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## UNIVERSITY OF WISCONSIN-MADISON MATHEMATICS RESEARCH CENTER

NUMERICAL SOLUTION OF VISCOPLASTIC FLOW PROBLEMS BY AUGMENTED LAGRANGIANS

Patrick Le Tallec
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ABSTRACT


#### Abstract

This report describes an application of Augmented Lagrangian techniques to the numerical solution of quasistatic flow problems in incompressible viscoplasticity, focusing on cases where the internal viscoplastic dissipation potential is not a differentiable function of the material deformation rate. The stresses of elastic origin are neglected, and the variational formulation of these problems is approximated via mixed finite elements of order 1. Convergence results are proved or recalled, both for the finite element approximation and for the augmented lagrangian algorithm. A detailed study of the local minimization problems which occur in the augmented lagrangian decomposition of the above problems is also presented, together with several numerical results. These results were obtained using the MODULEF finite element code on a VAX 780 at the Mathematics Research Center and cover successively the case of Norton, of Bingham and of Tresca type materials.


 AMS (MOS) Subject Classifications: $65 \mathrm{~K} 10,65 \mathrm{~N} 30,73 \mathrm{~F} 05,76 \mathrm{~A} 05$Key Words: viscoplasticity, convexity, incompressibility, finite elements, augmented lagrangians.

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## SIGNIFICANCE AND EXPLANATION

Augmented lagrangian methods, introduced around 1970 by M. R. Hestenes and M. J. D. Powell, are now classical numerical tools in scientific computation. They take into account the dual structure that most problens in continuum mechanics do present, involving usually both stresses and displacements (or velocities), to reformulate them as saddle-point problems, which can then be solved numerically by Uzawa type algorithms. These methods have already been used in situations like viscoplasticity by GLOWINSKI and MAROCCO [1975] and are described in detail in FORTIN and GLONINSKI [1982]. Compared to previous publications, this report:
(1) tries to present a clean and updated version of these techniques, (i1) uses a low order, convergent finite element for the approximation of incompreasible velocity fields,
(ii1) and studies in details each local minimization problem which appears during the algorithm.

The main mathematical tool used herein will be convex analysis. The goal of this report is to give a comprehensive presentation of all the theoretical aspects which are behind the application of augmented lagrangian techniques to viscoplasticity (existence theory, approximation, convergence of the algorithm, ...) so that the reader may be able to implement these techniques In any finite element code, to obtain reasonable numerical reaults with a minimal experimentation time, to assess the validity of his numerical results and to judge the efficiency of his numerical technique.

The responsibility for the wording and views expressed in the descriptive sumary lies with MRC, and not with the author of this report.

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## nUMERICAL SOLUTION OF VISCOPLASTIC FLOW PROBLEMS BY AUGMENTED LAGRANGIANS

## Patrick Le Tallec

## 1. INLRODUCTION AND FORMULATION OF THE CONTINUOUS PRORLPMS.

1. Introduction. We consider in this report the problem of computing the quasistatic Llow of incompressible viscoplastic materials subjected to given distributions of external loads. The constitutive law which modelizes the behavior of the considered viscoplastic materials and the configuration of the body are aupposed to be given. The unknown is the velocity field inside the body resulting from the application of the external loads.

The materials which are involved in such probiems include freshly mixed concrete, bitumen, frozen soile, different types of mud, polymers at high temperature or very hot metale. These materiale, when subjected to external loads, fow viscouely in a nonrevarible pattern and develop stressee which are mainiy of viscous origin. Most of these materials flow in an incompreseible or nearly incompressible way.

Herein, to compute the velocity field v, we use a variational formulation of the mechanical problem (Sec. 1), which neglects the stresses of elastic origin, we discretize the pace of kinematically amiasible incompressible velocity fields by mixed finite element: of order 1 (Bec. 2), and finally we solve the resulting discrete problem by augmented lagrangian techniques (Sec. 3). Convergence results are proved both for the Inite element approximation and for the augmented lagranglan algorithm, and the local problems which appear in the augmented lagrangian decomposition are studied in details in sec. 4. several numerical results are presented in Secs. 5 to 7, successively for Norton, Bingham and Tresca type materials. The basic assumption in this work is that the internal Aissipation potential associatad to the considered viscoplastic material is a convex,

## continuous but not necessarily differentiable function of the deformation rate tensor

 inside the body.1.2 The mechanical problem. Depending whether we consider a specific piece of material with very little motion or a specific domain with inconing and outcoming material, the configuration $\Omega$ given in the data of the problem will correspond either to the reference configuration or to the present configuration of the body. In this report, we will suppose that it corresponds to the reference configuration of the body; in other words, we will consider solids in small straing. The other case, associated to viscoplastic fluids flowing viscously, is identical within the replacement of the lagrangian coordinates $x$ by the eulerian coordinates $x$.

Within this convention, the unknown velocity field is determined by the two mechanical equations below (PERZYNA [1966]):
constitutive law (viscoplastic incompressible solid in small strains)

$$
\left\{\begin{array}{l}
(\sigma(x)+p I d) \text { e } \partial D_{1}(x, \dot{B}(v)), \\
\operatorname{Tr}(\dot{E}(\nabla))=0, \dot{E}(v)=1 / 2\left(\nabla \nabla+\nabla \nabla^{T}\right)
\end{array}\right.
$$

virtual work theorem (quasistatic case)

$$
\left\{\begin{array}{l}
\int_{\Omega} \sigma \cdot\left(\nabla_{w}+\nabla_{w}^{T}\right) / 2 d x=\int_{\Omega} f \cdot v d x+\int_{r_{2}} g^{\circ} w d a, \\
\text { for any } w \text { such that } w=0 \text { on } r_{1} .
\end{array}\right.
$$

These equations involve the Cauchy atress tensor field $\sigma(x)$ and hydrostatic pressure field $p(x)$. Here, the notations $\nabla w$ and $\partial D_{1}$ represent the gradient of the vector field and the subgradient of the convex function $D_{1}(x, \cdot)$, respectively. In addition, $r_{1}$ and $r_{2}$ denote the parts of the boundary of $\Omega$ where imposed velocities

1.3 Variational formulation. If we restrict the virtual work theorem to divergencefree test functions and if we aliminate the Cauchy atress tensor $\sigma$ using the constitutive law, then the mechanical equations above correspond, at least formally, to the variational problem:
(1.3) Minimize the dissipated energy rate $J(w)$ over the set $X$ of incompressible

$$
\text { where } J \text { and } \dot{x} \text { are respectively defined by }
$$

$$
(1.4) \quad J(w)=\int_{\Omega} D_{1}\left(\frac{1}{2}\left(\nabla_{w}+\nabla_{w}\right)\right) d x-\int_{\Omega} f \cdot w d x-\int_{r_{2}} g \cdot v d a
$$

(1.5)

$$
x=\left\{w e w^{1, p}(\Omega), \text { div } w=0, w=\dot{u}_{0} \text { on } \Gamma_{1}\right\}
$$

This variational problem is well-posed and we have:

EXISTENCE THEOREM: Let $\Omega$ be open bounded connected in $\mathrm{R}^{\mathrm{N}}(\mathrm{N}=2$ or 3 ) with IIpschitz continuous boundary $r$. We suppose that the interior of $r_{1}$ is not empty, that $:_{0}$ is the trace of a function of $w^{1, p}(\Omega)$, and satisfies

$$
\int_{r} \dot{u}_{0} \cdot v \text { da }=0
$$

whenever $r_{1}=r$. We assume moreover that the external hody forces $f$ and surface tractions $g$ are respectively in $L^{P^{*}}(\Omega)$ and $X^{P^{*}}\left(\Gamma_{2}\right)\left(p p^{*}=P+P^{*}\right)$, and that the convex internal dissipation potential $D_{1}$ satisfies:

$$
\text { (1.6) } \quad c_{1}|D|^{p}<D_{1}(D) \leqslant c_{2}+c_{3}|D|^{p}
$$

```
almost everywhere in }\Omega\mathrm{ for any symmetric, N x N matrix D with zero trace,
```

    \(1<P<+\infty\).
    Then, there exists a velocity field \(v\) hich minimizes the dissipated energy rate
    \(J(v)\) over the set \(K\) of kinematically admisgible velocity fields. This solution is
    unique if $D_{1}$ is strictly convex. Moreover, for each minimizer $v$, there exists a
deviatoric stress tensor field $\sigma_{D}$ in ( $\left.\mathrm{L}^{\mathrm{p}^{*}}(\Omega)\right)^{\mathrm{NxN}}$, a hydrostatic pressure field
$p$ in $L^{p^{*}}(\Omega)$ which satisfy the weak equilibrium equations and constitutive laws:
(1.7) $\quad \sigma_{D}$ e $\partial D_{1}(\dot{(i n}(\nabla)$ a.e. in $\Omega$.
$v=\left\{w e w^{1, p_{( }}(\Omega), w=0 \text { on } r_{1}\right\}^{A}$.

Proof: The proof of this result is very classical in convex analysis. The existence of a solution involves the Weierstrass theorem, its characterization by (1.7) uses duality arguments and the closed range theorem.

First, from (1.6) and from the Rorn's inequality on $V$ (GEYMONAT, SUQUET[1983]),
J satisfies

$$
\begin{aligned}
C_{1}\|v\|_{1, p}^{p}-\left(\|f\|_{p^{*}}+\|g\|_{p^{*}}\right)\|w\|_{1, p}<J(w) & <c_{2}(\text { mes }(\Omega))+c_{3}\|w\|_{1, p}^{p} \\
& +\left(\|f\|_{p^{*}}+\|g\|_{p^{*}}\| \|\| \|_{1, p}\right.
\end{aligned}
$$

for any in $K$. Therefore, $J$ is coercive, convex (strictiy convex if $D_{1}(\cdot)$ is) and continuous on $x$ for the $w^{1, p}(\Omega)$ topology. It is thus weakly lower semicontinuous on K. In addition, $K$, defined as the Kernel of the linear application $w \rightarrow\left\{\right.$ div $\left.w, v-\left.\dot{u}_{0}\right|_{r_{1}}\right\}$, is convex and closed in $w^{1, p}(\Omega)$. Since $K$ is also not empty, applying the Weierstrass theorem, there exists a minimizer $v$ of $J(\cdot)$ over K, which is unique if $J$ is strictly convex.

To further characterize such a minimizer $v$, we now introduce

$$
x=\left\{v e w^{1, p}(\Omega), v=0 \text { on } r_{1}, \text { div } v=0\right\}
$$

$$
Y=\left\{D e\left(L^{P}(\Omega)\right)^{N x N}, D^{T}=D, T r(D)=0 \text { a.e. in } \Omega\right\}
$$

$$
\Phi(w, D)=\int_{\Omega} D_{1}(D(\nabla+\nabla)-D) d x-\int_{\Omega} f \cdot(\nabla+\sigma) d x-\int_{\Gamma_{2}} g \cdot(\nabla+v) d a
$$

Above $D(\nabla+\sigma)$ represents the tensor $D(\nabla+w)=1 / 2\left(\nabla(\nabla+\sigma)+\nabla(\nabla+\sigma)^{T}\right)$ and $Y$ is in duality with the space

$$
Y^{*}=\left\{\tau e\left(\tau^{p^{*}}(\Omega)\right)^{N \times N}, \tau^{T}=\tau, \operatorname{Tr}(\tau)=0\right\}
$$

through the duality pairing

$$
\langle\tau, D\rangle=\int_{\Omega} \tau \cdot D d x=\int_{\Omega} \sum_{i, j=1}^{N} t_{i j} D_{i j} d x
$$

Obviously, from (1.6), $1(\cdot, \cdot)$ takes on finite values and is real, convex and continuous on XxY. Moreover, since $\nabla$ minimizes $J$ over $K$, 0 is a solution of the primal problem: Minimize $\quad(v, 0)$ on $X$. From a hasic theorem of convex analysis (EKELANDreak [1976, $p$ 52-53]), this implies that the dual problem: Maximize $-\$ *(0, \tau)$ over $y$ has a solution $\left(-\sigma_{D}\right)$ which satisfies
(1.8) $\left\{0,-\sigma_{D}\right\}$ e $\partial(0,0)$
that ts $\left\langle-\sigma_{D}, D\right\rangle\langle\Phi(v, D)-\Phi(0,0), v\{v, D\} e x x y$.
Writing ( 1.8 ) succestively for $\{v, D\}=\{v, D(v)\}$ and $\{v, D\}=\{0,-\mathrm{B}\}$, we obtain
(1.9) $L(v)=\int_{\Omega} \sigma_{D} \cdot D(v) d x-\int_{\Omega} f \cdot v d x-\int_{r_{2}} g \cdot v$ da $=0, v$ e ex,
(1.10) $\int_{\Omega} \sigma_{D} \cdot B d x \leqslant \int_{\Omega}\left\{D_{1}(D(v)+E)-D_{1}(D(v))\right\} d x, v \in e x$.

But (1.10) can only hold if (EKELAND-TEMAM [1976 p 21, p 271])
(1.11) $\quad \sigma_{D} e \partial D_{1}(D(v))$, a.e. in $\Omega$.

Now, to obtain (1.7) out of (1.9), (1.11), it is sufficient to observe that the divergence operator is a continuous surfection from $v$ onto $L^{p}(\Omega)$ (or onto $L^{p}(\Omega) / R$ if $\Gamma_{1}=\Gamma$ ). Therefore, from the closed range theorem, its transpose is a continuous homemorphism from $L^{\mathrm{p}^{*}}(\Omega)$ onto the orthogonal of its Rernel in $V^{*}$, that is onto $\mathrm{X}^{*}$. Since, from (1.9) $L(\cdot)$ is an element of $\mathrm{X}^{*}$, there exists then an element $p$ in $L^{p^{*}}(\Omega)$ such that

$$
L(w)=\langle p, \operatorname{div} v\rangle, v \geqslant \text { in } v,
$$

and our proof is complete.

