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Numerical solution to the optimal feedback control of continuous casting process

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Abstract Using a semi-discrete model that describes the heat transfer of a continuous casting process of steel, this paper is addressed to an optimal control problem of the continuous casting process in the secondary cooling zone with water spray control. The approach is based on the Hamilton–Jacobi–Bellman equation satisfied by the value function. It is shown that the value function is the viscosity solution of the Hamilton–Jacobi–Bellman equation. The optimal feedback control is found numerically by solving the associated Hamilton–Jacobi–Bellman equation through a designed finite difference scheme. The validity of the optimality of the obtained control is experimented numerically through comparisons with different admissible controls. Detailed study of a low-carbon billet caster is presented.

Keywords Continuous casting · Viscosity solution · Hamilton–Jacobi–Bellman equation · Finite difference scheme · Optimal feedback control

1 Introduction

Continuous casting is widely used in the steel industry for the casting of different grades of steel. In the continuous casting process, the aim is to solidify molten steel into a solid structure with as few defects as possible. A brief description of the process is as follows. The molten steel arrives at the continuous caster in a ladle (see Fig. 1). The ladle feeds the molten steel into the tundish, which acts as a reservoir of the molten steel. The tundish feeds the mould with liquid steel through a stopper rod and

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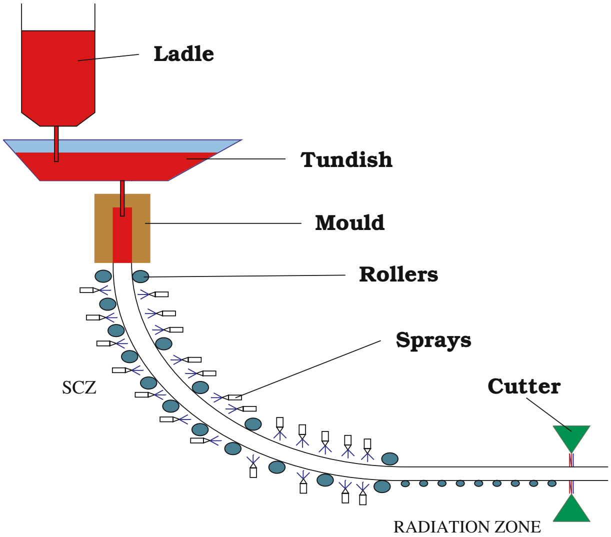

Fig. 1 The continuous casting process

submerged entry nozzle (SEN) system. The primary extraction of heat occurs in the water-cooled copper mould. Below the mould, heat is extracted from the strand by means of the water spray. This region is known as the secondary cooling zone (SCZ). In the SCZ, the strand is supported by rollers. After the SCZ, the strand cools off naturally by the air in the radiation zone. After the radiation zone the strand is cut and sent for further processing such as rolling into sheet metal.

The control of the temperature profile in the secondary cooling zone can contribute to improving the quality of the cast product. However, it was observed in Brimacombe [3] that improper cooling such as excessive reheating in the secondary cooling zone severely contributes to crack formation on the surface and in the interior of the strand. Therefore, one should require the water spray that, on the one hand, make the temperature profile close to the desired profile as much as possible at the end of the SCZ, and on the other hand, keep the changes of the temperature profile in the SCZ in a reasonable scope to avoid the improper cooling.

This leads to the optimal control problem in the SCZ to be considered in this paper. It is generally recognized that finding the closed-form solution to the optimal feedback control for a nonlinear system is formidable. In contrast to the efforts in analytic way, numerous works have been done for the numerical solution of optimal control problems. Basically these are direct and indirect methods. The indirect method gets the solution of optimal control problem by solving a two-point boundary value problem given by the necessary conditions of optimality, usually the Pontryagin maximum principle. However, the Pontryagin maximum principle usually gives only the optimal control in open-loop form if it does exist. Moreover, the indirect method that is mainly the multiple shooting method has happened the difficulty in "guess" of the initial data [18]. For the direct method, its simplifying the original problem leads to the fall of the reliability and accuracy [19], and exhibits a performance decay for increasing problem size.

Correspondingly, it has been realized from Pontryagin's time that the value function that satisfies some Hamilton–Jacobi–Bellman equation can give the optimal feedback solution to the optimal control problem. The new difficulty is that the Hamilton–Jacobi–Bellman equation may have no classical solution no matter how smooth its coefficients are. The fundamental turn comes when the viscosity solution of the Hamilton–Jacobi–Bellman equation was introduced in 1980s [7]: the value function is the unique solution of the associated Hamilton–Jacobi–Bellman equation. Many references for viscosity solution of Hamilton–Jacobi–Bellman equation are already available in literature, which cover both finite-dimensional [1] and infinite-dimensional optimal control problems [2,4,9,11,14]. Moreover, some substantial progresses have been made for those algorithms of solving numerically the finite-dimensional Hamilton–Jacobi–Bellman equations [5,6,8,20,21].

In this paper, we study an optimal control problem for the continuous casting of steel via the viscosity solution approach. This leads to the numerical solution of the optimal feedback control, which is different to the study in Miettinen et al. [15] where the optimal control problem of continuous casting with non-differentiable multi-objective optimization was investigated and solved by the interactive NIMBUS method.

