Transactions on Modelling and Simulation vol 15, © 1997 WIT Press, www.witpress.com, ISSN 1743-355X

Numerical Solution of the Lifting Surface Equation

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Abstract

The piecewise-constant vorticity method of Tuck[1] for solution of the lifting surface integral equation determines accurately integrated quantities such as the lift produced by planar lifting surfaces. Here a modification to this method is presented whereby the leading-edge singularity strength and leading-edge suction force, and hence the induced drag, may also be calculated accurately.

Introduction

Lifting surfaces may be wings on airplanes or birds, windmills, racing car downforce devices, aerodynamic aids such as tails or fins on airplanes or dragsters, frisbees or aerobees, paper planes, kites, control surfaces in air or water, hydrofoils, boomerangs or re-entry space vehicles. In all cases, forward motion induces a pressure difference between the upper and lower sides of a relatively thin surface which is dependent upon the geometry of that surface, and which can be obtained by solving an integral equation over the surface.

In particular, for a lifting surface $z = f(x, y)$ that is close to the plane $z = 0$ in an x-directed stream U, the pressure difference or loading is proportional to a bound vorticity $\gamma(x, y)$ which is determined for small f by solution of the lifting surface integral equation (LSIE)

$$\iint_B \gamma(\xi, \eta) W(x - \xi, y - \eta)\, d\xi\, d\eta = -4\pi U f_x(x, y) \tag{1}$$

over the projection B of the lifting surface onto the plane $z = 0$. The kernel function $W(X, Y) = Y^{-2}(1 + X/R)$, with $R = \sqrt{X^2 + Y^2}$, is the downwash induced by a unit horseshoe vortex (Ashley and Landhal[9], Tuck[1]). Equation (1) can be integrated once with respect to x and the resulting "constant" of integration used to satisfy the Kutta condition at each fixed y, requiring $\gamma = 0$ at the trailing edge of B. No analytic solutions of (1) exist.

A simple way to discretize and solve numerically the LSIE (1) is to assume that the loading $\gamma(x, y)$ is constant on each of a finite number of rectangular panels. A system of linear equations then results from collocation, which is solved directly for the vector of values of γ. This method has been used (Tuck[1], Tuck[2]) to produce very accurate (7-figure) values for the lift coefficient

$$C_L = \frac{2}{UB} \iint_B \gamma(x, y) dx dy.$$

However, close examination of the calculated loading in the vicinity of the leading edge (LE) reveals a highly localised inadequacy in the representation of the inverse square-root LE singularity (Standingford and Tuck[3]). All known numerical techniques for solving the LSIE (1), including the popular VLM or vortex lattice method (Lan[13]), exhibit this type of feature (see Lazauskas, Standingford and Tuck[6] for a survey) and yet the LE singularity strength is of direct aerodynamic significance. One method of fixing this problem for the VLM, presented by Carter and Jackson[15] is to assume a quadratic profile of $\sqrt{x - x_{LE}}\gamma(x, y)$ over the first 3 collocation points from the LE. We first turn to the two-dimensional version of the problem to seek an alternative remedy.

The airfoil equation

The airfoil equation

$$\int_0^1 \frac{\gamma(\xi)}{x - \xi} d\xi = f'(x) \tag{2}$$

is the two-dimensional equivalent of the LSIE (1), for a given function $f'(x)$, and integrates once to give

$$\int_0^1 \gamma(\xi) \log|x - \xi|\, d\xi = f(x). \tag{3}$$

An implicit constant of integration in $f(x)$ ultimately determines the unique solution of (2) satisfying the Kutta condition $\gamma(1) = 0$. For example, if the airfoil is a flat plate with $f'(x) = 1$, this solution has

$$\gamma(x) = \frac{1}{\pi}\sqrt{\frac{1 - x}{x}}.$$

Note the inverse square root LE singularity at $x = 0$, and a zero of square-root type at the trailing edge $x = 1$.

Although an explicit analytic solution can be written down as a quadrature (Tricomi[25], pp 173–180) for any $f'(x)$, the airfoil equation (2) may also be solved numerically to "remarkable accuracy" (James[4]) by the VLM. In that method, the unknown function $\gamma(x)$ is replaced by a finite but large number of Dirac delta functions whose strength is to be determined by collocation. This method thus models the flow by discrete line vortices, rather than by a smooth distribution of vorticity. The location of these vortices and collocation points is crucial to success of the VLM.