REMARK 1.1: We are not supposing here any differentiability of the internal dissipation potential $D_{1}(\cdot)$. The numerical techniques to be used later will have to he able to handle such a lack of differentiability.

REMARX 1.2: Even though the argument $x$ has been omitted in the potential $D_{1}$ for simplicity, the whole theory developed in this report applies for potentials which are measurable functions of $x$ on $\Omega$.

REMARK 1.3: For Norton and for Bingham materials, the internal dissipation potential is strictly convex and satisfies (1.7) with

$$
\begin{aligned}
& c_{2}=0, \quad c_{1}=c_{3}=\frac{1}{p}(k \sqrt{2})^{p}, \quad \text { (Norton) } \\
& c_{1}=\mu, \quad c_{2}=\sqrt{2} g, c_{3}=(\sqrt{2} g+\mu), \quad p=2 \quad \text { (Bingham). }
\end{aligned}
$$

But (1.7) is still valid, and therefore the above existence theorem still applies for materials associated to non-strictly convex and non-differentiable potentials such as

$$
D_{1}(D)=\frac{1}{p}(k \sqrt{2})^{p} \sup _{i, j}\left(\left|D_{i}-D_{j}\right|\right)^{p}
$$

where $D_{i}$ are the eigenvalues of the deformation rate tensor $D$. This corresponds to Tresca's type viscoplasticity.

## 2. THE DISCRETE PROBLEMS.

2. 1 The discrete spaces. The approximation of the set $K$ of kinematically adnissible velocity fields, which is needed for the numerical solution of the variational problem (1.4), can not be achieved by the basic finite element spaces used in general. For example, the space of divergence-free functions whose restriction to each triangle (tetrahedron if $N=3$ ) of a given regular triangulation of $\Omega$ is a first degree polynomial may only approximate a small part of the space of divergence-free elements of $w^{1, p}(\Omega)$. Therefore, it is a very inappropriate finite dimensional approximation of the set $K$ of kinematically admissible incompressible velocity fields. To obtain a satisfactory approximation of $K$, the set of approximate test functions must be enriched and the incompressibility constraint must be weakened.

As pointed out in BREZZI [1974] and sumuarized in GIRAULT-RAVIART [1979] in their study of the Stokes problem, a good approximation of $x$ is obtained as follows:
(i) we first decompose the domain $\Omega$ into a regular triangulation $T_{h}$ of $N_{h}$ polygons ( $N=2$ ) or polyhedrons $(N=3)$ which satisfy the classical assembly conditions described in CIARLET [1978p 51]:
(1i) we then define the space $v_{h}$ of approximate test functions by
(2.1)

$$
v_{h}=\left\{v_{h} \in c^{0}(\bar{\Omega}), v_{h}=0 \text { on } r_{1},\left.v_{h}\right|_{\Omega_{l}} e{P_{x}}\left(\Omega_{\ell}\right), v \ell=\left\{, N_{h}\right\}\right.
$$

wherf ${\underset{x}{x}}^{\left(\Omega_{\ell}\right)}$ is a given finite dimensional space of continuous interpolating functions defined on $\Omega_{\ell}$,
(iii) in addition, we introduce an appropriate finite element space $p_{h}$, included in $L(\Omega)$ and which gatisfies the so-called BREZZI (or inf-sup) ondition:

where $B$ is independent of the diameter $h$ of the triangulation $T_{h}$, and where $p$ is the exponent which appears in the definition of $K\left(p p^{*} x p+p^{*}\right)$;
(iv) we finally approximate $K$ by:
(2.3) $X_{h}=\left\{w_{h},\left(w_{h}-\dot{q}_{0}\right)\right.$ e $v_{h}, \int_{\Omega} q_{h} d i v v_{h} d x=0, q_{h}$ e $\left.p_{h}\right\}$.

Briefly speaking, this construction of $C_{h}$ amounts to impose the incompressibility constraint in an averaging sense only. In that way, more elements of $V_{h}$ can satisfy this constraint and the set $X_{h}$ is bigger. It can then better approximate $K$.

The choice of the polyhedrons $\Omega_{\ell}$, of the interpolating space $P_{x}\left(\Omega_{\ell}\right)$ and of the space $P_{h}$ of approximate pressures is free, provided that the BREZZI condition (2.2) is satisfied. In this report, we will use triangles (respectively tetrahedrons if $N=3$ ) as polygons $\Omega_{\ell}$, and define $P_{x}\left(\Omega_{\ell}\right)$ and $P_{h}$ by
(2.4)

$$
P_{x}\left(\Omega_{\ell}\right)=\left\{\text { wecc}^{0}\left(\Omega_{\ell}\right),\left.\right|_{\ell} ^{1} \text { e } P_{1}\left(\Omega_{\ell}^{i}\right), \quad i=1,2^{N}\right\}
$$

(2.5) $\quad p_{h}=\{q$ ecci( $\bar{\Omega}), \quad q_{\|_{\ell}}$ e $\left.P_{1}\left(\Omega_{\ell}\right), \quad v \ell=1, N_{h}\right\}$,
where $p_{1}\left(\Omega_{k}\right)$ is the space of first order polynomials defined over $\Omega_{k}$ and where $\left(\Omega_{\ell}^{i}\right)$ are the $2^{N}$ triangles

$x$ position of the degrees of
freedon for the pressures

- position of the degrees of
freedom for velocities

Figure 2.1 Decomposition of a triangle $\Omega_{\ell}$
in four equal subtriangles


Figure 2.2 Triangulation $T_{h}$
(pressures)


Figure 2.3 Triangulation $T_{h}^{2}$
(velocities)
(respectively tetrahedrons) included in $\Omega_{\ell}$ which are obtained by joining together the mideides of every edge of $\Omega_{\ell}$. With that definition of $P_{x}\left(\Omega_{\ell}\right)$, the space $V_{h}$ of approximate test functions is now simply the space of continuous vector functions with zero trace on $r_{1}$ and whose restriction to each triangle (reapectively tetrahedron) of $T_{h}^{2}$ is first degree polynomial, $T_{h}^{2}$ being the triangulation obtained by dividing each triangle (reapectively tetrehedron) of $T_{h}$ into four equal subtriangles (respectively eight subtetrahedrons). As for the space $p_{h}$ of approximate pressures, it becomes the space of continuous scalar functions whose restriction to each triangle (reapecively tetrahedron) of $T_{h}$ is a first degree polynomial (see GLOWINSKI (1984) for more details on those discrete spaces).

The above choice of approximate epaces ( 2.1 ), (2.3), (2.4), (2.5)) is far from being the only posisible one but it eatisfies the BREZZI condition (2.2) and leads to a very convenient approximate augmented lagrangian decomposition of our viacous flow problem (1.3). Moreover, it uses low order finite elements, which is adviseable in nonlinear problems where little regularity is to be expected. Finally, the sete ( $X_{h}$ ) conatructed by (2.1), (2.3), (2.4) and (2.5) form a converging sequence of finite dimensional approximation of $K$ and we have (BERCOVIER-PIRONNEAU [1977]):

$$
\begin{equation*}
v=e x, \lim _{h \rightarrow 0}\left\{\operatorname{Inf}_{w_{h}} \operatorname{IK}_{h} \quad \mid w-w_{h}^{\prime \prime}, p\right\}=0 \tag{2.6}
\end{equation*}
$$

REMARK 2.1: When the maximal diameter $h$ of the triangulation goes to vero, we also have (CIARLET [1978]):



But, since we are inposing the incompressibility constraint in an averaging senee only, Kh is not included in $K$.
2. 2 The discrete problems. The approximate incompressible viscous flow problem is simply obtained by replaing $K$ by $K_{h}$ in (1.3). But, since $K_{h}$ is not included in $X_{\text {, }}$ we first have to extend the internal dissipation potential $D_{1}($.$) . initially defined as a$ convex continuous coercive function on the space of symetric $N x N$ real matrices with zero trace, to convex continuous coercive function $D_{1}^{e}($.$) defined on the whole space of$ symetric $N x$ real matrices. Thiz extension must be convex and satisfy


In other words, the extension $D_{1}^{e}(\cdot)$ of $D_{1}(\cdot)$ coincides with $D_{1}(\cdot)$ on the space of symetric matrices with zero trace, penalizes the silghty compressible velocity fields and extends to $R^{N x N}$ the coercivity and the continuity of the function $D_{1}(\cdot)$. The Introduction of $D_{1}^{e}(\cdot)$ does not affect the solutions of the viscous flow problems (1.3) since $D_{1}^{e}(\cdot)$ and $D_{1}(\cdot)$ coincide for sumetric matrices with rero tracei it only provides a mathematical tool for calculating the dissipation potential for compressible velocity fields and therefore enables us to compute reasonable nearly incompreseible finite dimensional approximations of the solutions of (1.3). The introduction of such extensions $D_{1}^{(\cdot)}$ which satisfy (2.9) is easy. For example, for Norton and Binghan
materials, the expressions of $D_{1}(\cdot)$ given in (1.1) and (1.2) define much extensions. In other casea, one can take

$$
D_{1}^{e}(D)=D_{1}(D-\operatorname{se}(D) x d)+C_{1}|\operatorname{Dr}(D)|^{p}
$$

being careful not to choose too big values for $C_{p}$ in order to avoid locking" phenomena.

Once this extension defined, the discrete variational formulation of our incompresible vincous tlow problems (1.3) is

```
(2.10)
Minimize the dissipated energy rate J( }\mp@subsup{W}{h}{}\mathrm{ ) over the set }\mp@subsup{F}{h}{}\mathrm{ of approximate
kinmarically adnissible velocity fields,
```

where $K_{h}$ is the subset which is defined by (2.1), (2.3), (2.4), (2.5) and where the dissipated energy rate $J(\cdot)$, given by (1.4), ia extended to a function from $f_{h}$ into - by replacing the internal dissipation potential $D_{1}(\cdot)$ by its extencion $D_{1}(\cdot)$ introduced in (2.9).

THPORE 2. 1: Under the aseumptions of Theorem 1.1, for any fixed $h$, the discrete incompreanible viscous flow problem (2.10) has a solution $v_{h}$. Moreover, any solution $\nabla_{h}$ of (2.10) in associated to an approximate strese tensox field $o_{h}$ and to an approximate preseure field $P_{h}$ in $P_{h}$ guch that


In (2.11), the approximate deviatoric stress tensor ( $\sigma_{h}+P_{h}$ Id) belonge to the space
$Y_{h}$ of plecewise constant symetric matrix fields:

$$
\begin{equation*}
y_{h}=\left\{D_{h}: \Omega \rightarrow R^{N \times N}, D_{h}^{T}=D_{h},\left.D_{h}\right|_{\Omega_{l}^{1}} e\left(P_{0}\left(\Omega_{l}^{i}\right)\right)^{N \times N}, \psi 1=1,2^{N}, v \ell=1, N_{h}\right\} \tag{2.12}
\end{equation*}
$$

Proof: this theorem is the discrete equivalent of the existence Theorem 1.1. Up to (1.9), (1.10) its proof is identical, after replacement of $0_{1}(0)$ by $0_{1}(\cdot)$, of $x$ by $K_{h}$, of $Y$ by $y_{h}$ and of $x$ by
(2.13) $\quad X_{h}=\left\{w_{h}\right.$ e $v_{h}, \int_{\Omega} Q_{h}$ div $v_{h}=0$, voln e $\left.p_{h}\right\}$.

Now, since $y_{h}$ is made of plecewise constant matrix fields, (1.10) will alimo give

$$
-\left(\sigma_{D}\right)_{h} \text { e } \partial D_{1}\left(D\left(\nabla_{h}\right)\right) \text { a.e. in } \Omega
$$

To finish the proof of (2.11), we introduce the operator $B$ from $V_{h}$ into $P_{h}$ defined by
(2.14) $\left\langle B w_{h}, q_{h}\right\rangle=\int_{\Omega} q_{h}$ div $q_{h} d x$, q $q_{h}$ e $p_{h}$, *h $h_{h}$ e $v_{h}$.
whose Kernel is $X_{h}$, by definition. From the brezzl inequality (2.2), (see for example GIRAULT-RAVIART[1979, $p$ 41]), this operator is continuous surjection from $V_{h}$ onto $P_{h}$. Using the closed range theorem, its transpose ia mone-tone homeomorphitim from $P_{h}$ onto the orthogonal of $X_{h}$ in $v_{h}^{*}$. But, from (1.9), the element $L_{h}(\cdot)$ of $v_{h}^{*}$ defined by
(2.15) $\quad L_{h}\left(z_{h}\right)=\int_{\Omega}\left(\sigma_{D}\right)_{h} \cdot D\left(z_{h}\right) d x-\int_{r_{\Omega}} g \cdot u_{h} d a-\int_{\Omega} q \cdot u_{h} d x$
belongs to thi orthogonal subapace. Therefore, there exiats anigue pressure field $P_{h}$ in $P_{h}$ such that

which is exactly (2.11).
2.3 Convergence result. In order to check that the discrete problem (2.10) is a good approximation of the continuous viscous flow problem (1.3) when the maximal diameter $h$ of the triangulation $T h$ goes to zero, one must study the behavior of the sequence $\left(\nabla_{h}\right)$ of solutions of $(2,10)$ when $h$ goes to zero. We will prove in this paragraph that ( $\nabla_{h}$ ) converges weakiy towards solutions $v$ of the continuous problem and
that the diselpated energy rate $J\left(\nabla_{h}\right)$ converges towards $J(v)$. Moreover, under additional uniform convexity assumptions, such as those satisfied by Norton or by Bingham materials, there is strong convergence of $\left(\nabla_{h}\right)$ towards in $\boldsymbol{v}^{1, p}(\Omega)$. The next theorem mamarizes these convergence propertien, denoting by $q$ the maximum of $p$ ( $p$ is the exponent introduced in (1.6)) and 2 and by $X_{p}$ the apace

$$
Y_{p}=\left\{D e\left(L^{P}(\Omega)\right)^{N \times N}, D^{T}=D\right\}
$$

THEOREX 2.2: Under the assumptions of Theorem 1.1. the seguence ( $\mathrm{F}_{\mathrm{h}}$ ) of solutions of the discrete problem (2.10) decomposes itself into subsequences, ach of
them converging weakly in $w^{1 / P}(\Omega)$ towards a solution $v$ of the continuous Incompreseible viscous flow problem (1.3), when $h$ goes to zero. The diesipated energy. rate $J\left(\nabla_{h}\right)$ also convergee towards $J(\nabla)$. Moreover, if the extended internal dissipation potential $D_{i}^{i}$ is of the form
(2.16) $\quad D_{1}^{e}(D)=D_{1} \quad(D)=G_{0}(D)+G_{1}(D)$,
with $G_{1}$ convex and bounded below, Go convex, differentiable and satisfying
 then the whole sequence $\left(\sigma_{h}\right)$ converges strongly in $w^{1, P}(\Omega)$ towards the unigue solution $v$ of the continuous problem (1.3).