This paper is organized as follows. In the next section, Sect. 2, a brief overview of a semi-discrete model developed in Guo et al. [10] with concrete boundary conditions in different zones is presented, and an example of a low-carbon billet caster is given to demonstrate numerical solutions of Eq. 2.4 that are needed for the numerical solution of optimal control in Sect. 5. In Sect. 3, we formulate the optimal control problem in the SCZ. The dynamic programming principle for the value function of the optimal control problem is established. Section 4 is devoted to show that the value function is just the viscosity solution of the corresponding Hamilton–Jacobi–Bellman equation, and the optimal feedback control is thereby formulated by the value function under the smooth assumption. In the last section, Sect. 5, we design a finite difference scheme to the numerical solution of the associated Hamilton–Jacobi–Bellman equation of the semi-discrete model, and numerical solutions of the optimal feedback control are presented. Finally, the validity of the optimality of the obtained control is experimented numerically through comparisons with other admissible controls and trajectories.

2 Semi-discrete model

Suppose that the cross section of the billet is a rectangular $\Omega = [0,a] \times [0,b]$ which is moving along the z direction with a constant speed v. Let $P = P(x,y,z)$ be the temperature at the point (x,y,z). Set $W(x,y,t) = P(x,y,z), z = vt$. Then W satisfies the following nonlinear heat conduction equation with boundary condition [12]:

$$\begin{aligned}
&\rho(W)[c(W) + Lf(W)]\frac{\partial W}{\partial t} = \operatorname{div}(K(W)\nabla W), \quad (x,y) \in \Omega, 0 \le t \le t^*,\\
&-K(W)\frac{\partial W}{\partial n} = Q(x,y,t,W), (x,y) \in \Gamma,\\
&W(x,y,0) = W_{\text{mold}},
\end{aligned} \tag{2.1}$$

where Γ is the boundary of Ω. $c(W)$ denotes the specific heat, $\rho(W)$ the density, and $K(W)$ the thermal conductivity. W_{mold} is the pouring temperature at the beginning of the mould, n the outward normal unity vector of Γ and $t^* = t_1 + t_2 + t_3$, where $vt_1, v(t_2 - t_1), v(t^* - t_2)$ denote the length of the mould, the SCZ and the radiation zone, respectively. L is the latent heat and $f(W)$ is a function that describes the solid-phase fraction variation with temperature. All these parameter functions are assumed to be bounded, positive, and differentiable in W. Q is the heat flux on the boundary [3]:

$$Q(x,y,t,W) = \begin{cases} Q_t\left[-\dfrac{3(1-c_w)}{2+c_w}\left(\dfrac{r_d - r}{r_d}\right)^2 + \dfrac{3}{2+c_w}\right], & \text{in the mould;} \\ h(W - W_{H_2O}) + \sigma\varepsilon(W^4 - W_{\text{ext}}^4), & \text{in the SCZ;} \\ \sigma\varepsilon(W^4 - W_{\text{ext}}^4), & \text{in the radiation zone,} \end{cases} \tag{2.2}$$

where W_{H_2O} is the spray–water temperature, W_{ext} the ambient spray zone temperature, σ the Stefan–Boltzmann constant, ε the emission factor. h is the heat-transfer coefficient which is determined by the water spray in the SCZ and hence is the real control variable. c_w is a constant representing the ratio of the heat flux in the corner of the mould relative to the heat flux at the middle surface. r_{d} is half of the width of the mould. Q_{t} is assumed to be [3]

$$Q_{\text{t}} = \frac{6(\alpha - \beta\sqrt{t_c})}{1 + 2c_a}\left(1 + \sqrt{\frac{t}{t_c}}(c_a - 1)\right), \tag{2.3}$$

where α and β are constants, and t_c is the dwell time in the mould. c_a is the ratio of the heat flux at the mould exit to the heat flux at the top level of liquid steel (meniscus) in the mould. $r = x$ at $(x, 0, t)$ or (x, b, t) and $r = y$ at $(0, y, t)$ or (a, y, t).

Due to the symmetry of the cross section of the billet, only one quarter region $\Omega_0 = \{(x,y) |\ 0 < x < a/2, 0 < y < b/2\}$ is considered. Let us briefly overview the semi-discrete modeling of the continuous casting [10]. Denote by $T_{ij}(t) = T(i\Delta x, j\Delta y, t)$ for fixed Δx and $\Delta y, 0 \le i \le m, 0 \le j \le n, m \ge n$. The following semi-discrete approximation Eq. of 2.1 was developed in Guo et al. [10]:

$$\begin{aligned} \frac{\mathrm{d}T(t)}{\mathrm{d}t} &= \mathcal{F}(T(t))(\mathcal{A}T(t) + \mathcal{B}U(t)), \\ T(0) &= g(W_{\text{mold}}), \end{aligned} \tag{2.4}$$

where $T = (T_0, T_1, \ldots, T_n)^T, T_j = (T_{mj}, T_{(m-1)j}, \ldots, T_{jj}, \ldots, T_{jn})^T, j = 0, 1, 2, \ldots, n$, $U = (u_{1m}, \ldots, u_{10}, u_{20}, \ldots, u_{2n})^T$,

$$\begin{aligned} u_{1i}(t) &= Q(i\Delta x, 0, t, g^-(T_{i0}(t))), \\ u_{2j}(t) &= Q(0, j\Delta y, t, g^-(T_{0j}(t))), \\ &\quad i = 0, 1, \ldots, m, \quad j = 0, 1, \ldots, n. \end{aligned} \tag{2.5}$$

$$\mathcal{A} = \begin{pmatrix} A_0 & C_0 & 0 & \cdots & 0 & 0 & 0 \\ B_1 & A_1 & C_1 & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ 0 & 0 & 0 & \cdots & B_{n-1} & A_{n-1} & C_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & B_n & A_n \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B \\ 0 \\ \cdots \\ 0 \end{pmatrix}.$$