At one level higher in "smoothness", to solve the airfoil equation in a manner analogous to the three-dimensional method of Tuck[1], we assume a constant value $\gamma(\xi) = \gamma_j$ on each of n panels, which are Chebyschev spaced, resulting in the discrete set of linear equations

$$\sum_{j=1}^{n} \gamma_j \int_{\xi_{j-1}}^{\xi_j} \log|x_i - \xi| \, d\xi = f(x_i)$$

where the integral equation has been forced to hold at the n collocation points x_i, $i = 1 \ldots n$. The integral itself can be evaluated exactly over each panel, and the resulting algebraic equations

$$\sum_{j=1}^{n} \gamma_j A_{ij} = f(x_i) \tag{4}$$

require inversion of the influence matrix

$$A_{ij} = \Big[(x_i - \xi)(1 - \log|x_i - \xi|) \Big]_{\xi_{j-1}}^{\xi_j} \tag{5}$$

Solution of the set of equations (4) produces an accurate estimate for the overall lift (proportional to $\int \gamma \, dx$), which converges with $O(n^{-2})$ rate. However, inspection of the output values of the function $\sqrt{x}\ \gamma(x)$, which should be smooth near $x = 0$ shows instead a distinct "kink" which does not appreciably diminish in amplitude with an increase in the number n of panels used. This numerical artefact is largely local to the first few values of γ from the leading edge and hence the error it contributes to the predicted lift tends to zero rapidly with n, being proportional to the size of the panels, which for a Chebyshev grid are especially small in that vicinity. However, the effect on local properties near the LE can be significant.

To correct this numerical error, the representation of the strength of the inverse square root singularity in the loading function $\gamma(x)$ near the leading edge $x = 0$ must be improved.

One method that is quite successful but computationally expensive is "subpanelization", in which we subdivide each panel into many smaller subpanels, and then modify the numerical integration of the kernel in the integral equation to account for the variation of the relative loads on each of the subpanels, namely, an inverse square root interpolation to the centre of that subpanel, based on the reference value $\gamma_j = \gamma(\overline{\xi_j})$ at the centre of main panel j.

Rather than using large numbers of subpanels to achieve greater resolution of the LE behaviour, it is possible in two dimensions to specifically

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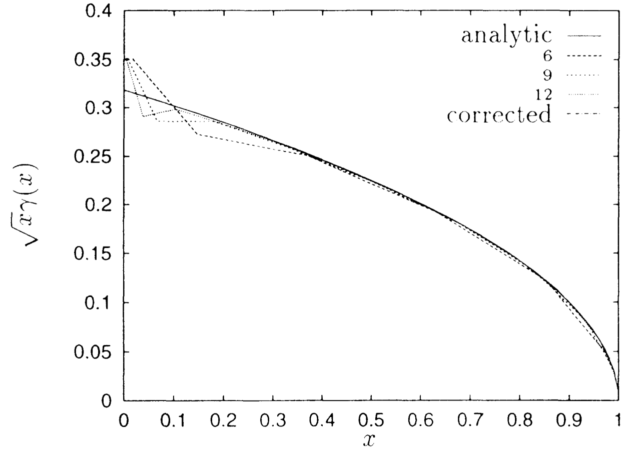

Figure 1: *Two-dimensional airfoil loading with square root singularity removed, with 6, 9 and 12 panels. The kink in the results near the leading edge does not reduce in size with increased numbers of panels. The corrected curve is also shown, and is indistinguishable from the analytic solution.*

include the singularity, by assuming an inverse square root load distribution over all of the n panels, resulting in the influence matrix

$$A_{ij} = \sqrt{\xi_j} \int_{\xi_{j-1}}^{\xi_j} \frac{\log|x_i - \xi|}{\sqrt{\xi}} d\xi \tag{6}$$

The integral in (6) can also be evaluated exactly, although with slightly more difficulty, regardless of the particular grid used. When the new matrix A_{ij} is inverted, the kink in the loading effectively disappears while the rate of convergence to the lift coefficient is maintained (See Figure 1).