Proof: The proof is an immediate generalization of the techniques used by GLOWINSKI-LIONS-TREMOLIERES [1981, $p$ 361] in their study of Bingham fluids. It requires three steps. Step $1\left(v_{h}\right)$ is bounded uniforaly in $h$. Let $v$ be alution of the continuous problem (1.3) and let $\varepsilon_{n}$ be the element of $x_{h}$ such that

$$
\begin{equation*}
1 \nabla-z_{h}^{\prime} 1, p=\operatorname{Inf}_{w_{h}} e_{h} 1 \nabla-z_{h}^{\prime \prime} 1, p \tag{2.18}
\end{equation*}
$$

 by extension, $J(\cdot)$ is continuous on $w^{1, P}(\Omega)$, this implies that, for $h$ sufficiently samil, we have
(2.19) $J\left(z_{h}\right)<J(\nabla)+1$.

But since $\sigma_{h}$ is a solution of (2.10), we get

$$
J\left(v_{h}\right) \leqslant J\left(\xi_{h}\right) \leqslant J(v)+1
$$

From the convexity of $J(\cdot)$, this implies
(2.20) $\left.J\left(\nabla_{h}-\dot{u}_{0}\right) / 2\right)<\left(J\left(-\dot{e}_{0}\right)+J(\nabla)+1\right) / 2=C_{6}$,
where, as usual, the notation $C_{i}$ represents strictly positive numbers independent of $:$ and h. From (2.9), $D_{1}^{e}$ is coercive, thus (2.20) implies

$$
\frac{c_{1}}{2^{p}} \int_{\Omega}\left|\left(\nabla\left(\nabla_{h}-\dot{i}_{0}\right)+\nabla\left(\nabla_{h}-\dot{ष}_{0}\right)^{T}\right) / 2\right|^{P} d \varepsilon<c_{6}+(\|f\|+\|g\|) \| v_{h}-\dot{i}_{0}^{1} 1, p
$$

which, from the Korn's inequality and since $p$ is strictly greater than 1 , can only hold if

$$
\nabla_{h} \|_{1, p} \leqslant c_{7}, \quad v h
$$

Step 2 weak convergence of ( $\nabla_{h}$ ). Since the sequence ( $\sigma_{h}$ ) is uniformly bounded in $\mathbf{1}^{1 / P}(\Omega)$, it decomposes itself into subsequences, each of them weakly converging in $\mathbf{w}^{1,} \mathrm{P}(\Omega)$. We still denote by $\left(\nabla_{h}\right)$ such a subsequence and denote by $i t a$ weak limit. Moreover, let $v$ be alution of (1.3) and let ( $g_{h}$ ) be the sequence of elements of $K_{h}$ defined by (2.18). Since $\sigma_{h}$ ginimizes $J$ over $K_{h}$, we have (2.21) $J\left(\nabla_{h}\right) \leqslant J\left(z_{h}\right), v h$.

Going to the limit in (2.21) as $h$ goes to zero, and using the weak lower samicontinuity of $J$ on the left-hand side, the strong continuity of $J$ on the right-hand aide, we obtain
(2.22) $J(\bar{v}) \leqslant \lim \inf J\left(\nabla_{h}\right) \leqslant \lim \sup J\left(\nabla_{h}\right) \leqslant \lim _{h \rightarrow 0} J\left(z_{h}\right)=J(v)$.

On the other hand, let $q$ be any element of $L^{p^{*}}(\Omega)$ and let $\left(\omega_{h}\right)$ be the sequence of elements of $P_{h}$ which strongly approximates $q$ in $L^{P^{*}}(\Omega)$. Since $v_{h}$ belongs to $K_{h}$, we have
(2.23) $\int_{\Omega} q_{h} d i v \bar{v} d x=\int_{\Omega} q_{h} d i v\left(\bar{\nabla}-\nabla_{h}\right) d x \quad$, vh. Going to the immit in (2.23) as $h$ goes to zero and using the atrong convergence of ( $\sigma_{h}$ ) and the weak convergence of $\left(\sigma_{h}\right)$ yielde

$$
\int_{\Omega} q \text { div } \bar{\nabla} d x=0, \quad \text { vą e } i^{p *}(\Omega)
$$

Moreover, from the weak continuity of the trace operator, $\overrightarrow{\mathbf{v}}-\dot{x}_{0}$ hae zero trace on
$r_{i}$. So, finally, $F$ belongs to $K$. But $\quad$ minimizes $J$ over $K$, therefore

$$
J(v) \leqslant J(\bar{\nabla}) \text {. }
$$

Combined with (2.22), this implies that $J(\bar{v})$ is equal to $J(v)$ and that all
inequalities in (2.22) are equalities. Therefore $\mathcal{F}$ it also a solution of (1.3) and the whole sequence $J\left(\nabla_{h}\right)$ converges towards $J(v)$.

Step 3 strong convergence of $\left(\nabla_{h}\right)$. From now on, we suppose that the extended internal diseipation potential $D_{1}$ satisfies (2.16) and (2.17). From (2.17), Go is strictly convex on $\underset{s}{N} \underset{\sim}{N}$, therefore, by addition, $D_{i}^{e}$ is strictiy convex. So is its restriction $D_{1}$ on the space of symetric matrices with zero trace. From Theorem 1.1, the solution $\sigma$ of (1.3) is then unique. Thus the only possible weak cluster point for the sequence $\left(\nabla_{h}\right)$ is and the whole sequence $\left(\nabla_{h}\right)$ of solutions of (2.10) converges weakly towards in $w^{1, p}(\Omega)$.

To prove its strong convergence, we first write the discrete weak equilibrium equations (2.11) and the continuous weak equilibrium equations (1.7) for the test function $v_{h}=v=\Sigma_{h}-\nabla_{h}$ where $z_{h}$ is the element of $k_{h}$ defined in (2.18). This gives

where the notation $D(v)$ represents as usual the symetric component $\left(\nabla w+v_{v}^{T}\right) / 2$ of the matrix $\nabla_{v}$ in $R^{N x N}$. By definition of the subgradient, the third Iine of (2.24) is equivalent to

$$
\text { (2.25) } \quad(\sigma+p I d) \cdot H<D_{1}(\dot{m}+m)-D_{1}(\dot{E}), v \in \operatorname{R}_{z}^{N x N} \text { with } \operatorname{Tr}(m)=0
$$

Since ( $\sigma+\mathrm{P}$ Id) has zero trace and since $D_{1}^{e}$ is an extension of $D_{1}$ which satiafies (2.9), (2.24) yields


or, in other words
(2.26) (a+p Id) e a $D_{1}^{e}$ (e).

Now setting

$$
t=\sigma+p I d-\frac{\partial G_{o}}{\partial D}(\dot{i}), \quad t_{h}=\sigma_{h}+P_{h} I d-\frac{\partial G_{o}}{\partial D}\left(\dot{F}_{h}\right),
$$



But, since $t$ and $t_{h}$ are subdifferentials of $G_{1}$, we have by definition


A suitable combination of (2.27) and (2.28) yields
(2.29)

$$
\begin{aligned}
& \int_{\Omega}\left(\frac{\partial G_{O}}{\partial D}\left(D\left(z_{h}\right)\right)-\frac{\partial G_{0}}{\partial D}\left(\dot{Z}_{h}\right)\right) \cdot D\left(z_{h}-\sigma_{h}\right) d x<\int_{\Omega}\left(\frac{\partial G_{O}}{\partial D}\left(D\left(z_{h}\right)\right)-\frac{\partial G_{O}}{\partial D}(\dot{D})\right) \cdot D\left(z_{h}-T_{h}\right) d x \\
& \quad+\int_{\Omega}\left(p-z_{h}\right) \operatorname{div}\left(z_{h}-T_{h}\right) d x+\int_{\Omega}\left\{t \cdot D\left(v-z_{h}\right)+G_{1}\left(D\left(z_{h}\right)\right)-G_{1}(\dot{E})\right\} \text { dis. }
\end{aligned}
$$

Both $T_{h}$ and $I_{h}$ are elements of $K_{h}$, so we can replace in (2.29)

$$
\int_{\Omega}\left(p-p_{h}\right) \text { div }\left(z_{h}-\nabla_{h}\right) d x \quad \text { by } \quad \int_{\Omega}\left(p-q_{h}\right) \operatorname{div}\left(z_{h}-\nabla_{h}\right) d x,
$$

where $G_{h}$ is the element of $P_{h}$ which approximates $p$ in $L^{p^{*}}(\Omega)$. Once this replacement done, we have from (2.29), (2.17) and the Korn's inequality

Since by construction $L_{h}$ and $g_{h}$ converge strongly respectively towards $\quad$ and $p$ in $W^{1, P}(\Omega)$ and $L^{P^{*}}(\Omega)$, since from step $2\left(v_{h}\right)$ is uniformly bounded in $n^{1, P(\Omega)}$ and since, from (2.9), the integral of $G_{1}$ is continuous on $Y_{p}$, the right-hand side of (2.30) converge towards 0 when $h$ goes to zero. Therefore $1 \sum_{h}-v_{h} 1$, $p$ mst also converge to zero, and from the triangular inequality, the sequence ${ }^{\text {f }}$ strongly converges towards $v$ in $W^{\prime \prime P}(\Omega)$ when $h$ goes to zero.

REMARK 2. 2: Three facts are crucial in our proof of convergence: the existence of an approximate pressure $P_{h}$, the existence of a sequence $\left(z_{h}\right)$ of elements of $K_{h}$ approximating $\nabla$ and the existence of a sequence $\left(\sigma_{h}\right)$ in $p_{h}$ approximating any
element $q$ of $\mathrm{L}^{\mathrm{p}^{*}}(\Omega)$. Although not necessary the BREZZI condition (2. 2) is a basic tool for proving the first two facts.

RGMARX 2.3: Norton and Bingham materials satisfy the uniform convexity assumptions (2.16) and (2.17) (SCHEURER (1977\}, GLOWINSKI-MAROCCO [1975]) and therefore, etrong convergence can be proved in both cases. Moreover, the speed of convergence of ( $\mathrm{v}_{\mathrm{h}}$ )


$$
a b<\frac{a^{q}}{q}+\frac{b^{q^{*}}}{q^{*}}
$$

In addition, since the dissipation potential is continuousiy differentiable for Norton materials, the strong convergence of $\left(\nabla_{h}\right)$ implies in this case the strong convergence of the discrete stresses ( $\sigma_{h}$ ) towards ( $\sigma$ ) in $\left(L^{p *}(\Omega)\right)^{N \times N}$.

## 3 AUGMENTED LAGRANGIANS

3.1 Formulation of the discrete problems as saddle-point problems.

In view of the numerical solution of the approximate viscous flow problem (2.10) by augmented lagrangian techniques, we must first reformulate (2.10) under a slightiy different form.

To do that, observe that, if we replace $\dot{u}_{0}$ by its $H_{0}^{1}(\Omega)$ projection over the space of continuous functions whose restriction to each subelement $\Omega_{\ell}^{i}$ is a first degree polynomial, we can rewrite (2.10) as
(3.1) Minimize $F\left(D\left(x_{h}\right)\right)+G\left(x_{h}\right)$ over $K_{h}$, with
(3.2) $\left.D\left(w_{h}\right)=1 / 4 \nabla_{w_{h}}+\nabla_{w_{h}}^{T}\right)$,
(3.3) $\begin{cases}F: & Y_{h}+R_{1} \\ & F\left(\mathcal{G}_{h}\right)=\int_{\Omega} D_{1}^{e}\left(\mathcal{E}_{h}\right) d x_{1}\end{cases}$

(3.5) $\quad Y_{h}=\left\{D_{h}: \bar{\Omega}+R^{N \times N}, D_{h}^{T}=D_{h},\left.D_{h}\right|_{\Omega_{l}^{1}} e\left(P_{0}\left(\Omega_{\ell}^{1}\right)\right)^{N \times N}, v i=1,2^{N}\right.$, v $\left.=1, N_{h}\right\}$.

If we follow the methodology of FORTIN-GLOWINSKI [1982], we can then replace (3.1) by
its augmented lagrangian formulation
$(3.6)\left\{\begin{array}{l}\text { Find a gaddle-point }\left\{\left\{v_{h}, F_{h}\right\}, \lambda_{h}\right\} \text { of the augmented lagranglan } \\ L_{R}\left(u_{h}, G_{h}, L_{h}\right)=F\left(G_{h}\right)+G\left(w_{h}\right)+\frac{R}{2} \| D\left(v_{h}\right)-G_{h}^{\prime}{ }_{0,2}^{2}-\left\langle\mu_{h}, D\left(x_{h}\right)-G_{h}\right\rangle \\ \text { over the set }\left(X_{h} \times Y_{h}\right) \times Y_{h},\end{array}\right.$
where $R$ is any positive number and where $\langle\cdot, \cdot\rangle$ denotes the ciassical $L^{2}(\Omega)$ scalar product over $\left(L^{2}(\Omega)\right)^{N 6 N}$. Observe that (3.6) imposes the incompressibility condition on the continuous variable $v_{h}$ ( $v_{h}$ must belong to the set $X_{h}$ of approximately Incompressible velocity fields) but minimizes the noniinear functional $F(\cdot)$ with respect to the piecewise constant variable $G_{h}$. In other words, there is a splitting of the difficulties of our problem (nonlinearity and incompressibility) between these two variables.