And g^- is the inverse function of the Kirchhoff transform [12]:

$$T = g(W) = \int_{W^0}^{W} K(\rho)\mathrm{d}\rho \tag{2.6}$$

for a given W^0. $F^*(T_j) = \mathrm{diag}\{F(T_{ij})\}, \mathcal{F}(T) = \mathrm{diag}\{F^*(T_j)\}$ are diagonal matrices and

$$F(T) = \frac{K(g^-(T))}{\rho(g^-(T))[c(g^-(T)) + Lf(g^-(T))]}. \tag{2.7}$$

For $1 \le j \le n-1$,

$$A_j = (-\tfrac{2}{\Delta x^2} - \tfrac{2}{\Delta y^2})I_{m+n-2j+1} + [A_{j1}, A_{j2}],$$

$$A_{j1} = \begin{pmatrix} 0 & \frac{2}{\Delta x^2} & 0 & \cdots & 0 & \mathbf{0_{1,m-j+1}} \\ \frac{1}{\Delta x^2} & 0 & \frac{1}{\Delta x^2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{m-j+1,1} & 0 & 0 & \cdots & \frac{1}{\Delta x^2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\Delta y^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$A_{j2} = \begin{pmatrix} \mathbf{0_{1,m-j+2}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\Delta y^2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\Delta y^2} & 0 & \cdots & 0 & 0 \\ \frac{1}{\Delta y^2} & 0 & \frac{1}{\Delta y^2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{2}{\Delta y^2} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} -\frac{2}{\Delta y}I_m & 0 & 0 & 0 \\ 0 & -\frac{2}{\Delta y} & -\frac{2}{\Delta x} & 0 \\ 0 & 0 & 0 & -\frac{2}{\Delta x}I_n \end{pmatrix},$$

$$A_n = -(\tfrac{2}{\Delta x^2} + \tfrac{2}{\Delta y^2})I_{m-n+1} + \begin{pmatrix} 0 & \frac{2}{\Delta x^2} & 0 & \cdots & 0 & 0 & 0 \\ \frac{2}{\Delta x^2} & 0 & \frac{2}{\Delta x^2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{2}{\Delta x^2} & 0 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} \frac{2}{\Delta y^2}I_{m-1} & 0 & 0 \\ 0 & \frac{2}{\Delta y^2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{2}{\Delta x^2} & 0 \\ 0 & 0 & \frac{2}{\Delta x^2}I_{n-1} \end{pmatrix}, \quad C_j = \begin{pmatrix} \frac{1}{\Delta y^2}I_{m-j-1} & 0 & 0 \\ 0 & \frac{1}{\Delta y^2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\Delta x^2} & 0 \\ 0 & 0 & \frac{1}{\Delta x^2}I_{n-j-1} \end{pmatrix},$$

$$B_j = \begin{pmatrix} \frac{1}{\Delta y^2} I_{m-j} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta y^2} & 0 & \frac{1}{\Delta x^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\Delta x^2} I_{n-j} \end{pmatrix}, \quad B_n = \begin{pmatrix} \frac{2}{\Delta y^2} I_{m-n} & & \mathbf{0} & \\ \mathbf{0} & \frac{2}{\Delta y^2} & 0 & \frac{2}{\Delta x^2} \end{pmatrix}$$

where $\mathbf{0}_{i,j}$ stands for the entry 0 at the position (i,j) in the matrices, and I_k denotes the $k \times k$ identity matrix.

It is seen that the Eq. 2.4 is a standard lumped control system with state variable T and input U. This is one of the advantages of the semi-discrete model of continuous casting process compared with the infinite-dimensional formulation (2.1). The control variable U appears only in the SCZ. Specifically, in the mould,

$$\begin{aligned} u_{1i}(t) &= Q_t \left[-\frac{3(1-c_w)}{2+c_w} \left(\frac{a/2 - i\Delta x}{a/2} \right)^2 + \frac{3}{2+c_w} \right], \\ u_{2j}(t) &= Q_t \left[-\frac{3(1-c_w)}{2+c_w} \left(\frac{b/2 - j\Delta y}{b/2} \right)^2 + \frac{3}{2+c_w} \right], \\ i &= 0,1,\ldots,m, \quad j = 0,1,\ldots,n. \end{aligned} \tag{2.8}$$

In the SCZ,

$$U = \sigma\varepsilon((g^-(T_0))^4 - W_{\text{ext}}^4) + (g^-(T_0) - W_{\text{H}_2\text{O}})h, h = (h_{1m},\ldots,h_{10},h_{20},\ldots,h_{2n})^T,$$

$$\begin{aligned} h_{1i}(t) &= h(i\Delta x, 0, t), \quad i = 0,1,\ldots,m, \\ h_{2j}(t) &= h(0, j\Delta y, t), \quad j = 0,1,\ldots,n. \end{aligned} \tag{2.9}$$

Note that $(g^-(T_0))^4 - W_{\text{ext}}^4$ denotes the column vector $((g^-(T_{i0}))^4 - W_{\text{ext}}^4, (g^-(T_{0j}))^4 - W_{\text{ext}}^4)^T$. Therefore, in the SCZ, the Eq. 2.4 becomes

$$\begin{aligned} \frac{\mathrm{d}T(t)}{\mathrm{d}t} &= \mathcal{F}(T(t))[\mathcal{A}T(t) + \mathcal{B}\sigma\varepsilon((g^-(T_0(t)))^4 - W_{\text{ext}}^4) \\ &\quad + \mathcal{B}(g^-(T_0(t)) - W_{\text{H}_2\text{O}})h(t)] \text{ for almost all } t \in (t_1, t_2), \\ T(t_1) &= S^0, \end{aligned} \tag{2.10}$$