For any given grid, we may now calculate the difference between the influence matrix $A_{ij} = A_{ij}^C$ assuming constant loading, as given by (5) and the more accurate influence matrix $A_{ij} = A_{ij}^B$ with the singularity built in, as given by (6). Hence a correction matrix $E_{ij} = A_{ij}^B - A_{ij}^C$ is obtained for any discretization. For a Chebyschev grid (cosine spacing) the correction matrix E_{ij} is a fixed constant (the size of the smallest panel) multiplied by a set of factors whose only parameter is the number of panels n. For example, for $n = 12$ the most significant corrected influence coefficients A_{ij} and their correction factors E_{ij} are:

CORRECTED MATRIX A_{ij}					MATRIX CORRECTION E_{ij}/ξ_1				
i/j	1	2	3	4	i/j	1	2	3	4
1	0.1019	0.1679	0.1831	0.1711	1	-0.0032	-0.0046	-0.0025	-0.0016
2	0.0602	0.2355	0.2178	0.1915	2	0.0014	-0.0013	-0.0037	-0.0019
3	0.0401	0.1409	0.3367	0.2495	3	0.0004	0.0020	-0.0010	-0.0033
4	0.0285	0.0937	0.1951	0.4108	4	0.0002	0.0007	0.0019	-0.0010

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Since the (two-dimensional) airfoil equation has an analytic solution and numerical methods are really only needed for lifting surfaces in three dimensions, the influence matrix correction E_{ij} is more useful when applied to the three-dimensional problem. Integrated once in the x direction, the kernel for the three-dimensional LSIE (1) may be expressed as $W(X,Y) = K_{XY} = Y^{-2}(X+R)$, where

$$K(X,Y) = X\log(Y+R) + \frac{1}{2}Y\log(X+R) - XY^{-1}(X+R)/2 \, .$$

Now the kernel, K_{XY} is to be integrated over a rectangular panel. We observe that the numerical scheme provides adequate accuracy in the spanwise direction Y and turn our attention to the X–integration of K_X. Integrating once with respect to Y, we obtain

$$K_X = \log(Y+R) - Y^{-1}(X+R) + 1$$

All of the terms here are analytic with respect to X except when $Y = 0$ and $X \to 0$. In this case there is a weak singularity in $\log(Y+R)$. If we let $Y = 0$, then this is reduced to the two-dimensional kernel and we might expect that a correction factor equal to that used in the two-dimensional case would be appropriate. We use the above formula for K_X as it stands only when $Y = y - \eta > 0$; if this is not so, the identity $\log(Y+R) = 2\log X - \log(Y-R)$ is used. Now when Y takes the same sign on both sides of the panel, the term $2\log X$ is either not present (both Y values positive) or else cancels out (both Y values negative). On the other hand, when the sign of Y changes from one side of the element to the other (this occurs when the collocation point lies in the same chordwise strip as the panel), the integration over the full panel takes the form

$$\log(Y^+ + R^+) - \log(Y^- + R^-) = \log(Y^+ + R^+) - \left(2\log X - \log\left|Y^- - R^-\right|\right) .$$

There is now a $-2\log X$ term present, so the appropriate three-dimensional correction to the influence matrix A_{ij} is exactly -2 times that for the corresponding two-dimensional kernel. On application of this correction, the LE kink in the three-dimensional results for γ disappears, as it did in two dimensions (see Figure 2).

Comparison of results

Most lifting-surface algorithms are designed to provide accurate results for doubly integrated quantities such as lift and pitching moment. It is somewhat harder to obtain accurate results for the spanwise variation of loading $\Gamma(y) = \int \gamma(x,y)\,dx$ (and consequently the induced drag), and harder again to determine the pointwise variation of loading $\gamma(x,y)$ itself, especially near the leading edge and the wingtips. The following table (reproduced in part from Hauptmann and Miloh[24] with the current results added) compares the lift and moment slope coefficients for a circular wing.

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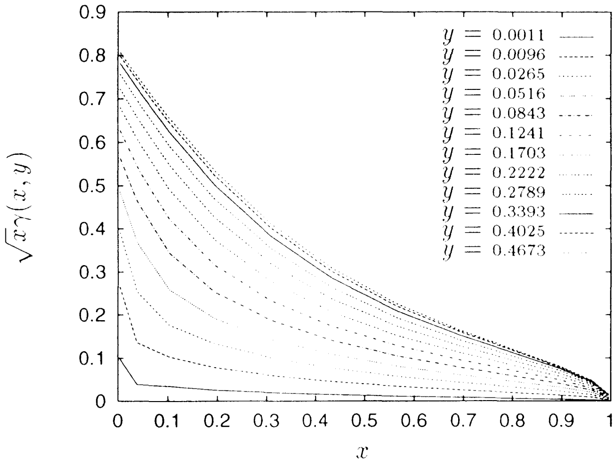