## THEOREM 3.1: The augmented lagrangian problem (3.6) and the approximate

incompressible viscous flow problem (2.10) are equivalent: to any solution $\boldsymbol{v}_{\mathrm{h}}$ of (2.10), one can associate a solution $\left\{\left\{\nabla_{h}, \eta_{h}\right\}, \lambda_{h}\right\}$ of (3.6) and conversely. Moreover, $H_{h}$ is equal to $D\left(F_{h}\right)$ and there exists an approximate pressurefield $p_{h}$ in $p_{h}$ such that the approximate stress tensor field $\left(-\lambda_{h}-P_{h} I d\right)$ satisfies the discrete equilibrium equations and constitutive laws (2.11).

Proof: Pirst, let $\nabla_{h}$ be a solution of (2.10) and let $\sigma_{h}$ and $p_{h}$ be the associated discrete stress and pressure fields. From (2.11), Theorem 2.1, we have


In particular, taking $v_{h}$ as $\left(z_{h}-v_{h}\right)$ where $j_{h}$ is any element of $K_{h}$, and aince, by construction of $K_{h}$, div $\left(\varepsilon_{h}-\nabla_{h}\right)$ is equal to zero in the dual of $P_{h}$, we have

$$
\int_{\Omega}\left(\sigma_{h}+P_{h} r d\right) \cdot D\left(z_{h}-v_{h}\right) d x=\int_{\Omega} f \cdot\left(x_{h}-v_{h}\right) d x+\int_{\Gamma_{2}} g \cdot\left(z_{h}-\nabla_{h}\right) d a, v z_{h} e r_{h}
$$

Substracting this to the first inequality of (3.7) yielda

$$
L_{R}\left(v_{h}, \dot{i}_{h},-\sigma_{h}-p_{h} I d\right)<L_{R}\left(z_{h}, \dot{i}_{h}+\epsilon_{h},-\sigma_{h}-R_{h} I d\right)-\frac{R}{2}\left\|D\left(z_{h}\right)-\dot{E}_{h}-G_{h}\right\|_{0,2}^{2}
$$

for any $\left\{z_{h}, G_{h}\right\}$ in $K_{h} \times X_{h}$. since in addition

$$
L_{R}\left(v_{h}, \dot{E}_{h},-\sigma_{h}-p_{h} \text { Id }\right)=L_{R}\left(\nabla_{h} \cdot \dot{E}_{h}, \mu_{h}\right)=J\left(\nabla_{h}\right), v \mu_{h} \text { e } Y_{h}
$$

$\left\{\left\{\nabla_{h}, \dot{F}_{h}\right\},-\sigma_{h}-F_{h} I d\right\}$ is indeed a saddle-point of the augmented lagrangian $L_{R}(\cdot, \cdots, *)$
over $\left(K_{h} \times Y_{h}\right) \times Y_{h}$.
Conversely, let $\left\{\left\{\nu_{h}, F_{h}\right\}, \lambda_{h}\right\}$ be a solution of the augmented lagrangian problem (3.6). Then, we must have

$$
L_{R}\left(v_{h}, E_{h}, \lambda_{h}\right) \geqslant L_{R}\left(\nabla_{h}, \mathcal{R}_{h}, \mu_{h}\right), v \mu_{h} e y_{h}
$$

which can only hold if we have
(3.8) $\quad F_{h}=D\left(\nabla_{h}\right)$.

Taking (3.8) into account, the second addin point inequality yields
(3.9) $L_{R}\left(\nabla_{h}, D\left(\nabla_{h}\right), \lambda_{h}\right) \leqslant L_{R}\left(\psi_{h}, G_{h}, \lambda_{h}\right), v\left(v_{h}, G_{h}\right\}$ e $X_{h} \times Y_{h}$.

In particular, by taking $G_{h}$ as $D\left(w_{h}\right)$, (3.9) implies

$$
J\left(\nabla_{h}\right) \leqslant J\left(z_{h}\right), v k_{h} \in{K_{h}}_{h}
$$

and $\nabla_{h}$ is indeed a solution of the original minimization problem (2.10).
To further characterize any solution $\left\{\left\{\nabla_{h}, \nabla_{h}\right\}, \lambda_{h}\right\}$ of the augmented lagrangian problem (3.6), we again use (3.8) and (3.9). From (3.8), $H_{h}$ is necessarily equal to $D\left(v_{h}\right)$. On the other hand, introducing the space $x_{h}$ defined in (2.13), (3.9) can be rewritten as

$$
\left.L_{R}\left(\nabla_{h}, D\left(v_{h}\right), \lambda_{h}\right)<L_{R}\left(\nabla_{h}+\alpha_{h}, D\left(v_{h}\right)+G_{h}, \lambda_{h}\right), v_{h}, E_{h}\right\} e x_{h} \times{v_{h}}_{h}
$$

Equivalently, if we consicter $L_{R}\left(\nabla_{h}+\cdots, D\left(\nabla_{h}\right)+*, \lambda_{h}\right)$ as a convex function of the pair \{ $\left.W_{h}, G_{h}\right\}$ on the space $X_{h} \times Y_{h}$ we can write (3.9) as

$$
\{0,0\} \text { e } \partial L_{R}\left(v_{h}+0, D\left(v_{h}\right)+0, \lambda_{h}\right) \text { in } x_{h}^{*} \times y_{h}^{*}
$$

A direct calculation characterizes the elements of this subgradient as the pairs
$\left\{g_{h}, \mu_{h}\right\}$ of $X_{h} \times Y_{h}$ such that


Setting $g_{h}$ and $\mu_{h}$ to zero in (3.10), we simply obtain the variational system (1.9)(1.11) with $\left(\sigma_{D}\right)_{h}=-\lambda_{h}$. As seen in the proof of Theorem 2.1, this in turn implies (2.11) with $\sigma_{h}=-\lambda_{h}-P_{h}$ Id, and our proof is complete.

REMARK 3.1: In order to accelerate the convergence of the algorithm to be used for the solution of the augmented lagrangian problem (3.6), it is usually better to replace, in the definition of the augmented lagrangian, the classical $L^{2}$ ( $\Omega$ ) scalar product by an equivalent weighted scalar product of the type

$$
\langle C, D\rangle=\int_{\Omega} r(x) C \cdot D d x
$$

Here $r(x)$ is a strictly positive scalar function of $L^{\infty}(\Omega)$, bounded away from zero, which can be arbitrarily chosen. Proper choices for this function will be discussed later. With this new scalar product, $L_{R}(*, *, *)$ becomes

$$
\begin{aligned}
& L_{R}\left(v_{h}, G_{h}, \mu_{h}\right)=\int_{\Omega} D_{1}^{e}\left(G_{h}\right) d x-\int_{\Omega} f \cdot v_{h} d x-\int_{r_{2}} g \cdot v_{h} d a \\
& +\frac{R}{2} \int_{\Omega} r(x)\left|G_{h}-D\left(w_{h}\right)\right|^{2} d x-\int_{\Omega} r(x) \mu_{h} \cdot\left(D\left(w_{h}\right)-G_{h}\right) d x
\end{aligned}
$$

```
    3.2 Numerical algorithm The fundamental interest of the equivalent augmented
lagrangian formulation (3.6) is the existence of a very cheap and gimple algorithm for its
numerical solution. This algorithm combines an Uzawa algorithm for the solution of the
saddle-point problem and a block-relaxation technique for the solution of the minimization
problems associated to the primal variable {\mp@subsup{w}{h}{},\mp@subsup{G}{h}{}}\mathrm{ . Dropping the aubscript h from}
all variables for simplicity, this algorithm is
(3.11) Let { [左, 且直} begiven in }\mp@subsup{Y}{h}{}\times\mp@subsup{K}{h}{
```



```
by block-relation, 1.e. by setting
\[
\varepsilon_{0}^{n}=\theta^{n-1}
\]
and by computing sequentially \(\nabla_{k}^{n}\) and \(\varepsilon_{k}^{n}\) by solving
（3．12）\(L_{R}\left(\nabla_{k}^{n}, q_{k-1}^{n}, \lambda^{n}\right)<L_{R}\left(w, \eta_{k-1}^{n}, \lambda^{n}\right)\) ，v \(\in\) ex \(_{n}\) ．
\[
\begin{equation*}
L_{R}\left(\nabla_{k}^{n}, \dot{q}_{k}^{n}, \lambda^{n}\right) \leqslant L_{R}\left(\nabla_{k}^{n}, G, \lambda^{n}\right), \forall \in e Y_{h^{\prime}} \tag{3.13}
\end{equation*}
\]
Once \(\left\{\nabla^{n},{ }^{n}\right\}\) is known，the Lagrange multiplier \(\lambda\) is updated by
```

（3．14）$\lambda^{n+1}=\lambda^{n}-R\left(D\left(v^{n}\right)-E^{n}\right)$ ．

Many variants exist for this algorithm and are described for example in FORTIN－GLONINSRI ［1982］．Usually，the block－relaxation（1．e．the loop（3．12）－（3．13）on k）is only carried out for one to five iterations．

Observe that the above algorithm only considers one variable at a time and therefore takes full advantage of the splitting of the difficulties achieved by the saddle－point
formulation (3.6). First, in (3.12), the matrix field $\mathrm{En}_{\mathrm{k}-1}$ and the multiplier $\lambda^{n}$ are supposed to be know, and the algorithm minimizes the augmented lagrangian $\mathcal{L}_{R}$ with respect to the velocity field in $K_{h}$. As function of the velocity field, $L_{R}$ is quadratic and corresponds to the energy dissipated by an incompressible Stokesian fluid, flowing viscously under the action of the external loads (f,g\}. In other words, (3.12\} is a classical linear stationary Stokes problem, discretized by mixed finite element methods. Many numerical techniques are available for fts solution, and we refer to TAYLOR-HOOD [1973], GIRAULT-RAVIART [1979] or GLOWINSKI-PIRONNEAU [1979] for the practical description of such techniques. In our numerical experiments, we will choose a conjugate gradient method operating on the hydrostatic pressure space $P_{h}$, which only reguires the inversion of sparse, fixed, positive definite, symetric matrices and therefore only uses little computer running time and memory core (FORTIN-GLOWINSKI [1982 p57]). In any case, most finite element codes now propose efficient subroutines for the solution of the stokes problem, which can be blindly used for solving (3.12).

Then, the algorithm supposes the velocity field $\nabla_{k}^{n}$ and the multiplier $\lambda^{n}$ given, and in (3.13) minimizes $L_{R}$ with respect to the matrix field $G$ in $Y_{h}$. The incompressibility condition and the spatial derivatives of $G$ are not involved in (3. i3): this is an unconstrained local convex minimization problem whose numerical solution, described in details in the next section, reduces to the solution in parallel of independent convex minimization problems set on $\boldsymbol{R}^{N}(\mathbb{N}=2$ or 3$)$.

Pinally, after a few resolutions of (3.12) and (3.13), the algorithm updates the multiplier $\lambda^{n}$ by the explicit formula (3.14), so that the constraint $D\left(\nabla^{n}\right)=E^{n}$ can be better satisfied by the solution of (3.12)-(3.13), and then returns to (3.12) and (3.13).
3.3 Convergence of the algorithm (3.11)-(3.14). We now study the convergence properties of the above Uzawa algorithm, considering the basic particular case where only one iteration of block-relaxation is done at each step of the Uzawa algorithm. In our
etudy, it will be most inportant to work on $X_{h}$ with the precies woighted $L^{2}(\Omega)$ norm which is used in the construction of the augented lagrangian $L_{R}$ (see Rearark 3.1).

Then, if we denote by $\{\nabla, E, \lambda\}$ the solution of the augmented lagrangian problem (3.6), by $\lambda^{n}$ the multiplier calculated in (3.14), by $\boldsymbol{z}^{n}$ the matrix field ar calculated in (3.13) and by $\nabla^{n}$ the vector field $\boldsymbol{\beta}_{1}^{n}$ calculated in (3.12), we can prove

CONVERGENCE THEORE 3.2: Under the assumption of the existence Theorem 1.1 ( $D_{1}$ convex, continuous, coercive), the secuence $\left\{\lambda^{n}\right\}$ is bounded in $y_{h}$, the difference $\left(D\left(\nabla^{n}\right)-n^{n}\right)$ convergen to zero in $Y_{h}$, and the guantity $F\left(n^{n}\right)+G\left(\nabla^{n}\right)$ convergen tovarde the dissipated energy rete $J(\nabla)$. If in addition (2.16) ie catiefied together with the E1rat line of (2.17) ( $D_{1}(\cdot)$ uniformily convex on the bounded sets of $Y_{h}$ ), then the
 intarnal diesipation potential $D,\left(^{\circ}\right)$ ie continuousiy diffarentiable, and if ite gradient is invartible with a coercive and wpachitz continuous invarse, that is if $D_{1}(\cdot)$ satisfien

for any $G_{1}$ and $G_{2}$ in $y_{h}$, then we can prove that the sequence $\left\{\nabla^{n}, \nabla^{n}, \lambda^{n}\right\}$ converges innearly towards $\{\nabla, F, \lambda\}$ in $K_{h} \times Y_{h} \times Y_{h}$ with an asymptotic conatant bounded by

$$
c_{15}=\left(1-2 R C_{14} /\left(1+C_{13} R\right)^{2}\right)^{1 / 2}
$$

Proof: Since $\{v, B, \lambda\}$ is a solution of the sadde-point problem (3.6), the following extremality relation is satisfied:

$$
F\left(B^{n}\right)+G\left(\nabla^{n}\right)+\left\langle\lambda, E^{n}-D\left(v^{n}\right)\right\rangle \geqslant F(n)+G(\nabla)
$$

Moreover, by construction, the solutions $v^{n}$ and $a^{n}$ of (3.12) and (3.13) aatiafy the extremality relations

$$
\begin{aligned}
& G(v)-G\left(\nabla^{n}\right)+\left\langle R\left(D\left(\nabla^{n}\right)-H^{n-1}\right)-\lambda^{n}, D\left(v^{n}\right)\right\rangle \geqslant 0, v \in e K_{h}, \\
& F(G)=F\left(R^{n}\right)+\left\langle R\left(H^{n}-D\left(\nabla^{n}\right)\right)+\lambda^{n}, H-H^{n}\right\rangle \geqslant 0, v \text { e } \gamma_{h^{\prime}} \\
& \text { reapectively. By addition, setting } v=\nabla, \text { and } G=B, \text { we get: }
\end{aligned}
$$

$$
\text { (3.17) } \begin{aligned}
F(B)+G(\nabla)-R I D\left(\nabla^{n}\right) & -E^{n} \|^{2}-R\left\langle A^{n-1}-\theta^{n} \cdot D\left(\nabla^{n}\right)\right\rangle \\
& \left.+\left\langle\lambda^{n}, D\left(\nabla^{n}\right)-B^{n}\right\rangle\right\rangle F\left(\nabla^{n}\right)+G\left(\nabla^{n}\right) .
\end{aligned}
$$

Adding (3.17) to (3.16), we then obtain
(3.18) -RID $\left.\nabla^{n}\right)-H^{n} \|^{2}-R\left\langle B^{n-1}-B^{n}, D\left(\nabla-\nabla^{n}\right)\right\rangle+\left\langle\lambda^{n}-\lambda, D\left(\nabla^{n}\right)-B^{n}\right\rangle \geqslant 0$.

