

For initial condition 2:

$$\boldsymbol{2\ terms}: \quad u = \sin\left(\frac{\pi x}{n}\right) + \frac{\pi^3 t\mu \cos\left(\frac{\pi x}{n}\right)}{n^3} - \frac{\pi t\epsilon \cos\left(\frac{\pi x}{n}\right)\sin\left(\frac{\pi x}{n}\right)}{n}$$

$$\boldsymbol{3\ terms}: \quad u = \sin\left(\frac{\pi x}{n}\right) + \frac{\pi^3 t\mu \cos\left(\frac{\pi x}{n}\right)}{n^3} - \frac{\pi t\epsilon \cos\left(\frac{\pi x}{n}\right)\sin\left(\frac{\pi x}{n}\right)}{n} - \frac{5\pi^4 t^2 \epsilon\mu \cos\left(\frac{2\pi x}{n}\right)}{2n^4} + \frac{\pi^2 t^2 \epsilon^2 \sin\left(\frac{\pi x}{n}\right)}{4n^2} - \frac{\pi^6 t^2 \mu^2 \sin\left(\frac{\pi x}{n}\right)}{2n^6} + \frac{3\pi^2 t^2 \epsilon^2 \cos\left(\frac{2\pi x}{n}\right)\sin\left(\frac{\pi x}{n}\right)}{4n^2}$$

The four terms approximate solution for the second initial condition gives 13 terms. Graphical plot of the Adomian decomposition solution for the two initial conditions are presented in Fig 5 and Fig 6.

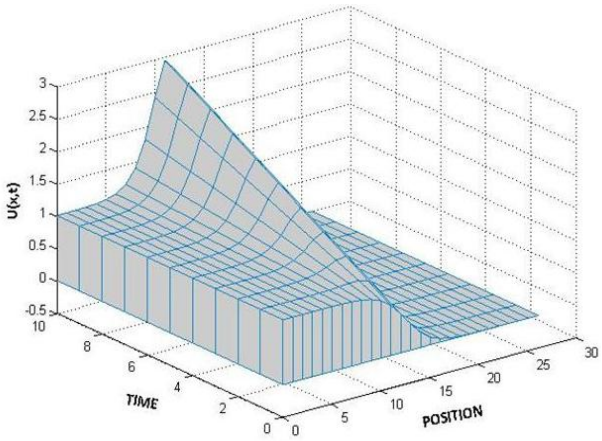

Figure 5: Adomian decomposition solution of the KDV equation (initial condition 1) after 2000 time steps (200 seconds).

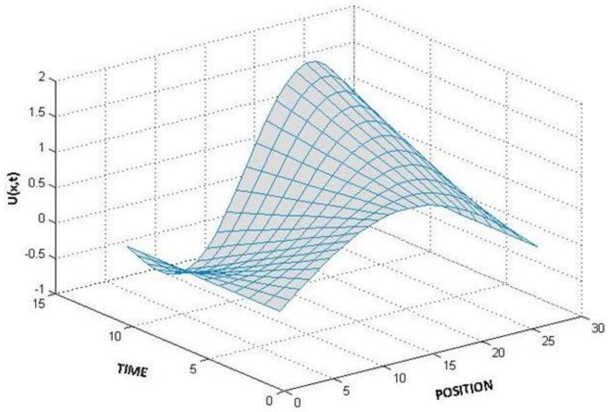

Figure 6: Adomian decomposition solution of the KDV equation (initial condition 2) after 2000 time steps (200 seconds).

4 Discussion of Results.

Table 1 and 2 shows the solution of the KDV equation for $\Delta x = 0.4$ and $\Delta t = 0.1$ using the Analytic method, explicit method, implicit method and the Adomian decomposition method for the time interval $0 \leq t \leq 1$. The choice of time interval is because the analytic solution is liable to the restriction that the time remains close to the initial time $t = 0$ to obtain reliable results. Fig 7 – 10 compares the results obtained from the different methods with time increment. The analytic solution deviates from the other methods as time increases since the analytic solution is oblivious of the boundary conditions. The closeness between the results from numerical explicit and implicit method and the Adomian decomposition method shows that the decomposition method is a powerful tool for solving the KDV equation.

More accurate results are not necessarily obtained from increasing the number of terms used in the Adomian decomposition method. The common approach is to use the Adomian-Malakian convergence acceleration procedure proposed by (25).