Figure 2: *Effect of correcting the LE kink for a 3D square wing.*

	C_L/α_W	$-C_M/\alpha_W$
Present solution	1.790	0.4661
Hauptman and Miloh[24]	1.790750	0.46882
Jordan[23]	1.790023	0.46617
Prandtl lifting line	2.444	0.611

We also compare the resolution of the leading edge singularity strength with those of Guermond[16] and Jordan[23]. Figure 3 shows the spanwise variation of the leading edge singularity strength. This is by far the hardest quantity to determine correctly by any numerical lifting surface method. Jordan's infinite-series analytic solution predicts a finite value for the LE singularity strength at the wingtip, but with an infinite slope as a function of the spanwise co-ordinate, so that the strength drops very rapidly as we move away from the wingtip. For finite numbers of panels, the present method (and Guermond[16]) suggests incorrectly that the LE singularity strength is zero at the wingtip. However, it then rises rapidly to a maximum close to the wingtip, and as the precision of our computation is increased by taking more panels, this maximum moves closer to the wingtip itself, and the results approach those of Jordan.

Induced drag on planar surfaces

The induced drag of a lifting surface (see Thwaites[7], page 454) may be evaluated as the kinetic energy in the Trefftz plane, far downstream and perpendicular to the free stream direction $+x$. The resulting formula for the induced drag of a general (non-planar) lifting surface is a double integral involving the chordwise-integrated loading function Γ. In general, it is difficult to evaluate this double integral numerically. Authors such as Katz and

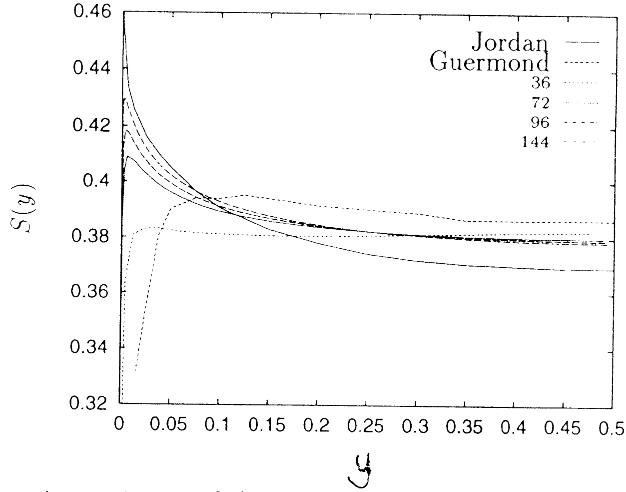

Figure 3: *Spanwise variation of the leading edge singularity strength for a circular planform wing. Results are expressed as the spanwise suction force* $S(y) = \pi/4Q^2(y)$, *where* $Q(y)$ *is the leading edge singularity strength in the loading* $\gamma(x,y)$, *for* $n_x = n_y = 36, 72, 96$ *and* 144.

Plotkin[8] present Riemann-based algorithms assuming that Γ has a discrete span-wise representation, but to date we have found these slow to converge with the number of spanwise panels.

In the case of a single planar lifting surface of span s, integration by parts results in the following integral, given in Ashley and Landahl[9], page 136, equation 7-44.

$$D_i = -\frac{\rho}{4\pi}\int_0^s\int_0^s \frac{d\Gamma}{dy}\frac{d\Gamma}{dy_1}\log|y-y_1|\,dy_1dy.$$

Assume that $\Gamma(y)$ may be accurately represented as a Fourier series $\Gamma(y) = U_\infty s\sum_{n=1}^{N} A_n \sin(n\theta)$, where $y = \frac{s}{2}\sin\theta$. Then the induced drag coefficient is given in Ashley and Landahl[9] as

$$C_{D_i} = \frac{\pi}{4}A\!\!\!R\sum_{n=1}^{N} nA_n^2.$$

In the case of a non-planar wing, an equivalent method has not been found. Whether there are computationally efficient ways to calculate C_{D_i} for non-planar geometries is an interesting question. However, by considering the balance of forces on the body, it should not be necessary to directly evaluate C_{D_i} at all. The force perpendicular to the flat wing provided by the pressure jump between its bottom and top sides must balance the drag and suction forces, such that $C_S = C_L\sin\alpha - C_D$.