Combining (3.18) with the construction (3.14) of $\lambda^{n+1}$ finally yields

$$
\begin{equation*}
\left.\left\|\lambda^{n}-\lambda\right\|^{2}-\left\|\lambda^{n+1}-\lambda\right\|^{2}\right\rangle R^{2} \| D\left(\nabla^{n}\right)-A^{n} i^{2}+2 R^{2}\left\langle\theta^{n-1}-A^{n}, D\left(\nabla-\nabla^{n}\right)\right\rangle \tag{3.19}
\end{equation*}
$$

On the other hand, using (3.13) and (3.14) at iteration ( $n-1$ ), we can estimate the right-hand side of (3.19) by standard algebraic manipulations. Exactly as in FORTINGLOWINSKI [1982 p117, equations (5.17) to (5.24)], setting $F_{0}=0, \rho=R$ and inverting the sign of $\lambda$, we have the following estimate
which, combined with (3.19), gives
(3.20) $\left.\left(\left\|\lambda^{n}-\lambda I^{2}+R^{2} \mid n^{n-1}-M\right\|^{2}\right)-\left(\left\|\lambda^{n+1}-\lambda\right\|^{2}+R^{2} \mid I^{n}-m \|^{2}\right)\right\rangle$

$$
R^{2} 1 D\left(\nabla^{n}\right)-\nabla^{n} i^{2}+R^{2} \mid E^{n}-E^{n-1} 1^{2} .
$$

The positive eequence $\left|\lambda^{n}-\lambda I^{2}+R^{2}\right| I^{n-1}-\| \|^{2}$ is therefore decreasing and thus convergen to alimit. Thie implies that the right-hand aide of (3.20) must converge to zero and we finally obtain

$$
\begin{aligned}
& \left|\lambda^{n}-\lambda \|^{2}+R^{2}\right| A^{n-1}-n^{2} \text { is bounded, } \\
& 11 m \mid D\left(\nabla^{n}\right)-E^{n} \|^{2}=0 \text {, } \\
& { }^{n+\infty}\left|E^{n}-E^{n-1}\right|=0 \text {. } \\
& n++\infty
\end{aligned}
$$

Thene convergence reaults, used hack in (3.16) and (3.17) obviously inply the converqence of $F\left(E^{n}\right)+G\left(\nabla^{n}\right)$ towarde $F(E)+G(\nabla)$.

Now, if $D_{1}(*)$ is uniformly convex on the bounded eete of $Y_{h}$, the convergence of the energy rate and the boundednes of in imply the convergence of the argument fin and $v^{n}$ respectively towards $B$ and $\nabla$.
 dual, we can prove the 1 inear convergence of the sequence $\left\{\nabla^{n}, f^{n}, \lambda^{n}\right\}$ by applying a result of LIONSHERCIER (1979, Prop. 4, p 970] which will be applicable here as soon as (3.15) it satisfied tee FORTIN-GLONINSKI [1982, p 300] for more details).

RPMARK 3.1: Condition (3.15) is satisfied at least locally for Norton materials. It is not matisfied in the general case, but linear convergence of the sequence $\left\{\boldsymbol{v}^{n}\right.$, $\left.i^{n}\right\}$ can still be observed numerically in almost any case.

RHARX 3.2: The asymptotic constant $C_{15}$ appears numerically not to be optimal. Neverthelest, its expression as a function of $C_{13}$ and $C_{14}$ will indicate the right
(ii) take $R$ close to $1 / \mathrm{C}_{13}$, which will usually be close to 1 if $r(\cdot)$ is properly chosen.

REMARK 3.3: If $D_{1}$ is quadratic and if we have equality between (3.21) and (3.22), then for $R=1, \nabla^{n}$ converges in 2 iterations and $\boldsymbol{o}^{n}$ converges linearly with asymptotic constant .5 (FORTIN-GLOWINSKI [1982, p 119]). In other cases, with proper choices of $R$ and of $r$, we usually observe linear convergence of $\left(\nabla^{n}, G^{n}\right)$ with an asymptotic constant around .7.

REMAPK 3.4: Linear convergence compares unfavorably to the quadratic convergence expected for conjugate gradient or for Newton algorithms. But Newton method requires $D_{1}$ to be twice differentiable, the factorization of quite a few finite element matrices, and its convergence rate can be very slow for weakly convex dissipation potentials. In general, it is not a good method for solving (2.10). On the other hand, a conjugate gradient method with preconditioning, of the type

- take $n_{0} \operatorname{in} K_{n}^{\prime}$
- solve $\left\langle D\left(g_{0}\right), D(v)\right\rangle=\left\langle J^{\prime}\left(\varepsilon_{0}\right), v\right\rangle, v w e K_{h}$
- set $\mathbf{E}_{0}=9_{0}$
- for $n=0$. until eatisfied do

$$
\rho_{n}=\operatorname{ArgMin} J\left(\min _{n}-\rho E_{n}\right)
$$

$n_{n+1}=n_{n}-p_{n} E_{n}$
solve $\left\langle D\left(g_{n+1}\right), D(v)\right\rangle=\left\langle J^{\prime}\left(E_{n+1}\right), \nabla\right\rangle, v \operatorname{ver}_{n}$
$y_{n}=\left\langle\left(\Phi_{n+1}\right) . D\left(g_{n+1}-g_{n}\right)\right\rangle /\left\langle D\left(g_{n}\right), g_{n}\right\rangle$.
$E_{n+1}=g_{n+1}+Y_{n} z_{n}$
end loop on $n$;
where $\left\langle\bullet, \bullet\right.$ is an adequate weighted $L^{2}(\Omega)$ scalar product on $y_{h}$, will only be efficient if $D,(*)$ is differentiable, if the scalar product on $y_{h}$ is correctly chosen and if a vary efficient stokes solver is available for computing $g_{n+1}$. If this is the case, the conjugate gradient method will be twice as fast as the Uzawa algorithm (3.11)(3.14). If this is not the case, Algorithm (3.11)-(3.14) appears to be one of the only reasonable numerical method for solving (2.10).

## 4. THE PROELEMS IN DEFORMATION RATES.

## 4. 1 The local problems. Problem (3.13) appears as one step of the algorithm

 proposed herein for the numerical solution of the viscous flow problems in quasistatic viscoplasticity, once these problems have been approximated by simplicial finite elements of order one and decomposed under an augmented lagrangian form. Recall that here, this problem consists in$$
\text { (3.13) Minimizing } L_{R}(\nabla, *, \lambda) \text { over } Y_{h}
$$

with

$$
\begin{aligned}
& L_{r}(\nabla, G, \lambda)=\int_{\Omega} D_{1}^{e}(E) d x-\int_{\Omega} f \cdot v d x-\int_{\Gamma_{2}} g \cdot \nabla d a \\
& +\frac{R}{2} \int_{\Omega} r(x)\left|\sigma\left(\nabla v+\nabla v^{T}\right) / 2\right|^{2} d x-\int_{\Omega} r(x) \lambda \cdot\left[\left(\nabla \nabla+\nabla \nabla^{T}\right) / 2-G\right] d x, \\
& Y_{h}=\left\{D: \Omega \rightarrow R_{B}^{N x N}, D_{\Omega_{l}} e\left(P_{0}\left(\Omega_{l}^{1}\right)\right)^{N x N}, v i=1,2^{N}, v i=1, N_{h}\right\},
\end{aligned}
$$

and that it is the only nonstandard step in this algorithm, the other steps consisting of linear stokes problems and explicit variables updating, respectively.

Since all the elements of $y_{h}$ are matrix fields which are constant on each $\Omega_{i}$, and since the functional $L_{R}$ does not involve any distributional derivative of $G$, Problem (3.13) can equivalently be written as
(4.1) $v i=1,2^{N}, v i=1, N_{n}$ Minimize $J_{2}^{1}(G)$ over ${ }_{8}^{N(N N}$, with
(4.2) $\quad J_{l}^{1}(G)=D_{1}^{e}(G)+\frac{r R}{2}|G|^{2}-\left.r G \cdot\left(R\left(\nabla \nabla+\nabla \nabla^{T}\right) / 2-\lambda\right)\right|_{\Omega_{i}}$.

Here, we are simply using the fact that the minimum value of the sum of independent terms is equal to the sum of the minimum value of each term. Then, Problem (3.13) reduces to
the solution in parallel of $N_{h} \times 2^{N}$ local independent convex minimization problempact on $\mathrm{F}_{\mathrm{NXN}}^{(\mathrm{N}=2}$ or 3).

Using a general purpose minimization algorithm for the solution of each local problem (4.1) 1s not adviseable here for two main reasons:
(1) such an algorithm is very difficult to implanent because it must be able to handle general nondifferentiable convex disalpation potentials $0_{1}$,
(ii) much an algorithm is unually expensive in computer running time.

An easier and more efficient strategy consists in adapting each time the minimization algorithm to the epecific class of potentiala $D_{1}$ which is under consideration. Doing that, we have most of the times been able to reduce each local problem (4.1) to onedimensional convex minimization problem set on $R_{4}$. The remainder of this report will deacribe the derivation of euch officient numerical techniques in the case of Norton materials, of Bingham materiale, and of Tresca type materiale in plane atreses, respectively. But before, we will derive a very ueful aimplification of the local problem (4.1).
4.2 Reduction of the local problemg. We begin by recalling several well-known reault of matrix theory, which will enable us to reduce the local probleme (4.1) which are net on $\mathbf{R}^{\mathrm{NxN}}$ to local convex minimization probleme set on $\mathrm{R}^{\mathrm{N}},(\mathrm{N}=2$ or 3 ).

Loman 4.1 (VON NEUMANN[1937]). Let $A$ and be two matricee in $\mathrm{R}^{\mathrm{N} \times \mathrm{N}}$ with eingular values $\alpha_{1} \geqslant \alpha_{2} \geqslant \cdots \geqslant \alpha_{N} \geqslant 0$ and $\beta_{1} \geqslant \beta_{2} \geqslant \cdots \geqslant \beta_{n} \geqslant 0$. Then

$$
\operatorname{Tr}(\operatorname{A} B) \leqslant \sum_{i=1}^{N} a_{i} \beta_{i}
$$

Lemma 4.2. Let $A$ and $B$ be two symmetricmatrices in $\boldsymbol{R}_{B}^{N \times N}$ with eigenvalues $A_{1} \geqslant A_{2} \geqslant \cdots \geqslant A_{N}$ and $B_{1} \geqslant B_{2} \geqslant \cdots \geqslant B_{N}$ Then

$$
A \cdot B=\operatorname{Tr}(A B)<\sum_{i=1}^{N} A_{i} B_{i}
$$

Proof: The result follows from the decomposition
$\operatorname{Tr}(A B)=\operatorname{Tr}\left[\left(A-A_{N} \operatorname{Id}\right)\left[B-B_{N} \operatorname{Id}\right)\right]+A_{N} \operatorname{Tr}(B)+B_{N} \operatorname{Tr}(A)-N_{N} A_{N}$
and from the application of Lemma 4.1 to the first term of the right-hand side.

Lemma 4.3 Let $D^{d}$ be a diagonal matrix with diagonal terms $D_{1} \geqslant D_{2} \geqslant \cdots \geqslant D_{N}$ and $A$ be a symmetric matrix with eigenvalues $A_{1} \geqslant A_{2} \geqslant \cdots \geqslant A_{N}$ then

$$
\operatorname{Max}_{\mathbf{P P}^{T}=I d}\left[\operatorname{Tr}\left[\mathbf{P}^{T} D^{d} P A\right]\right]=\operatorname{Tr}\left[Q D^{d} Q^{T} A\right]=\sum_{i=1}^{N} D_{i} A_{i},
$$

where 9 is an orthogonal matrix which diagonalizes $A$ with

$$
\left(Q^{T} A Q\right)_{i i}=A_{i}
$$

Proof: For $P$ given, let $B$ be the matrix defined by

$$
B=P^{T} D^{d} P
$$

and whose eigenvalues are $D_{i}$. Then, from Leman 4.2, we have:

$$
\operatorname{Tr}\left[P^{T} D^{d} P A\right]=\operatorname{Tr}(A B)<\sum_{i=1}^{N} A_{i} D_{i}
$$

On the other hand, we also have

$$
\operatorname{Tr}\left[Q_{0}^{d} Q^{T} A\right]=\operatorname{Tr}\left[Q^{T} Q_{0}^{d} Q^{T} A Q\right]=\operatorname{Tr}\left[D_{Q^{T}}^{d Q}\right)=\sum_{i=1}^{N} A_{i} D_{i}
$$

and the result follows.

