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Pontoppidan, K

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NUMERICAL SOLUTION OF WAVEGUIDE PROBLEMS
USING FINITE DIFFERENCE METHODS

K. Pontoppidan

Introduction   In a paper by Davies and Muilwyk [1] the finite difference method was described and used for waveguide problems. The basic idea is to subdivide the waveguide cross section by a grid into a number of small squares. A difference formula is set up for each mesh point of the grid and the resulting matrix eigenvalue problem is solved by successive overrelaxation. Only the dominant mode is found using this method. Beaubien and Wexler [2] modified the method in order to be able to obtain convergence towards any desired mode.

In both of the abovementioned methods, the boundary is approximated by line segments which join the outermost mesh points. For waveguides for which the boundary coincides with the grid lines, the methods are very accurate. However, for waveguides with curved boundaries, the approximation of the boundary reduces the accuracy significantly.

The aim of this paper is to present a modified version of the finite difference method by which it is possible, also for curved boundaries, to accurately determine the dominant mode and a number of the higher order modes. The determination of higher order modes is based on orthogonalization with respect to all lower order modes which therefore must be stored during the computations. As a consequence of this, only relatively coarse meshes can be used. In order not to loose accuracy in case of curved boundaries as described above, additional points on the boundary are used together with the ordinary mesh points.

Finite difference solution   A uniform, lossless, homogeneously filled waveguide of arbitrary cross section is considered. The electromagnetic fields in the waveguide can be determined from the solution ϕ to the Helmholz equation

$$\nabla^2\phi + k^2\phi = 0 \quad , \tag{1}$$

where k is the cut-off wave number. The boundary conditions are

$$\phi = 0 \quad \text{for TM-modes} \quad \text{and} \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{for TE-modes.} \tag{2},(3)$$

The cross section of the waveguide is subdivided into squares by a grid as shown in fig. 1 and the field is to be determined in each mesh point of the grid. The distance between the grid lines is h. For interior points, the Helmholz equation is approximated by a finite difference formula. For the point o and its four neighbours 1-4 in fig. 1, this is

$$\frac{\phi_1/h_1 + \phi_2/h_2}{h_1 + h_2} + \frac{\phi_3/h_3 + \phi_4/h_4}{h_3 + h_4} - \left(\frac{1}{h_1 h_2} + \frac{1}{h_3 h_4}\right)\phi_o + \tfrac{1}{2}k^2\phi_o = 0 \quad . \tag{4}$$

For points for which none of the neighbours are on the boundary, (4) is reduced to

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_o + (kh)^2\phi_o = 0 \quad . \tag{5}$$

For boundary points, which means points of intersection between the

K. Pontoppidan is with the Laboratory of Electromagnetic Theory, Technical University of Denmark.
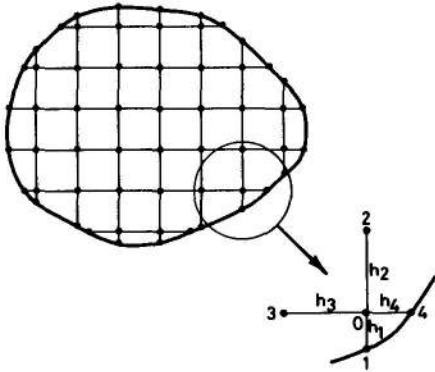
Fig. 1. Subdivision of wave-
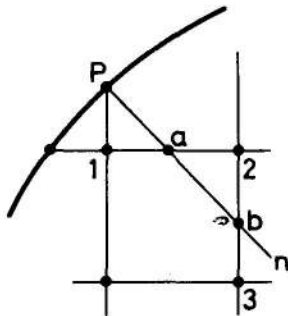guide cross section

Fig. 2. Boundary point with
corresponding normal

boundary and the grid, such as point P in fig.
2, the field value is determined by the boun-
dary conditions. For TM-modes this is $\phi_P = 0$.
To satisfy the boundary condition for TE-modes
the following method has been applied. The
points a and b are found as points of inter-
section between the normal n to the boundary
and the grid lines. The field values ϕ_a and
ϕ_b in a and b are determined by linear inter-
polation between ϕ_1 and ϕ_2 and between ϕ_2 and
ϕ_3, respectively. The field along n is ap-
proximated by a second order polynomial which
has the values ϕ_a and ϕ_b in a and b and for
which the derivative in P is zero. Analyti-
cally, the field ϕ_P is expressed by

$$\phi_P = \frac{\phi_a h_b^2 - \phi_b h_a^2}{h_b^2 - h_a^2} \quad , \tag{6}$$

where h_a and h_b are the distances from P to a
and b, respectively.