where h is the control variable determined by the water spray in the SCZ. The $g^-(S^0)$ is the temperature profile of the end section of the mould. S^0 can be found through solving Eqs. 2.4 and 2.8 because the pouring temperature W_{mold} is usually known. In the radiation zone,

$$U = \sigma\varepsilon((g^-(T_0))^4 - W_{\text{ext}}^4).$$

For any bounded measurable function h, the solution to (2.10) is understood to be the solution of the following integral equation [17], p. 345

$$\begin{aligned} T(t) = S^0 + \int_{t_1}^{t} & \mathcal{F}(T(\rho))[\mathcal{A}T(\rho) + \mathcal{B}\sigma\varepsilon((g^-(T_0(\rho)))^4 - W_{\text{ext}}^4) \\ & + \mathcal{B}(g^-(T_0(\rho)) - W_{\text{H}_2\text{O}})h(\rho)]d\rho \quad \text{for all } t \in [t_1, t_2]. \end{aligned} \tag{2.11}$$

Proposition 1 *For any bounded measurable $h(\cdot)$, the following two assertions hold.*

(1) *There exists a unique continuous solution T to Eq. 2.11 with respect to (t_1, S^0) in $[t_1, t_2]$.*

(2) $\|T^1(t) - T^2(t)\| \leq C_2 \left[\|S^1 - S^2\| + \int_{t_1}^{t_2} \|h^1(\rho) - h^2(\rho)\| d\rho\right]$ *for all $t \in [t_1, t_2]$, where C_2 is a constant and T^i is the solution to (2.11) corresponding to h^i and initial condition $T^i(t_1) = S^i, i = 1, 2$.*

Proof Under assumptions on these parameter functions of (2.1), for any fixed t, the function on the right hand side of (2.10) is differentiable in T and is bounded in any compact set of T and $t \in [t_1, t_2]$. It follows from Theorem 36 on p.347 and Proposition C.3.4 on p.351 of Sontag [17] that (2.11) admits a unique local continuous solution. A simple argument shows that this local solution can be expanded to the whole interval $[t_1, t_2]$. This is (1). As for (2), by assumptions on these parameter functions of (2.1) and (1), it is easy to get that

$$\|T^1(t) - T^2(t)\| \leq M_1 \left(\|S^1 - S^2\| + \int_{t_1}^{t_2} \|h^1(\rho) - h^2(\rho)\| d\rho\right) + M_2 \int_{t_1}^{t} \|T^1(\rho) - T^2(\rho)\| d\rho$$

for some constants $M_i, i = 1, 2$. The assertion (2) then follows from the Gronwall's inequality. □

Now we use an example of a low-carbon billet caster to demonstrate numerical solutions of Eq. 2.4 that are needed for the optimal control computation in Sect. 5. The section is $10 \times 10\,\text{cm}^2$ and the length of the mould and the SCZ are 0.7 m and 5 m, respectively.

Let W be the temperature profile as in Eq. 2.1, $W_{ij}(t) = W(i\Delta x, j\Delta y, t), i = 0, 1, \ldots, m, j = 0, 1, \ldots, n$. Since we consider only one quarter of the section, we can require that $m\Delta x = a/2, n\Delta y = b/2$. Let

$$\begin{aligned} W(t) &= (W_0(t), W_1(t), \ldots, W_n(t))^T, \\ W_j(t) &= (W_{mj}(t), \ldots, W_{jj}(t), \ldots, W_{jn}(t))^T, j = 0, 1, 2, \ldots, n. \end{aligned} \tag{2.12}$$

Then $W_j(t)$ denotes the temperature profile at these grid points of the jth layer of the section at time t. For a one quarter of the section [0.5 cm] × [0.5 cm], we take $\Delta x = \Delta y = 1$ cm. Hence, $m = n = 5$ in (2.12). There are total of 36 grid points in the region. The lowest temperature is the corner point W_{00} since the water sprays from two sides to this point and the highest temperature is the center point W_{55} since it is the center of the section. Other parameters are listed in Table 1.

Numerical solutions to (2.4) in the sequel are obtained by the classical Runge–Kutta method for all 36 grid points, and all values of T are transformed back to W through $W = g^{-}(T)$.

In order to solve numerically the Eq. 2.4, we need the nonlinear functions $F(T)$ in (2.7) and $g(W)$ in (2.6). $g(W)$ can be found by solving the following ordinary differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}W} = K(W), T(W^0) = 0 \tag{2.13}$$

Table 1 Parameters(=P-) used for the numerical simulation

P-	value	*P*-	value
a	0.1 m	b	0.1 m
Δx	0.01 m	W_{H_2o}, W_{ext}	$25°C$
Δy	0.01 m	σ	$\frac{5.67\times10^{-8}W}{m^2K^4}$
t_0	21 s	ε	0.8
t_f	171 s	c_a, c_w	0.5
v	1/30 m/s	α	$968.1KW.m^{-2}$
β	$\frac{80.1\,KW}{m^2.s^{1/2}}$	t_c	21 s
W^0	34.286^0C	W_m	$1,580°C$

Table 2 Look-up table for K and H

W[°C]	K[W/m°C]	H[J/m^3]
34.286	60.0	1.0
502.9	–	2.0×10^9
708.5	31.85	3.14×10^9
800.0	25.38	4.257×10^9
900.0	26.488	–
1,000.0	27.596	5.0×10^9
1,200.0	–	6.0×10^9
1,300.0	30.92	–
1,400.0	–	6.857×10^9
1,508.6	33.23	7.286×10^9
1,554.3	202.38	9.571×10^9
1,600.0	205.44	9.857×10^9

and $F(T)$ can be constructed using the enthalpy derivative-temperature relation and the thermal conductivity-temperature relation.

$$F(T) = \frac{K(g^-(T))}{H'(g^-(T))}, \tag{2.14}$$

where $H = H(W) = \int_{W^0}^{W} \rho(s)[c(s) + Lf(s)]\mathrm{d}s$ is the enthalpy-temperature relation of the steel in question. It is seen that in order to find $g(W)$ and $F(W)$, we need $K(W), H(W)$ that can be obtained through the look-up table (Table 2) and the interpolation. Obtained results for these functions are depicted in Fig. 2.