We are now ready to prove the main result of this section, which reduces the local problems to convex minimization problems set on $R^{N}(N=2$ or 3$)$.

THEOREM 4.1: If the internal dissipation potential $\mathcal{D}_{1}(\cdot)$ is convex and isotropic, then the solution $H$ of the local problems (4.1) is given by
(4.3)

$$
\begin{aligned}
& \mathbf{H}_{\Omega_{\ell}^{i}}=\boldsymbol{Q}_{\ell} \mathbf{R}_{\ell}^{d} \boldsymbol{Q}_{\ell}^{T} \\
& A_{\ell}^{i}=r\left(\frac{R}{2}\left(\nabla \nabla+\nabla \nabla^{T}\right)-\left.\lambda\right|_{\left.\right|_{\Omega}} ^{i}\right.
\end{aligned}
$$

where $Q_{q}$ is an orthogonal matrix of $\mathbf{R}^{N \times N}$ whose columns are normed eigenvectors of the matrix $A_{i}^{i}$ (the first column corresponding to the biggest eigenvalue and so on) and where $H_{l}^{d}$ is the diagonal matrix of $R^{N \times N}$ solution of
(4.4) Minimize $\left\{D_{1}\left(D^{d}\right)+\frac{R r}{2}\left|D^{d}\right|^{2}-\prod_{i=1}^{N} A_{i}\left(D^{d}\right)_{i i}\right\}$ on $D^{N}$. Above $D^{N}$ denotes the space of diagonal matrices of $\mathrm{R}^{N \times N}$ and ( $A_{1}$ ) denotes the set of eigenvalues of $A_{\ell}^{i} \quad\left(A_{1} \geqslant A_{2} \geqslant \cdots \geqslant A_{N}\right)$.

$$
\text { Proof. First observe that any matrix } G \text { of } \mathbf{R}_{s}^{N X} \text { can be decomposed into }
$$

(4.5)
$\mathbf{G}=\mathbf{P}^{T} D^{d} \mathbf{P}$
where $P$ and $D^{d}$ are two independent matrices of $A^{N \times N}$, $P$ being orthogonal ( $P^{T}=$ Id) and $D^{d}$ being the diagonal matrix whose diagonal elements are the eigenvalues of C. Then, if we denote by $0^{N}$ (respectively $D^{N}$ ) the space of orthogonal (respectively diagonal) matrices of $\mathbf{R}^{\mathrm{NxN}}$, the local problem (4.1) becomes

$$
\text { (4.6) vi, vi, Minimize } J_{\ell}^{i}\left(P, D^{d}\right) \text { over } 0^{N} \times 0^{N} \text {. }
$$

with

$$
J_{\ell}^{1}\left(P, D^{d}\right)=J_{\ell}^{1}(G)=D_{1}^{e}\left(P^{T} D^{d} P\right)+\frac{R r}{2}\left|P_{D}^{T} P\right|^{2}-\left(P^{T} D^{d} P\right) \cdot A_{l}^{i}
$$

But since $D_{1}^{e}(\cdot)$ and $|\cdot|$ are isotropic $\left(D_{1}^{e}\left(P^{T} D^{d}\right)=D_{1}^{e}\left(D^{d}\right)\right)$, we can rewrite $J_{i}^{i}$ as

$$
J_{l}^{i}\left(P, D^{d}\right)=D_{1}^{e}\left(D^{d}\right)+\frac{R r}{2}\left|D^{d}\right|^{2}-\left(P^{T} D^{d} P\right) \cdot \Lambda_{\ell}^{i} .
$$

Now, let $G_{\ell}^{d}$ be the solution of (4.4) (unique since the function to minimize is strictly convex) and let $P_{H}$ be the permutation matrix which reorders the diagonal elements of $H_{l}^{d}$ in the decreasing order. Since $A_{1} \geqslant A_{2} \geqslant \cdots \geqslant A_{N}$ and since $D_{1}^{e}(\cdot)$ is isotropic, we have

$$
D_{1}^{e}\left(\mathbf{P}_{H}^{T} H_{\ell}^{\mathrm{d}} \mathbf{P}_{H}\right)+\frac{r \mathrm{R}}{2}\left|\mathbf{P}_{H}^{\mathrm{T}} \mathbf{H}_{\ell}^{\mathrm{d}} \mathbf{P}_{H}\right|^{2}-\sum_{i=1}^{N} A_{i}\left(\mathbf{P}_{H}^{\mathrm{T}} \mathbf{B}_{\ell}^{\mathrm{d}} \mathbf{P}_{\mathrm{h}}\right)_{i} \leqslant \quad D_{1}^{e}\left(\mathbf{B}_{\ell}^{\mathrm{d}}\right)+\frac{R X}{2}\left|\mathbf{H}_{\ell}^{\mathrm{d}}\right|^{2}-\sum_{i=1}^{N} A_{i}\left(\mathbf{H}_{\ell}^{\mathrm{d}}\right)_{i} .
$$

But $\mathbf{H}_{\ell}^{d}$ is the only solution of (4.4), thus $P_{H}^{T} H_{\ell}^{d} P_{H}$ has to be equal to $H_{\ell}^{d}$, which means that the diagonal elements of $\mathbb{H}_{\ell}^{d}$ are already placed in a decreasing order. From Lemma 4.3, this implies
(4.7) $\sum_{i=1}^{N} A_{i}\left(H_{\ell}^{d}\right)_{i}=\left(Q_{\ell} H_{\ell}^{d} Q_{\ell}^{T}\right) \cdot A_{\ell}$.

Moreover, since $H_{\ell}$ is solution of (4.4), we also have

$$
D_{1}^{e}\left(B_{l}^{d}\right)+\frac{r R}{2}\left|H_{l}^{d}\right|-\sum_{i=1}^{N} A_{i}\left(B_{l}^{d}\right)_{i} \leqslant D_{1}^{e}\left(D^{d}\right)+\frac{R X}{2}\left|D^{d}\right|^{2}-\sum_{i=1}^{N} A_{i} D_{i}, Y D^{d} e D^{N}
$$

and in particular, if $P_{D}$ is the permutation matrix reordering $D^{d}$

$$
D_{1}^{e}\left(H_{l}^{d}\right)+\frac{R x}{2}\left|B_{l}^{d}\right|-\sum_{i=1}^{N} A_{i}\left(H_{l}^{d}\right)_{i} \leqslant D_{1}^{e}\left(P_{D}^{T} D^{d} P_{d}\right)+\frac{R x}{2}\left|P_{D}^{T} D^{d} P_{D}\right|^{2}-\left(P^{T} D^{d} P_{D}\right)_{i}^{A_{i}}, \cup D^{d}
$$

Now, from (4.7), Lemma 4.3, and the isotropy of $D_{1}^{e}(\cdot)$, this implies

$$
D_{1}^{e}\left(H_{\ell}^{d}\right)+\frac{R r}{2}\left|\mathbf{H}_{\ell}^{d}\right|^{2}-\left(Q_{\ell} \mathbf{E}_{\ell}^{d} Q_{\ell}^{T}\right) \cdot\left(\mathbf{A}_{\ell}^{i}\right) \quad<\quad D_{1}^{e}\left(D^{d}\right)+\frac{R r}{2}\left|D^{d}\right|^{2}-\left(P^{T} D^{d} P\right) \cdot\left(\mathbf{A}_{\ell}^{i}\right),
$$

for any $D^{d}$ in $D^{N}$ and any $P$ in $O^{N}$. So, finally, we get

$$
J_{\ell}^{i}\left(Q_{\ell}, H_{\ell}^{d}\right) \leqslant J_{\ell}^{1}\left(P, D^{d}\right) \quad \forall\left\{P, D^{d}\right\}, \text { in } \delta^{N} \times D^{N}
$$

In other words, $\left\{Q_{\ell}, H_{\ell}^{d}\right\}$ is a solution of (4.6). By construction, this implies that H , given by (4.3), is a solution of the original local problems (4.1). Our proof is therefore complete because, since $J_{\ell}^{i}$ is strictly convex, such a solution is unique. $\square$

REMARK 4.1. It is well known that the internal dissipation potential is isotropic and convex for all standard isotropic viscoplastic solids and all standard viscoplastic fluids. Theorem 4.1 can therefore be applied in most practical situations.

REMARK 4.2: In Theorem 4.1, the relation (4.3) simply expresses that $\|_{H_{l}}$ and $A_{l}^{i}$ have the same eigenvectors. In essence, this is the discrete equivalent of the well known result which states that principal stresses and principal strains are parallel in isotropic elasticity or visoplasticity.

## 5. NORTON VISCOPLASTICITY

5.1 The local problems. Norton viscoplasticity correaponds to one of the easiest possible case where the internal dissipation potential $D_{1}(\cdot)$ is given by
(5.1) $\quad D_{1}(G)=\frac{1}{p}(k \sqrt{2})^{p}|G|^{p}=\frac{1}{p}(k \sqrt{2})^{p}\left(\sum_{i, j=1}^{N} G_{i j}^{2}\right)^{p / 2}$.

Each local minimization problem (4.1) now becomes
(5.2) Minimize $J_{\ell}^{i}(G)$ over $R_{s}^{N \times N}$ with

$$
\left\{\begin{array}{l}
J_{l}^{1}(G)=\frac{1}{P}(k \sqrt{2})^{P}|G|^{P}+\frac{R r}{2}|G|^{2}-A_{l}^{i} \cdot G, \\
A_{l}^{i}=r\left(\frac{R}{2}\left(\nabla_{\sigma}+\nabla \nabla^{T}\right)-\lambda| |_{\Omega_{l}^{1}},\right.
\end{array}\right.
$$

# THEOREM 5.1: The solution $\mathbf{G}_{\ell}^{i}$ of the local minimization problem (5.2) is of the 

form
(5.3) $\quad B_{\ell}^{i}=n_{l}^{i} \times /\left|A_{\ell}^{i}\right|$.
where $x$ is the solution of the one-dimensional convex minimization problem
(5.4) Minimize $\left\{\frac{1}{p}(k \sqrt{2}\rangle^{p} y^{p}+\frac{R r}{2} y^{2}-\left|A_{l}^{i}\right| y\right\}$ over $R_{+}$.

Prof: Suppose that the norm $\left|H_{l}^{i}\right|$ of the minimizer $H_{l}^{i}$ of $J_{l}^{i}$ over $R_{s}^{N \times N}$ is known. Then, $\mathbf{G}_{\ell}^{i}$ minimizing $J_{\ell}^{i}$ over $\mathbb{R}_{S}^{N x N}$ will in particular minimize $J_{l}^{i}(\cdot)$ over the set of matrices of $\boldsymbol{r}_{s}^{N \times N}$ with fixed norm $\left|B_{l}^{i}\right|$. But this last minimization problem reduces to the maximization of the scalar product $A_{l}^{i}$. $G$ over a sphere of $R_{s}^{N}$ and its solution is given by

$$
\begin{equation*}
\mathrm{B}_{\ell}^{i}={A_{l}}_{i}\left|\mathrm{H}_{l}^{i}\right| /\left|\lambda_{l}^{i}\right| . \tag{5.3}
\end{equation*}
$$

By plugging (5.3) into the expression of $J_{l}^{i}$, the minimization of $J_{l}^{i}$ over $R_{B}^{N x N}$ finally reduces to finding the unknown norm $\left|B_{l}^{i}\right|$ which has to mininimize $J_{l}^{i}\left(X_{l}^{i} y /\left|A_{l}^{i}\right|\right)$ over the set $R_{+}$of positive numbers. This last problem is precisely (5.4) and our proof is complete.
a) the computation of the solution $\left|B_{l}^{i}\right|$ of the one dimensional convex minimization problem (5.4)
b) the computation of $H_{l}^{i}$ by the explicit formula (5.3).

The numerical solution of (5.4) can be achieved for example by using the one dimensional
Newton algorithm below

$$
\begin{aligned}
& \text { data: } x^{\circ}=\text { solution of (4.4) at previous iteration; } \\
& \text { initialisation: } x+x^{0} \text {; } \\
& j+1 ; \\
& \text { repeat : } \quad j+j+1 \text {; } \\
& \mathrm{g}+(\mathrm{k} \sqrt{2})^{\mathrm{p}} x^{\mathrm{p}-1}+\mathrm{rRx}-\left|\mathrm{A}_{\ell}+\mathrm{A}_{\ell}^{\mathrm{T}}\right| / 2 ; \\
& \text { test : if }|g| \text { below tolerance exit; } \\
& \left.\mathrm{dg}+(\mathrm{p}-1)(k \sqrt{2})^{p} x^{p-2}+\mathrm{Rr}\right) \\
& x+\max \left(10^{-30}, x-(d g)^{-1} g\right) \text {; } \\
& \text { exit } \quad\left|H_{\ell}\right|=x \text {. }
\end{aligned}
$$

Putting together all the steps which permit the numerical solution of the decomposed approximated viscous flow problem (3.6) by the algorithm (3.11)-(3.14), we finally obtain the simple and easy to code computer flow chart of Figure 5.1.


Fig. 5.1: Computer flow chart for solving Norton viscous flow problems
5.2 Numerical resulta. The first numerical test considers a horizontal cylindrical hose, with external radius $1 .$, which is glued on a rigid core on ita internal face, and which is subjected to its own weight. This situation may represent for example the cooling process of the plastic coating of an electrical wire, for which manufacturers must verify that the deformations undergone by the coating during cooling remain small. In a firtt approximation, strains are assumed to remain plane, and the coating is supposed to be made of an homogeneous Norton material with $k=.47, p=1.4$, and volumic weight . 1 .