Now, the computations proceed as follows.
An initial mesh size h is chosen (normally 1/4
or 1/8 of a dominant length in the cross sec-
tion). The boundary shape is given by a func-
tion expression and the distance from each
boundary point to the nearest interior point
is calculated and stored. All initial field
values are set equal to unity except the boundary points for TM-modes which
are set equal to zero. For the initial eigenvalue, k = 1 is used.

The finite difference formula (4) or (5) for all interior points
forms a system of equations which is solved by successive overrelaxation.
After each completed relaxation process, the boundary values are adjusted
according to (6) in the case of TE-modes. For TM-modes, the boundary
points are left unchanged equal to zero. The equations cannot be satisfied
since the eigenvalue is not correct. A new and better eigenvalue is ob-
tained by the Rayleigh quotient

$$k^2 = - \frac{\int_A \phi \nabla^2 \phi \, da}{\int_A \phi^2 \, da} \quad , \tag{7}$$

where A represents the area of the cross section. The integrals in (7) are
evaluated numerically from the current field values. The relaxation is re-
peated with the new eigenvalue and the process is continued until two con-
secutive eigenvalues only differ by a certain prescribed amount.

Orthogonalization   The method described above is known to converge to-
wards the smallest eigenvalue and the corresponding eigenfunction. Higher
order modes, however, may be found in the same way if they, in addition,
are required to be orthogonal to all previously determined lower order
modes [3].

Let ϕ_m' represent the current field of mode number m after one relaxa-
tion and suppose, that all lower order modes have been determined. ϕ_m' is
usually not orthogonal to the lower order modes and therefore a new ϕ_m is
calculated in all points of the mesh by

$$\phi_m = \phi_m' - \sum_{n=1}^{m-1} \frac{\int_A \phi_m'\phi_n \, da}{\int_A \phi_n^2 \, da} \phi_n \quad , \tag{8}$$

for which the orthogonality condition is satisfied.

The determination of higher order modes is performed as follows. Having calculated the first order mode to the required accuracy, the second order mode is determined in the same way and by applying (8) after each relaxation. This procedure is continued until all required modes are determined.

When the problem is solved for the first coarse mesh, a finer mesh is generated by halving the distance between grid lines. In doing this, the number of mesh points is increased by a factor four and the fields in the new mesh points are determined by linear interpolation from the already known values. Again, the relaxation procedure is repeated for each mode until the required accuracy is obtained. The mesh-halving technique can be applied as far as storage requirement permits.

Computed results   The computer program has been tested for waveguides with known solution. When three consecutive eigenvalues have been found, an extrapolated eigenvalue is determined by Aitken extrapolation. Especially for TM-modes, this gives a very significant improvement of the result.

Tables 1 and 2 show the eigenvalues obtained for the three lowest order modes in a circular waveguide of diameter 2a. It is seen that the

Table 1, Circular TE-modes

$h/2a$	TE_{11}			TE_{21}			TE_{01}		
	ka	error,%	extr.	ka	error,%	extr.	ka	error,%	extr.
1/8	1.8731	1.7		3.2354	6.0		3.7950	1.0	
1/16	1.8599	1.0		3.1407	2.8		3.8231	0.2	
1/32	1.8516	0.6	1.8372	3.0920	1.2	3.0405	3.8327	0.03	
1/64	1.8475	0.3	1.8436	3.0755	0.7	3.0670			3.8377
exact	1.8412			3.0542			3.8317		

Table 2, Circular TM-modes

$h/2a$	TM_{01}			TM_{11}			TM_{21}		
	ka	error,%	extr.	ka	error,%	extr.	ka	error,%	extr.
1/8	2.3836	0.9		3.7437	2.3		4.9739	3.1	
1/16	2.3987	0.25		3.8071	0.6		5.0863	1.0	
1/32	2.4033	0.06	2.4052	3.8254	0.16	3.8328	5.1229	0.25	
1/64	2.4044	0.017	2.4049	3.8302	0.04	3.8319			5.1405
exact	2.4048			3.8317			5.1356		

error is almost proportional to h for TE-modes and proportional to h^2 for TM-modes. The lesser rate of convergence for TE-modes is probably due to the more complicated boundary condition.