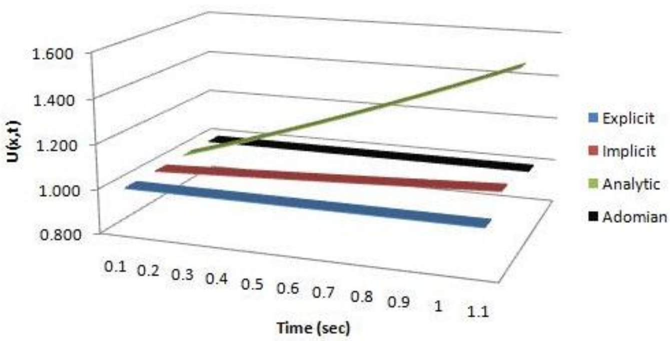

Figure 7: Results for initial condition 1 with x = 14, 0 ≤ t ≤ 1 using the Analytic, Explicit, Implicit and Adomian methods.

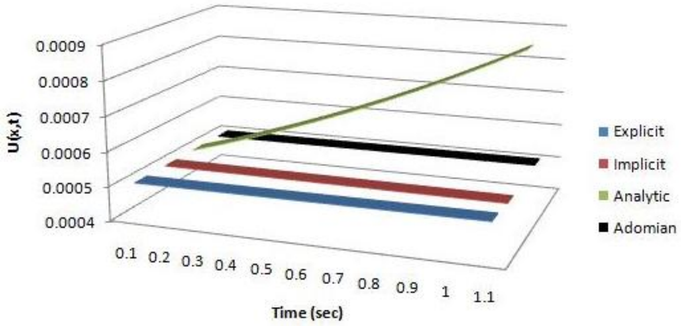

Figure 8: Results for initial condition 1 with x = 44, 0 ≤ t ≤ 1 using the Analytic, Explicit, Implicit and Adomian methods.

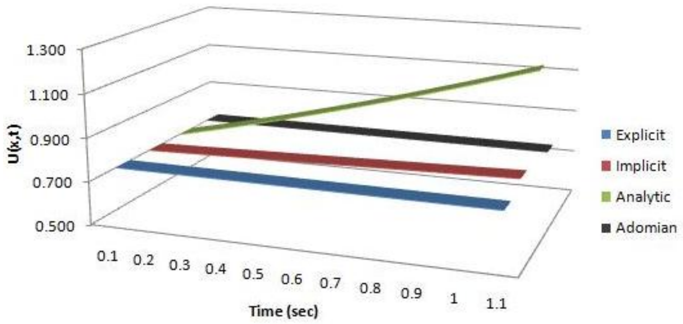

Figure 9: Results for initial condition 2 with x = 14, 0 ≤ t ≤ 1 using the Analytic, Explicit, Implicit and Adomian methods.

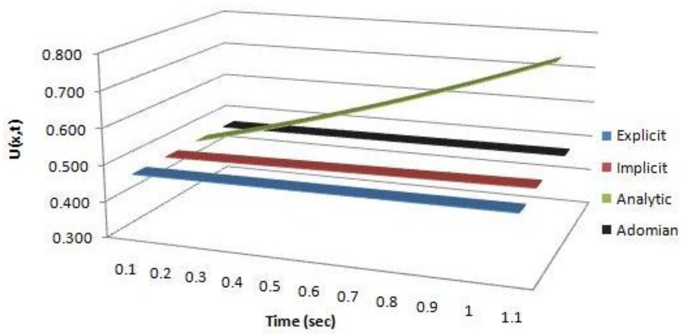

Figure 10: Results for initial condition 2 with x = 44, 0 ≤ t ≤ 1 using the Analytic, Explicit, Implicit and Adomian methods

Table 1: Results from initial condition 1 for $\Delta\boldsymbol{x} = \mathbf{0.4}$ and $\Delta\boldsymbol{t} = \mathbf{0.1}$ after 10 time steps for comparison.