Por symetry reasons, only the right half of the section is considered, 225 nodes are used to approximate the velocity, 65 nodes are used for the pressure. only one blockrelaxation iteration is done at each Uzawa step and after 60 iterations of the Uzawa algorithm, the error $\| \mathrm{E}^{\mathrm{n}}-\mathrm{D}\left(\mathrm{v}^{\mathrm{n}}\right)$ ) has decreased by a factor of $10^{-7}$ (in fact, 30 iterations were more than sufficient to obtain a very accurate velocity field). The total CPU time was approximatively 3 mn on the VAX 780. Figure 5.2 represents the computed velocity field (magnified 7 times). The shape of the hose after one second of flow (the deformations being multiplied by 40 ) is indicated on Figure 5.3 , together with the mesh used for the pressure.

In the entire computation, the parameter $R$ was equal to 1 . The weight $r(x)$ was equal to 1 during the first 20 iterations, then updated by the formula $r(x)=\left|A^{20}\right|^{p-2}$ (see Remark 3.2 for justification), kept that way until iteration 40 where it was finally updated by $r(x)=\left|a^{40}\right|^{p-2}$.

## 6. BINGHAM VISCOPLASTICITY

6. 1 The local problems. Bingham viscoplasticity corresponds to an internal diseipation potential $D_{1}(\cdot)$ of the type

$$
\begin{equation*}
D_{1}(\epsilon)=\mu|\epsilon|^{2}+g \sqrt{2}|\epsilon| \tag{6.1}
\end{equation*}
$$

Each local minimization problem (4.1) now becomes




(6.2) Minimize $\left\{\left.\left(\frac{r R}{2}+\mu\right)|G|^{2}+\sigma \sqrt{2}|\epsilon|-A_{l}^{i} \cdot G \right\rvert\,\right.$ over $R_{s}^{N \times N}$
whose solution $H_{l}^{i}$ is simply given by the explicit formula
(6.3) $\quad \mathbf{u}_{l}^{i}=\operatorname{Max}\left\{0,1-\sqrt{2} g /\left|\Lambda_{l}^{i}\right|\right\} \quad \lambda_{l}^{i} /(r R+2 u)$.

In practice, the whole program solving the flow problem (3.6) for Bingham fluids still corresponds to the computer flow chart of fig. 5.1, each local problem being now solved by (6.3).
6. 2 Numerical result. We consider herein a Bingham fluid flowing viscously through a cavity. Fluid enters at the upper right of the cavity and exits at the upper left, with an imposed velocity of 3.0. No alip boundary conditions are imposed elsewhere. Dimensions of the cavity are 1. for the main square and .1 for the entrance and exit tubes. The fluid viscosity $u$ is assumed to be .09 and velocities are supposed to remain plane.

The finite element mesh uses 419 nodes for velocities and 166 nodes for pressures. One block-relaxation is done at each Uzawa step and we take $R=.1$ and $r(x)=1.0$. Figures 6.1 and 6.2 represent velocity obtained after 40 iterations, the plasticity threshold $g \sqrt{2}$ being respectively of $\mu$ and of $10 \mu$. As expected, the domain where the fluid is at rest is bigger in the latter case. The computation time for each case was approximatively 10 mn on the vax 780.

## 7. TRESCA TYPE VISCOPLASTICITY IN PLANE STRESSES.

7. 1 The local problems. In plane stresses, the body under consideration is supposed to be very thin along $x_{3}$ and is loaded in its plane so that, in a first approximation, all stresses along $x_{3}$ are equal to zero. It is then possible to eliminate the $x_{3}$ direction and to reduce our original problem to a two-dimensional one whose domain will be the middle plane section of the body and whose unknowns will be the in-plane velocities.

These inplane velocities need no longer be incompressible since any reduction of the plane section can be compensated by a corresponding thickening of the body. Therefore, in Sections 2 and $3, k_{h}$ is everywhere replaced by $v_{h}+\dot{u}_{0}$ and in particular (3.12) becomes

Minimize

$$
L_{R}(\cdot, G, \lambda) \text { over } V_{h}+\dot{m}_{0},
$$

which is a classical linear elasticity problem with a zero first Lamé coefficient. As for (3.13), its formulation is unchanged and it still reduces to the local problem (4.1).

In plane stresses, a Tresca type viscoplastic material corresponds to the internal
dissipation potential
(7.1) $\quad D_{1}(G)=\frac{1}{p}(k \sqrt{2})^{P}\left\{\operatorname{Max}\left(\left|G_{1}\right|,\left|G_{2}\right|,\left|G_{1}+G_{2}\right|\right)\right\}^{p}$
where $G_{1}$ and $G_{2}$ are the eigenvalues of the $2 \times 2$ symetric matrix $G$. This potential Is a symmetric convex function of the eigenvalues of $G$, therefore is isotropic and convex (HILL[1970]). Moreover it satisfles the inequalities (1.6) for coerciveness and continuity. On the other hand, this potential is not strictly convex and not differentiable, which clearly appears in Fig. 7.1, where the level ines of $D_{1}$ are drawn.

The viscous flow problem associated to this potential (7.1) can still be solved by the Uzawa algorithm (3.11)-(3.14): (3.12) is a linear elasticity problem and (3.13) reduces to local problems whose solution $t$ is given by (see Theorem 4.1).


Above, as before, $A_{1}$ and $A_{2}\left(A_{1}>A_{2}\right)$ denote the eigenvalues of $A_{l}^{i}$ and $Q_{l}$ is an orthogonal transformation matrix which diagonalizes $A_{\ell}^{i}$ and orders the diagonal elements of the resulting matrix in a decreasing order.

Thus, for tresca type viscoplasticity in plane stresses, the numerical solution of each local problem (4.1) reduces to

Step 1 compute $A_{l}^{i}=\left[\begin{array}{ll}A_{11} & A_{12} \\ A_{12} & A_{22}\end{array}\right] \quad$ by $\quad A_{l}^{i}=\left.r\left(\frac{R}{2}\left(\nabla \nabla+\nabla_{V}^{T}\right)-\lambda\right)\right|_{\mid \Omega_{l}^{i}}$;
Step 2 compute the eigenvalues $A_{1}$ and $A_{2}$ by

$$
\begin{aligned}
& A_{1}=\left[\left(A_{11}+A_{22}\right)+\sqrt{\left(A_{11}-A_{22}\right)^{2}+4 A_{12}^{2}}\right] / 2, \\
& A_{2}=\left[\left(A_{11}+A_{22}\right)-\sqrt{\left(A_{11}-A_{22}\right)^{2}+4 A_{12}^{2}}\right] / 2,
\end{aligned}
$$

Step 3 compute $H_{1}$ and $H_{2}$ by $j\left(H_{1}, H_{2}\right) \leqslant j\left(D_{1}, D_{2}\right),\left\{\left(D_{1}, D_{2}\right\} e R^{2} ;\right.$

Step 4: Compute $H_{\ell}^{i}=Q_{\ell} H_{\ell}^{d} Q_{\ell}^{T}$ by

$$
\begin{aligned}
& \left(\boldsymbol{H}_{l}^{i}\right)_{11}=\left(H_{1}-H_{2}\right)\left(A_{11}-A_{2}\right) /\left(A_{1}-A_{2}\right)+H_{2} \\
& \left(H_{l}^{i}\right)_{22}=\left(H_{1}-H_{2}\right)\left(A_{22}-A_{1}\right) /\left(A_{1}-A_{2}\right)+H_{1}, \\
& \left(H_{l}^{i}\right)_{12}=\left(H_{1}-H_{2}\right) A_{12} /\left(A_{1}-A_{2}\right)
\end{aligned}
$$

Consequently, the computer flow chart associated to the Uzawa algorithm (3.11)-(3.14) for the numerical. solution of viscous flow problems in rresca type viscoplasticity in plane stresses is the one described in Figure 5.1, but with the solution of the local problems being achieved by the four steps above. Among these steps, only one, Step 3, is not explicit and its solution is described in the next paragraph.
7.2 Solution of Step 3. By definition of the subgradient, Step 3 is equivalent to (7.3)
$\{0,0\}$ e $\partial j\left(H_{1}, H_{2}\right)$.
Therefore, to solve step 3, we first begin by computing the subgradient of $f(\cdot, \cdot)$ over $\mathbf{m}^{2}$. Since we know from Theorem 4.1 that, at the solution of (7.3), $\mathrm{H}_{1}$ is qreater or equal to $H_{2}$, we reatrict ourselves to the half plane $D_{1} \geqslant D_{2}$. Then, direct calculation gives:

Case 1: if $\left\{D_{1}, D_{2}\right\}$ e $K_{1}=\left\{\{x, y\}\right.$ e $\left.\left.R^{2}, x\right\rangle y, y>0\right\}$, then:
$\left\{\begin{array}{l}f\left(D_{1}, D_{2}\right)=\frac{1}{P}(k \sqrt{2})^{P_{( }}\left(D_{1}+D_{2}\right)^{P}+\frac{r R}{2}\left(D_{1}^{2}+D_{2}^{2}\right)-A_{1} D_{1}-A_{2} D_{2}, \\ \partial f\left(D_{1}, D_{2}\right)=\left\{\left(u_{1}, u_{2}\right), u_{i}=(k \sqrt{2})^{P}\left(D_{1}+D_{2}\right)^{P-1}+E R D_{1}-A_{1}\right\} .\end{array}\right.$
Case 2: if $\left\{D_{1}, D_{2}\right\}$ e $k_{2}=\left\{\{x, y\}\right.$ e $\left.\left.R^{2}, x\right\rangle 0, y=0\right\}$, then:

$$
\left\{\begin{array}{l}
f\left(D_{1}, D_{2}\right)=\frac{1}{P}(k \sqrt{2})^{P}\left(D_{1}\right)^{P}+\frac{r}{2} D^{2} D_{1}^{2}-A_{1} D_{1} \\
\left.\partial j\left(D_{1}, D_{2}\right)=f\left(u_{1}, u_{2}\right), u_{1}=k \sqrt{2}\left(D_{1}\right)^{P-1}+r R D_{1}-A_{1},-A_{2} \leqslant u_{2}<(k \sqrt{2})^{P}\left(D_{1}\right)^{P-1}-A_{2}\right\}
\end{array}\right.
$$

Case 3: if $\left\{D_{1}, D_{2}\right\}$ e $K_{3}=\left\{\{x, y\} e R^{2}, x>-y, y<0\right\}$, then

$$
\left\{\begin{array}{l}
f\left(D_{1}, D_{2}\right)=\frac{1}{P}\left(k \sqrt{2} D_{1}\right)^{P}+\frac{r R}{2}\left(D_{1}^{2}+D_{2}^{2}\right)-A_{1} D_{1}-A_{2} D_{2} \\
\partial f\left(D_{1}, D_{2}\right)=\left\{\left(u_{1}, u_{2}\right), u_{i}=(k \sqrt{2})^{P}\left(D_{1}\right)^{p-1} \delta_{11}+r R D_{1}-A_{1}\right\}
\end{array}\right.
$$

Case 4: if $\left\{D_{1}, D_{2}\right.$ : e $K_{4}=\left\{\{x, y\} e R^{2}, x=-y, y<0\right\}$, then

$$
\left\{\begin{array}{r}
\partial\left(D_{1}, D_{2}\right)=\frac{1}{P}\left(-k \sqrt{2} D_{2}\right)^{P}+\frac{r R}{2}\left(D_{1}^{2}+D_{2}^{2}\right)-A_{1} D_{1}-A_{2} D_{2} \\
\partial j\left(D_{1}, D_{2}\right)=\left\{\left(u_{1}, U_{2}\right), r R D_{1}-A_{1} \leqslant u_{1} \leqslant(k \sqrt{2})^{P}\left(D_{1}\right)^{P-1}+r R D_{1}-A_{1}\right. \\
\left.r R D_{2}-A_{2}-(k \sqrt{2})^{P}\left(-D_{2}\right)^{P-1} \leqslant u_{2} \leqslant r R D_{2}-A_{2}\right\}
\end{array}\right.
$$



Figure 7.1: level lines of $D_{1}$. $\quad(\mathrm{P}=2)$


Case 5: if $\left\{D_{1}, D_{2}\right\}$ e $x_{5}=\left\{\{x, y\}\right.$ e $\left.R^{2},-y>x>0\right\}$, then

$$
\begin{aligned}
& \left\{\begin{array}{l}
j\left(D_{1}, D_{2}\right)=\frac{1}{P}(k \sqrt{2})^{P_{1}}\left(-D_{2}\right)^{p}+\frac{R_{f}}{2}\left(D_{1}^{2}+D_{2}^{2}\right)-A_{1} D_{1}-A_{2} D_{2}, \\
\partial_{j}\left(D_{1}, D_{2}\right)=\left(\left(u_{1}, u_{2}\right), u_{i}=-(k \sqrt{2})^{p}\left(-D_{2}\right)^{p-1} \delta_{i 2}+R R D_{i}-A_{i}\right\} .
\end{array}\right. \\
& \text { Case 6: if }\left\{D_{1}, D_{2}\right\} e K_{6}=\left\{(x, y\} e R^{2}, x=0, y<0\right\}, \text { then }
\end{aligned}
$$

$$
\begin{aligned}
& \left\{\begin{array}{l}
\left.j\left(D_{1}, D_{2}\right)=\frac{1}{P}(k \sqrt{2}) P_{\left(-D_{2}\right.}\right)^{p}+\frac{r R}{2} D_{2}^{2}-A_{2} D_{2} \\
\partial j\left(D_{1}, D_{2}\right)=\left\{\left(u_{1}, u_{2}\right),-(k \sqrt{2})^{P}\left(-D_{2}\right)^{P-1}-A_{1}\left\langle u_{1}<-A_{1}, u_{2}=-(k \sqrt{2})^{P} P_{\left(-D_{2}\right.}\right)^{P-1}+r R D_{2}-A_{2}\right\} .
\end{array}\right. \\
& \text { Case 7: if } \left.\left\{D_{1}, D_{2}\right\} e k_{7}=\left\{\{x, y\} e R^{2}, 0\right\rangle x \geqslant y\right\}, \text { then }
\end{aligned}
$$

$$
\left\{\begin{array}{l}
j\left(D_{1}, D_{2}\right)=\frac{1}{p}\{k \sqrt{2})^{p}\left(-D_{1}-D_{2}\right)^{p}+\frac{R r}{2}\left(D_{1}^{2}+D_{2}^{2}\right)-A_{1} D_{1}-A_{2} D_{2} \\
\partial j\left(D_{1}, D_{2}\right)=\left\{\left(u_{1}, u_{2}\right), u_{i}=-(k \sqrt{2})^{p}\left(-D_{1}-D_{2}\right)^{p-1}+r R D_{i}-A_{i}\right\}
\end{array}\right.
$$

Observe that the sets $K_{1}$ form a partition of the half plane $D_{1} \geqslant D_{2}$. Then, by definition of (7.3) and since its solution $\left\{\mathrm{H}_{1}, \mathrm{H}_{2}\right\}$ belongs to this halfplane (Theorem 4.1), we have:
(7.4) $\left\{H_{1}, H_{2}\right\}=\underset{i=:}{7}\left\{\left\{x_{i}, y_{i}\right\}\right.$ e $x_{i},\{0,0\}$ e $\left.\partial j\left(x_{i}, y_{i}\right)\right\}$.