Figure 3 shows numerical solutions of the temperature profile in the mould. There are total of 36 curves representing the temperature profile at 36 grid points. The lowest curve represents the temperature change of the corner point in the mould and the top curve represents the temperature change of the center point in the mould.

With $T(t_1)$ as the initial value for the SCZ, we can now solve the Eqs 2.10 and 2.9. Figure 4 presents the temperature profile without control in the SCZ (i.e., $h = 0$) and Figure 5 is the result when these components of the control $h(t)$ are taken the function of the following

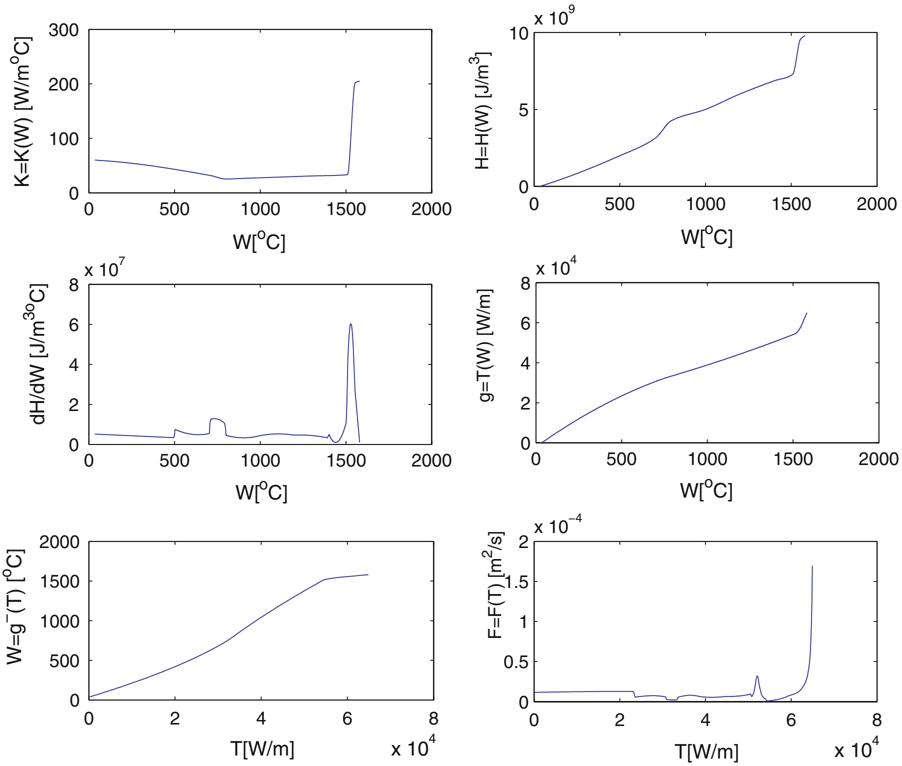

Fig. 2 Six functions required in the simulation

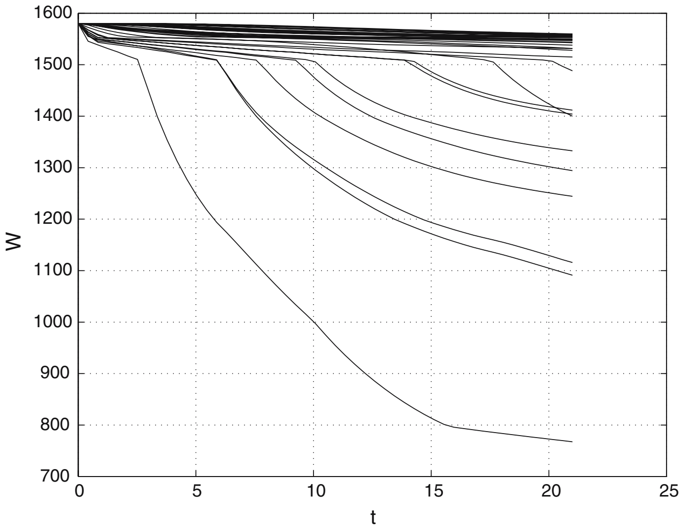

Fig. 3 Temperature profile in the mould

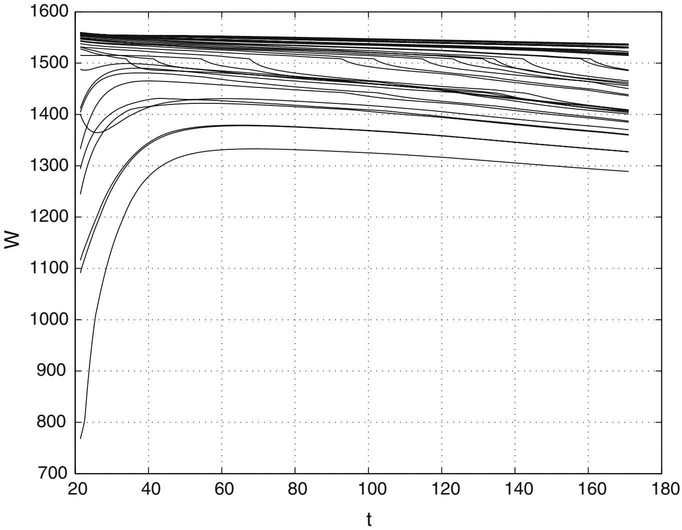

Fig. 4 Temperature profile in the SCZ for $h = 0$

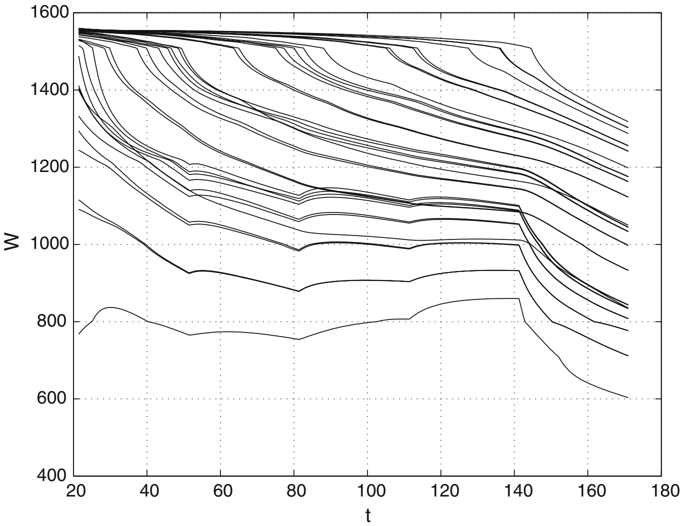