Figs. 3a and 3b show the error for different mesh sizes for the dominant modes in the circular waveguide. For comparison, the results obtained by Davies and Muilwyk are also shown. For the TE_{11}-mode, the existence of the additional boundary points used in the present theory gives an accuracy which, without these boundary points, is obtained only after further 2-3 mesh-halvings. Since each mesh-halving involves four times as many mesh points, both computing time and core storage are significantly reduced. For the TM_{01}-mode, the improvement of the accuracy is even more pronounced because of the more rapid convergence in this case. In fig. 4 the convergence of a higher order mode is shown, namely the second order
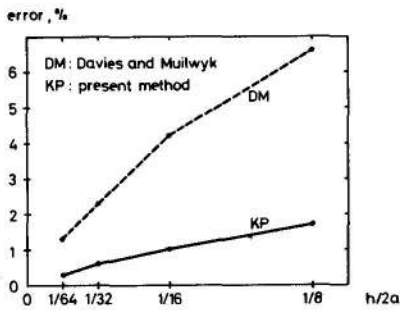
error, %

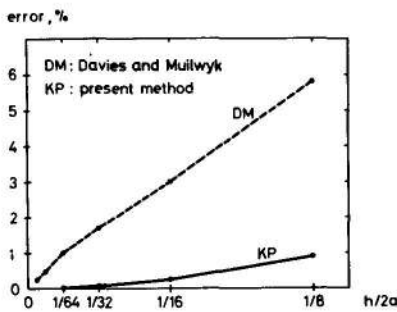Fig. 3a. Convergence, for
circular TE_{11}-mode

error, %

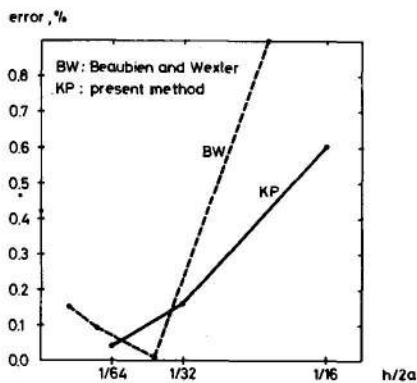Fig. 3b. Convergence, for
circular TM_{01}-mode

error, %

Fig. 4. Convergence, for
circular TM_{11}-mode

TM_{11}-mode in a circular waveguide. The results obtained by Beaubien and Wexler are shown, too, for comparison. Especially for small mesh lengths, the convergence is improved. The increasing error (which is due to the fact, that the correct value has been passed) observed in the result by Beaubien and Wexler is probably again due to their less accurate fitting of the boundary.

Test computations for rectangular cross sections give, as expected, results which are similar to those obtained by the two above-mentioned methods.

The number of modes that can be determined by this method is limited by the available computer core storage and the required accuracy. If about four modes are required, the computing times per mode are roughly one minute for TE-modes and 20 sec. for TM-modes. The computations have been made on an IBM 7094 computer and a maximum of about 10.000 locations have been used for the storage of up to six modes.

Conclusion   Using the method presented here, it is possible to determine the dominant mode as well as a number of the higher order modes in waveguides of arbitrary cross section. The method requires little computer time and the accuracy for cross sections with curved boundaries is improved compared to other methods. The limitation of the method is that for the determination of a single higher order mode, it is necessary to determine and store all lower order modes.

[1] J.B. Davies and C.A. Muilwyk: "Numerical solution of uniform hollow waveguides with boundaries of arbitrary shape", Proc. IEE, Vol. 113, No. 2, 1966.

[2] M.J. Beaubien and A. Wexler: "An accurate finite-difference method for higher order waveguide modes", IEEE Trans. Microwave Theory and Techniques, Vol. MTT-16, No. 12, 1968.

[3] F.E. Borgnis and C.H. Papas: "Randwertprobleme der Mikrowellenphysik", Springer, 1955.