Leading edge suction

It may be shown (see for example Milne-Thompson[10], page 125, or Siekmann[14]) that there is a non-zero suction force that acts tangent to a sharp (cusped) point on a profile in two-dimensional flow. This force may be regarded as the product of the infinite pressure required to make the fluid negotiate a 180^o turn, times the zero area of an infinitesimal body element on which it acts. It may be shown that the magnitude of this LE suction force is proportional to the square of the coefficient of the inverse square root LE singularity produced in the pressure field at the cusp.

It has not always been clear (see for example Billington[11]) that this result is directly portable to three-dimensional flow. However, it has been shown (Tuck[12], Lan[13] and others) that if the suction force is to exist for a three-dimensional thin wing, then it must be given by $S = \frac{\pi}{4}\int_0^s Q(y)^2 dy$, where $Q(y)$ is the singularity strength

$$Q(y) = \lim_{x \to x_{LE}} \gamma(x, y)\sqrt{x - x_{LE}}.$$

Hence for a small angle of attack, we expect the LE suction force coefficient to be given by

$$\frac{C_S}{\alpha^2} = \frac{\pi}{4}\int_0^s \left(\frac{Q(y)}{\alpha}\right)^2 dy \tag{7}$$

Evaluating the integral in (7) is made very easy when the integrand is represented as a Fourier series such that

$$\left(\frac{Q(y)}{\alpha}\right)^2 = \sum_n B_n \sin(n\theta),$$

where $y = \frac{s}{2}\cos\theta$. In this case, the LE suction force is given by $C_S = 2S/A = \frac{\pi^2}{4}B_1$.

Results

To verify the present computational method, C_L/α, C_{D_i}/α^2 and C_S/α^2 are calculated independently for an elliptic planform wing of varying aspect ratio. These quantities are plotted in Figure 4. Note that we should find $C_L/\alpha = C_{D_i}/\alpha^2 + C_S/\alpha^2$. While the unextrapolated results are reasonable ($n_y = n_x = 18$ gives at least 3 figure accuracy for planforms with $AR > 1$), there is a noticeable decrease in accuracy as $AR \to 0$, especially for elliptic and delta planforms. Nonetheless, the error $C_L/\alpha - C_{D_i}/\alpha^2 - C_S/\alpha^2$ converges toward zero with rate n_y^{-1}. The problem of planar surfaces of curved planform has been tackled from a different direction by Guermond[16,17], who conformally maps the planform into a rectangular one before discretization.

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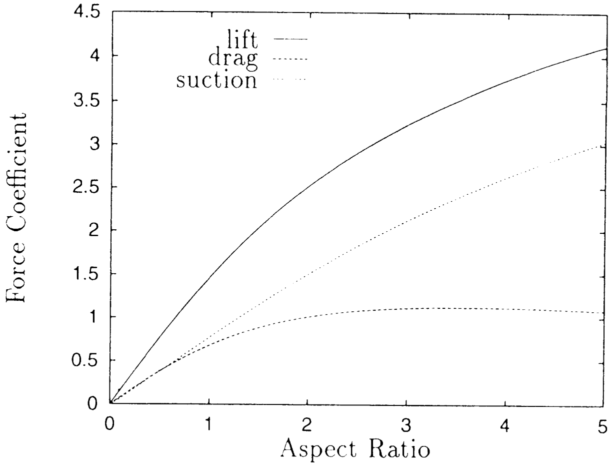

Figure 4: *Forces on elliptic wings*

Discussion

The power of the current technique is not fully realised for planar wings because it is relatively simple to calculate the induced drag C_{D_i} directly from the Trefftz-plane double integral. However, for lifting-surface geometries that are non-planar, or with multiple components such as endplates or biplane wings, or in ground effect, such direct evaluation of C_{D_i} is computationally difficult. By comparison, the evaluation of the LE suction force C_S is essentially geometry-independent, once the pointwise loading γ has been accurately calculated by solution of the non-planar equivalent of the LSIE (1). An immediate consequence is that the induced drag of wings with endplates or in ground effect can be confidently tackled, and we are doing this in further work in progress (see Standingford and Tuck[26], Standingford[27]).

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