Fig. 5 Temperature profile in the SCZ for h given in (2.15)

$$h_{1i}(t), h_{2j}(t) = \begin{array}{ll} 700, & 0.7 \leq vt \leq 1.7, \\ 500, & 1.7 < vt \leq 2.7, \\ 300, & 2.7 < vt \leq 3.7, \\ 200, & 3.7 < vt \leq 4.7, \\ 700, & 4.7 < vt \leq 5.7, \end{array} \quad i = 0, 1, \ldots, m, j = 0, 1, \ldots, n. \tag{2.15}$$

It is seen that when $h = 0$, the outer layer is reheated in the beginning of the SCZ.

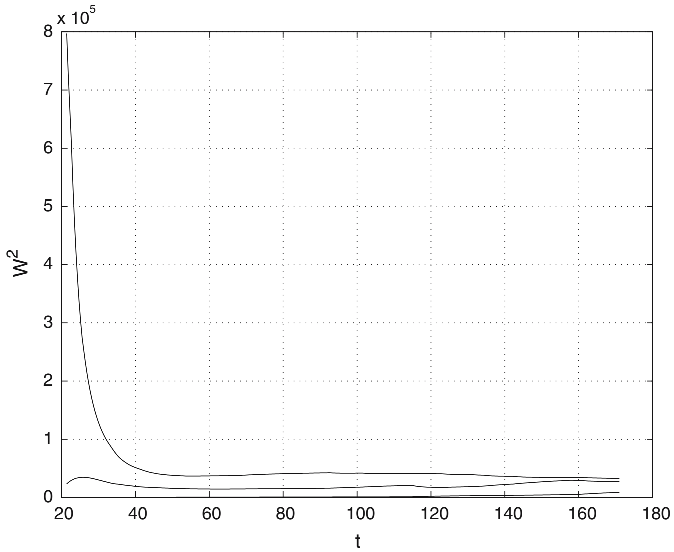

Fig. 6 $\|W_j - W_{j-1}\|^2, j = 1, ..., 5$ for $h = 0$

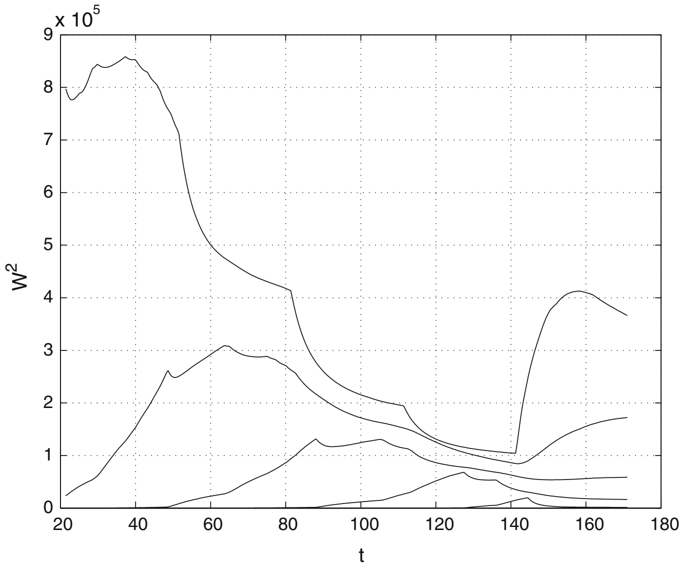

Fig. 7 $\|W_j - W_{j-1}\|^2, j = 1, ..., 5$ for h given in (2.15)

When h is given as in (2.15), although the temperature of the outer layer decreases at the beginning of the SCZ, it is reheated later in the SCZ. Their differences of the temperature between connected different layers for both cases are shown in Figs. 6 and 7, respectively.