t = 0.0 x	**Analytic**	**Explicit**	**Implicit**	**Adomian**	**t = 0.2 x**	**Analytic**	**Explicit**	**Implicit**	**Adomian**
0	0.999955	0.999955	1.000000	0.999955	**0**	1.095520	1.000000	1.000000	0.999955
14	0.987872	0.987872	0.987872	0.987872	**14**	1.082600	0.988074	1.000961	0.988075
26	0.401312	0.401312	0.401312	0.401312	**26**	0.445677	0.402724	0.403922	0.402719
36	0.012128	0.012128	0.012128	0.012128	**36**	0.013578	0.012145	0.012116	0.012145
44	0.000500	0.000500	0.000500	0.000500	**44**	0.000560	0.000501	0.000499	0.000501
52	0.000020	0.000020	0.000000	0.000020	**52**	0.000023	0.000000	0.000000	0.000020
t = 0.4 x	**Analytic**	**Explicit**	**Implicit**	**Adomian**	**t = 0.6 x**	**Analytic**	**Explicit**	**Implicit**	**Adomian**
0	1.197420	1.000000	1.000000	0.999956	**0**	1.305460	1.000000	1.000000	0.999957
14	1.183680	0.988274	1.014401	0.988279	**14**	1.290920	0.988470	1.028206	0.988482
26	0.494476	0.404147	0.406569	0.404126	**26**	0.548039	0.405581	0.409255	0.405533
36	0.015200	0.012162	0.012104	0.012162	**36**	0.017016	0.012179	0.012092	0.012178
44	0.000627	0.000501	0.000499	0.000501	**44**	0.000702	0.000502	0.000498	0.000502
52	0.000026	0.000000	0.000000	0.000020	**52**	0.000029	0.000000	0.000000	0.000020
t = 0.8 x	**Analytic**	**Explicit**	**Implicit**	**Adomian**	**t = 1.0 x**	**Analytic**	**Explicit**	**Implicit**	**Adomian**
0	1.419280	1.000000	1.000000	0.999958	**0**	1.538360	1.000000	1.000000	0.999959
14	1.404010	0.988664	1.042389	0.988686	**14**	1.522440	0.988854	1.056967	0.988890
26	0.606693	0.407025	0.411981	0.406940	**26**	0.670756	0.408479	0.414748	0.408347
36	0.019048	0.012195	0.012080	0.012195	**36**	0.021321	0.012212	0.012067	0.012211
44	0.000786	0.000503	0.000497	0.000503	**44**	0.000881	0.000503	0.000497	0.000503
52	0.000032	0.000000	0.000000	0.000021	**52**	0.000036	0.000000	0.000000	0.000021

Table 2: Results from initial condition 2 for $\Delta\boldsymbol{x} = \mathbf{0.4}$ and $\Delta\boldsymbol{t} = \mathbf{0.1}$ after 10 time steps for comparison.

t = 0.0 x	**Analytic**	**Explicit**	**Implicit**	**Adomian**	**t = 0.2 x**	**Analytic**	**Explicit**	**Implicit**	**Adomian**
0	0.000000	0.000000	0.000000	0.000000	**0**	0.000000	0.000000	0.000000	0.000004
14	0.748511	0.748511	0.748511	0.748511	**14**	0.824909	0.747315	0.756014	0.747314
26	1.000000	1.000000	1.000000	1.000000	**26**	1.095570	0.999997	1.013363	1.000000
36	0.822984	0.822984	0.822984	0.822984	**36**	0.905428	0.824111	0.832049	0.824111
44	0.464723	0.464723	0.464723	0.464723	**44**	0.515395	0.465715	0.467671	0.465714
52	0.000000	0.000000	0.000000	0.000000	**52**	0.000000	0.000000	0.000000	0.000004
t = 0.4 x	**Analytic**	**Explicit**	**Implicit**	**Adomian**	**t = 0.6 x**	**Analytic**	**Explicit**	**Implicit**	**Adomian**
0	0.000000	0.000000	0.000000	0.000009	**0**	0.000000	0.000000	0.000000	0.000013
14	0.907505	0.746121	0.763671	0.746118	**14**	0.996439	0.744928	0.771485	0.744921
26	1.197470	0.999988	1.027086	1.000000	**26**	1.305510	0.999974	1.041183	1.000000
36	0.994207	0.825237	0.841317	0.825238	**36**	1.089380	0.826364	0.850795	0.826366
44	0.570964	0.466711	0.470659	0.466704	**44**	0.631753	0.467710	0.473687	0.467695
52	0.000000	0.000000	0.000000	0.000009	**52**	0.000000	0.000000	0.000000	-0.000013
t = 0.8 x	**Analytic**	**Explicit**	**Implicit**	**Adomian**	**t = 1.0 x**	**Analytic**	**Explicit**	**Implicit**	**Adomian**

0	0.000000	0.000000	0.000000	0.000018	**0**	0.000000	0.000000	0.000000	0.000022
14	1.091760	0.743737	0.779463	0.743724	**14**	1.193420	0.742548	0.787609	0.742528
26	1.419340	0.999954	1.055671	1.000000	**26**	1.538420	0.999928	1.070565	1.000000
36	1.190890	0.827490	0.860489	0.827493	**36**	1.298550	0.828616	0.870408	0.828620
44	0.698073	0.468713	0.476758	0.468685	**44**	0.770206	0.469720	0.479870	0.469676
52	0.000000	0.000000	0.000000	-0.000018	**52**	0.000000	0.000000	0.000000	-0.000022

5 Conclusion

The applicability of the KDV equation in numerous fields such as fluid dynamics, plasma physics and solid state physics have stimulated interest in methods of solving the equation. The implicit Goda scheme is desirable to the explicit Zabusky & Kruskal scheme accuracy wise but requires more computational time as a system of linear equation is solved at each time step. The Adomian decomposition method should be extended to other non-linear partial differential equation as the results obtained in this study validate the reliability of the method in non-linear equations

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