Therefore the solution of (7.3) is simply the solution of one of the local subproblems (the one which admits a solution for the given data of $A_{1}$ and $A_{2}$ )

$$
\{0,0\} \text { e } \partial j\left(x_{i}, y_{i}\right), \quad\left\{x_{i}, y_{i}\right\} \text { e } x_{i}
$$

Then, once that for each subproblem the conditions which guarantee the existence of solutions are explicited and that the algebraic expressions of these solutions are computed, $\left\{\mathrm{H}_{1}, \mathrm{H}_{2}\right\}$ is simply obtained by:
(i) finding which subproblem has solution for the given values of $A_{1}$ and $A_{2}$, by checking successively the admissibility requirements (conditions for existence of solutions) of each subproblem;
(ii) setting $\left\{\mathrm{H}_{1}, \mathrm{H}_{2}\right\}$ equal to the corresponding solution.

Here, these computations are easy to carry out since we have just computed the algebraic expressions of $\partial f(\cdot, \cdot)$ on each subset $R_{1}$. For example, for $i=1$, we have
local subproblem:

$$
\left\{\begin{array}{l}
(k \sqrt{2})^{P}(x+y)^{p-1}+r R x-A_{1}=0 \\
(k \sqrt{2})^{P}(x+y)^{p-1}+r R y-A_{2}=0 \\
x>y>0,
\end{array}\right.
$$

admissibility requirement (necessary and sufficient condition for existence of solutiona):

$$
A_{2}>\left[\left(A_{1}-A_{2}\right) / r R\right]^{p-1}
$$

solution

$$
\left\{\begin{array}{l}
x_{1}=\left(z+\left(A_{1}-A_{2}\right) / r R\right) / 2, y_{1}=\left(z-\left(A_{1}-A_{2}\right) / r R\right) / 2, \\
2 \text { minimizes }\left\{\frac{2}{p}(k \sqrt{2})_{t} P^{P}+\frac{r R}{2} t^{2}-\left(A_{1}+A_{2}\right) t\right\} \text { over } R_{+}
\end{array}\right.
$$

All computations done, the solution $\left\{\mathrm{H}_{1}, \mathrm{H}_{2}\right\}$ of Step 3 is finally given by:

$$
\begin{aligned}
& \text { for } A_{2}>\left(\left(A_{1}-A_{2}\right) / r R\right)^{P-1}\left\{\begin{array}{l}
H_{1}=\left[z+\left(A_{1}-A_{2}\right) / r R\right] / 2, \\
H_{2}=\left[z-\left(A_{1}-A_{2}\right) / r R\right) / 2, \\
z \text { minimizes }\left\{\frac{2}{P}(k \sqrt{2})_{t} P_{+} \frac{R r}{2} t^{2}-\left(A_{1}+A_{2}\right) t\right\} \text { over } R_{+},
\end{array}\right. \\
& \text {for }\left(\left(A_{1}-A_{2}\right) / r R\right)^{p-1}>A_{2}>0 \quad\left\{\begin{array}{l}
H_{1} \text { minimizes }\left\{\frac{1}{P}(k \sqrt{2})^{P_{t}}{ }^{p}+\frac{\operatorname{Rr}}{2} t^{2}-A_{1} t\right\} \text { over } R_{+}, ~
\end{array}\right. \\
& \text { for } A_{1}>\left(-A_{2}+\left(-A_{2} / R x\right)^{P-1}\right)>0\left\{\begin{array}{l}
H_{1} \underline{\text { minimizes }\left\{\frac{1}{P}(k \sqrt{2})_{t} P^{P}+\frac{R_{r}}{2} t^{2}-A, t\right\} \text { over } R_{+},} \\
H_{2}=A_{2} / R r i
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \text { and }\left(A_{1} / \Sigma R\right)^{p-1}+A_{1} \geqslant-A_{2} \quad\left\{\begin{array}{l}
H_{2}=-H_{1} ;
\end{array}\right. \\
& \text { for }-A_{2}>A_{1}+\left(A_{1} / r R\right)^{p-1}>0\left\{\begin{array}{l}
H_{1}=A_{1} / r R \\
\left(-H_{2}\right) \operatorname{minimizeg}\left(\frac{1}{p}(k / \overline{2}) P_{t} P_{+R r t}{ }^{2}+A_{2} t\right) \text { over } R_{+},
\end{array}\right. \\
& \text {for } \left.0 \geqslant A_{1}\right\rangle-\left(\left(A_{1}-A_{2}\right) / r R\right)^{P-1}\left\{\begin{array}{l}
H_{1}=0 \\
\left\langle-H_{2}\right\rangle \text { minimizes }\left\{\frac{1}{p}(k \sqrt{2})^{P_{t}}{ }^{p}+\frac{R x_{2}}{2} t^{2}+A_{2} t\right\} \text { over } R_{+}:
\end{array}\right. \\
& \text {for }-\left(\left(A_{1}-A_{2}\right) / r R\right)^{p-1}>A_{1}\left\{\begin{array}{l}
H_{1}=\left(2+\left(A_{1}-A_{2}\right) / r R\right) / 2, \\
H_{2}=\left(z-\left(A_{1}-A_{2}\right) / r R\right) / 2, \\
(-z) \text { minimizes }\left\{\frac{2}{p}\left(k / \overline{2}_{2}\right)_{t}{ }^{\rho}+\frac{r R_{t}}{2} t^{2}\left(A_{1}+A_{2}\right) t\right\} \text { over } R_{+} .
\end{array}\right.
\end{aligned}
$$

In the above formulas, the minimization over $R_{+}$is numericaliy achieved by uaing the one dimensional Newton algorithm described in Section 5 of this report.
7. 3 Numerical regult. We now consider a perforated square thin plate (width = .), subjected to an uniform traction of .52 per unit area on two of its opposite faces. This plate is supposed to be made of a tresca material with $p=1.5$ and $k=1 / \sqrt{2}$.

For symmetry reasons, only one fourth of the plate is considered. On this fourth, 126 nodes are used for approximating the velocity field. One block-ralaxation iteration is done per Uzawa step, and the parameter $R$ and the weight $r(x)$ are respectively given by $R=1$ and $f(x)=|G| P^{-2}$, $H$ being the deformation rate tensor corresponding to a computation done on the same geometry but with $D_{1}(G)={ }_{p}^{1}(k \sqrt{2})^{p}|G|^{p} \quad$ (compressible Norton material in plane atraina).

After 50 iterations, the error $10\left(\nabla^{n}\right)$ - $A^{n}$ is decreased by a factor of $10^{-4}$, and the total dissipated energy rate is equal to $\mathbf{- 2 . 7 3 7}$ for the whole plate. The corresponding velocitites are indicated on Fig 7.2. It must be noticed that, due to the little number of boundary conditions imposed on $v$, this case is particularly unstable for most numerical methods.

8 POSSIALE EXTENSIONS OF THE METHOD.
Many extensions can be considered for the numerical method described in this report. For example.
(1) different finite elements can be considered in the approximation $x_{h}$ of the set of kinematically admiasible incompressible velocity fields. Any finite element which is used with some success in the approximation of the stokes problem can be employed here. Nevertheless, if the gradients of the elements of $X_{h}$ are not piecewise constant, a numerical integration rule will be necessary to compute the dissipation $F(\epsilon)$, which leads to an additional truncation arror and which slightly complicates problem (3.6). Moreover, the space $Y_{h}$, which is then the pace of functions which are characterized by


Figure 7.3
their values at the integration points, must be gufficiently large to contain, within an isomorphism, the image of $K_{h}$ by the operator $D(*)$;
(ii) an inertia term can be added in the formulation of the virtual work theorem. Through an implicit time discretization, the resulting problem will then reduce to a sequence of augmented lagrangian problems (3.6) (one per time step), the functional $G$ being now replaced by

$$
G(w)=-\int_{\Omega} f \cdot w d x-\int_{\Gamma_{2}} g^{*} v d a+\frac{1}{2 D T} \int_{\Omega} \rho\left|v-\nabla_{n}\right|^{2} d x .
$$

Each problem (3.6) can atill be solved by Algorithm (3.11) - (3.14). Problem (3.12) will again correspond to a linear Stokes type problem, associated to fixed, symutric, positive definite finite elament matrices. Problem (3.13) remains unchanged;
(1i1) a convection term $\rho\left(\nabla^{*} \nabla\right) \nabla$ can also be added in the formulation of the virtual work theorem. Since the operator in $v$ will no longer be aelf-adjoint, no augmented lagrangian $L_{R}$ can then be introduced. Nevertheless, Algorithm (3.11) - (3.14) is still applicable there (FORTIN-GLOWINSKI [1982, p 71]. Problem (3.13) is unchanged, and (3.12) becomes

$$
\int_{\Omega} r(x)(R(D(v)-\nabla)-\lambda) \cdot D(v) d x+\int_{\Omega} \rho(v \cdot \nabla) v \cdot w d x=\int_{\Omega} f \cdot v d x+\int_{\Gamma_{2}} v \cdot v d a, v=e X_{h} .
$$

For mall convection terms, $v$ can be replaced in the convection term by the solution $\boldsymbol{v}_{\mathrm{k}}^{\mathrm{n}}$ at the previous iterate, and (3.12) then reduces to an ordinary Stokes problem. For large convection terms, one can use optimal control techniques ([GLONINSKI-LE TALLEC [1983]). All this is degcribed in details by TANGUY [1983] which uses augmented lagrangian techniques in a very similar situation;
(iv) finally, our problem can be coupled to an heat diffusion problem if we suppose for example, that the internal disaipation potential $D_{1}(x, G)$ is a function of the temperature $t$ at $x$. If convection phenomenon are not dominant, temperatures and velocities can be efficiently computed by block-relaxation: assuming the velocity to be given, one computes the temperature by solving the energy equations then, assuming the
temperature to be given, the velocity is determined by solving (3.6), the process being repeated until convergence. Observe that, despite a possible change of the temperature field between two successive resolutions of (3.6), the finite element matrices do not have to be changed because the temperature is only a parameter in the local problems (4.1). This results in considerable economy in computer running time.

## References

BERCOVIER, M., PIRONNEAJ, O. [1977]: Estimation d'erreurs pour la resolution d'un probleme de Stokes en elements finis conformes de Lagrange. CRAS Paris, T286A, p 1085-1087.

BREZZI, F. [1974]: On the existence, uniqueness and approximation of saddlepoint problems arising from Lagrange multipliers. RAIRO Anal. Numer. 8R2, p 129-151.

CIARLET, P. G. [1978]: The finite element method for elliptic problems. Amsterdan, North Holland.

EKELAND, I., TGMAN, R. [1978]: Convex analyais and variational problems. Amaterdan, North Holland.

FORTIN, M., GLOWINSXI, R. [1982] (eds): Methodes de lagrangien augmente. Dunod-Bordas, Paris.

GEYMONAT, G., SUQUET, P. [1983]: Espaces fonctionnels pour les milieux de. Norton-Hoff (to appear).

GIRAULT, V., RAVIART, P. A. [1979]: Finite element approximation of the Navier-Stokes equations. Berlin, Heidelberg, New York. Springer Verlag.

GLOWINSKI, R. [1984]: Numerical methods for nonlinear variational problems. Berlin, Heidelberg, New York. Springer Verlag.

GLOWINSKI, R., LE TALLEC, P. [1983]: Numerical solution of partial differential equation problems in nonlinear mechanics by quadratic minimization methods. Conference on large scale acientific computing. Madison May 16-18, 1983. Academic Press.

[^0]HILL, R. [1970]: Constitutive inequalities for isotropic elastic solids under finite strain. Proc. Roy. Soc. London A314 p 457-472.

LIONS, P. L., MERCIEP, B. [1979]: Splitting algorithma for the sum of two nonlinear operators. SIMM J. Numer. Anal., 16, p 964-979.

PFRZYNA [1966]: Fundamental problems in plasticity. Adv. Appl. Mech. 9, p 243-377.
SCHEURER, B. [1977]: Pxistence et approximation de points-selles pour certains prohlemes non lineaires. RAIRO, serie rouge, Anal. Numer.., 11, p 369-400.

Tanguy, P. [1983]: Numerical Simulation of a Pseudo 3-D Turbulent Flow in a Kaplan Turbine Diffuser. Technical Report TR-83-10. Department of Chemical Engineering, University of Nova Scotia.

TAYLOR, C., HOOD, P. [1973]: A numerical solution of the Navier-Stokes equations using the finite element technique. Computers and Fluids, 1, p 73-100.

VON NELMANN, J. [1937]: Some matrix inequalities and metrization of matric-space, Tomsk Univ. Rev. 1, p 286-300. Reprinted in collected works, Vol iv, Pergamon, Oxford, 1962.

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