3 Optimal control formulation and dynamic programming principle

In this section, we formulate an optimal control problem for the continuous casting, which is motivated by avoiding the crack formation in the SCZ. By the transformation $W = g^-(T)$ (see (2.6)), we can formulate the optimal control problem in the setting of T. For a given ideal temperature profile $W^* = (W_0^*, W_1^*, \ldots, W_n^*)^T$ of a one quarter end section of the SCZ, the optimal control problem is to find an optimal control $h^* = (h_{1m}^*, \ldots, h_{10}^*, h_{20}^*, \ldots, h_{2n}^*)^T$ such that

$$J(t_1, S^0, h^*) = \inf_{h(\cdot)\in\mathcal{U}(t_1,t_2)} J(t_1, S^0, h) \tag{3.1}$$

subject to (2.10), where

$$\begin{aligned} J(t_1, S^0, h) &= \int_{t_1}^{t_2} \left[\|h(t)\|^2 + \sum_{j=1}^{n} \max\left(G_j(g^-(T(t))), 0 \right) \right] \mathrm{d}t + \|g^-(T(t_2)) - W^*\|^2 \\ &= \int_{t_1}^{t_2} \left[\sum_{i=0}^{m} h_{1i}^2(t) + \sum_{j=0}^{n} h_{2j}^2(t) + \sum_{j=1}^{n} \max\left(G_j(g^-(T(t))), 0 \right) \right] \mathrm{d}t \\ &\quad + \sum_{j=1}^{n} \left[\sum_{i=j+1}^{m} |g^-(T_{ij}(t_2)) - W_{ij}^*|^2 + \sum_{k=j}^{n} |g^-(T_{jk}(t_2)) - W_{jk}^*|^2 \right] \end{aligned} \tag{3.2}$$

with

$$\begin{aligned} & G_j(g^-(T(t))) \\ &= \|g^-(T_j(t)) - g^-(T_{j-1}(t))\|^2 - c_j \\ &= \sum_{i=j}^{m} |g^-(T_{ij}(t)) - g^-(T_{i(j-1)}(t))|^2 + |g^-(T_{(j-1)(j-1)}(t)) - g^-(T_{jj}(t))|^2 \\ &\quad + \sum_{k=j}^{n} |g^-(T_{jk}(t)) - g^-(T_{(j-1)k}(t))|^2 - c_j, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{3.3}$$

where $c_j > 0$ are given constants.

$$\mathcal{U}(t_1, t_2) = L^\infty(t_1, t_2; \mathbf{E}), \tag{3.4}$$

$$\begin{aligned} \mathbf{E} = \big\{ h = (h_{1i}, h_{2j}) \,|\, 0 \le h_{1i} \le d_{1i},\ 0 \le h_{2j} \le d_{2j},\ i = 0, 1, 2, \cdots, m, \\ j = 0, 1, 2, \cdots, n \big\} \subset \mathbb{R}^{m+n+2}, \end{aligned}$$

where $\mathrm{d}_{1i}, \mathrm{d}_{2j}$ are given constants.

Noteworthily, $\sum_{j=1}^{n} \max(G_j(g^-(T(t))), 0)$ in the cost functional (3.2) can be understood as a relaxed state constraint [16] and its function is to keep these differences of temperature profiles between two connected layers be in a given scope as much as possible.

Let $\langle\cdot,\cdot\rangle$ denote the Euclidean inner product and $\|\cdot\|$ the inner product induced Euclidean norm in Euclidean space of appropriate dimension. Let $\mathbb{R}^{(m+1)(n+1)}$ be the state space. We say that h is an admissible control if $h(\cdot) \in \mathcal{U}(t_1, t_2)$.

It is seen that there are four objectives of the optimization problem formulated above: (a) to control in an optimal manner the rapid reheating in the forepart of the SCZ; (b) to make the temperature profile $g^-(T(t_2))$ of the end section of the SCZ close to a desired temperature profile W^* as much as possible; (c) to keep these differences of temperature profiles between two connected layers in a given scope; and (d) to keep the amount of water spray in a reasonable scope. These objectives will guarantee to some extent that the quality problem such as crack formation both in surface and interior of the strand does not occur.

Now define the value function as following

$$V(\tau,S)=\inf_{h(\cdot)\in\mathcal{U}(\tau,t_2)} J(\tau,S,h) \tag{3.5}$$

for all $(\tau,S)\in\Theta=[t_1,t_2]\times\mathbb{R}^{(m+1)(n+1)}$. Here by $J(\tau,S,h)$, we understand to be the cost functional (3.2) and (3.3) with (2.10) where the condition $T(t_1)=S^0$ is replaced by $T(\tau)=S$ and t_1 is replaced by τ. $\mathcal{U}(\tau,t_2)$ is similarly defined as $\mathcal{U}(t_1,t_2)$.

Proposition 2 *The value function V is continuous with respect to* $(\tau,S)\in\Theta$.

Proof For any $S^1,S^2\in\mathbb{R}^{(m+1)(n+1)}$, $\tau^1,\tau^2\in[t_1,t_2]$, and any given $\delta>0$, suppose τ^2,S^2 are fixed. Let $\tau^m=\max(\tau^1,\tau^2)$. By Proposition 1, one can find a smooth $h\in\mathcal{U}(t_1,t_2)$ such that

$$V(\tau^2,S^2)\geq\int_{\tau^2}^{t_2}\left[\|h(\rho)\|^2+\sum_{j=1}^{n}\max\left(G_j(g^-(T^2(\rho))),0\right)\right]\mathrm{d}\rho+\|g^-(T^2(t_2))-W^*\|^2-\delta,$$

where T^2 is the solution to (2.11) produced by h and the initial condition $T^2(\tau^2)=S^2$. Suppose T^1 is the solution to (2.11) produced by the same h and the initial condition $T^1(\tau^1)=S^1$. Then

$$\begin{aligned}
&V(\tau^1,S^1)-V(\tau^2,S^2)\\
&\leq\int_{\tau^1}^{t_2}\left[\|h(\rho)\|^2+\sum_{j=1}^{n}\max\left(G_j(g^-(T^1(\rho))),0\right)\right]\mathrm{d}\rho+\|g^-(T^1(t_2))-W^*\|^2\\
&\quad-\int_{\tau^2}^{t_2}\left[\|h(\rho)\|^2+\sum_{j=1}^{n}\max\left(G_j(g^-(T^2(\rho))),0\right)\right]\mathrm{d}\rho-\|g^-(T^2(t_2))-W^*\|^2+\delta\\
&\leq\int_{\tau^1}^{\tau^2}\|h(\rho)\|^2\mathrm{d}\rho+\int_{\tau^1}^{\tau^m}\sum_{j=1}^{n}\max\left(G_j(g^-(T^1(\rho))),0\right)\mathrm{d}\rho\\
&\quad+\int_{\tau^m}^{t_2}\sum_{j=1}^{n}\max\left(G_j(g^-(T^1(\rho))),0\right)\mathrm{d}\rho+\|g^-(T^1(t_2))-W^*\|^2\\
&\quad-\int_{\tau^m}^{t_2}\sum_{j=1}^{n}\max\left(G_j(g^-(T^2(\rho))),0\right)\mathrm{d}\rho-\|g^-(T^2(t_2))-W^*\|^2+\delta\\
&\leq\int_{\tau^m}^{t_2}\left[\sum_{j=1}^{n}\frac{G_j(g^-(T^1(\rho)))+|G_j(g^-(T^1(\rho)))|}{2}-\frac{G_j(g^-(T^2(\rho)))+|G_j(g^-(T^2(\rho)))|}{2}\right]\mathrm{d}\rho\\
&\quad+\mathcal{M}|\tau^1-\tau^2|+\|g^-(T^1(t_2))-W^*\|^2-\|g^-(T^2(t_2))-W^*\|^2+\delta
\end{aligned}\tag{3.6}$$

for some constant $\mathcal{M}$. Since T^1 is continuous, it follows that

$$T^1(\tau^2) - S^2 = T^1(\tau^2) - T^1(\tau^1) + S^1 - S^2 \to 0 \quad \text{as } \tau^1 \to \tau^2, S^1 \to S^2.$$

By Proposition 1, it has

$$T^1(t) - T^2(t) \to 0 \quad \text{for all } t \in [\tau^m, t_2] \quad \text{as } \tau^1 \to \tau^2, S^1 \to S^2.$$

This together with the dominant convergence theorem yields

$$\overline{\lim}_{\tau^1 \to \tau^2, S^1 \to S^2}[V(\tau^1, S^1) - V(\tau^2, S^2)] \le 0.$$

The inverse inequality

$$\underline{\lim}_{\tau^1 \to \tau^2, S^1 \to S^2}[V(\tau^1, S^1) - V(\tau^2, S^2)] \ge 0$$

can be proved similarly. Therefore, $V(\tau, S)$ is continuous in (τ, S). The proof is complete. □

Theorem 1 [**Dynamic Programming Principle**] *For any initial condition* $(\tau, S) \in \Theta$ *and* $r \in [\tau, t_2]$,

$$V(\tau, S) = \inf_{h(\cdot)\in\mathcal{U}(\tau,r)} \left\{ \int_\tau^r \left[\|h(t)\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(T(t))), 0 \right) \right] \mathrm{d}t + V(r, T(r)) \right\}. \tag{3.7}$$

Proof First, by the definition of the value function (3.5), for any $\delta > 0$ one can choose an admissible control $h^1(\cdot) \in \mathcal{U}(r, t_2)$ such that

$$\int_r^{t_2} \left[\|h^1(t)\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(T^1(t))), 0 \right) \right] dt + \|g^-(T^1(t_2)) - W^*\|^2 \le V(r, T(r)) + \delta \tag{3.8}$$

in which the state T is produced by the admissible control $h \in \mathcal{U}(\tau, r)$ with initial condition $T(\tau) = S$, and T^1 is the state corresponding to $h^1(\cdot)$ such that $T^1(r) = T(r)$. Such a control $h^1(\cdot)$ is called δ-optimal.

Define an admissible control $\tilde{h}(\cdot) \in \mathcal{U}(\tau, t_2)$ by

$$\tilde{h}(t) = \begin{cases} h(t), & \tau \le t \le r, \\ h^1(t), & r < t \le t_2 \end{cases}$$

and let $\tilde{T}(\cdot)$ be the state corresponding to $\tilde{h}(\cdot)$. In view of (3.8), it has

$$\begin{aligned} V(\tau, S) &\le J(\tau, S, \tilde{h}) \\ &= \int_\tau^{t_2} \left[\|\tilde{h}(t)\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(\tilde{T}(t))), 0 \right) \right] \mathrm{d}t + \|g^-(\tilde{T}(t_2)) - W^*\|^2 \\ &\le \int_\tau^r \left[\|h(t)\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(T(t))), 0 \right) \right] \mathrm{d}t + V(r, T(r)) + \delta. \end{aligned}$$

Secondly, for the $\delta > 0$, choose an admissible control $h(\cdot) \in \mathcal{U}(\tau, t_2)$ such that for any $r \in [\tau, t_2]$,

$$\begin{aligned}\delta + V(\tau, S) &\geq J(\tau, S, h) \\ &= \int_\tau^r \left[\|h(t)\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(T(t))), 0\right)\right] \mathrm{d}t + J(r, T(r), h) \\ &\geq \int_\tau^r \left[\|h(t)\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(T(t))), 0\right)\right] \mathrm{d}t + V(r, T(r))\end{aligned}$$

(3.7) thus follows from the arbitrariness of δ. □

4 Hamilton–Jacobi–Bellman equation and viscosity solution

Starting from the dynamic programming principle, we can derive the associated Hamilton–Jacobi–Bellman equation that is stated as Theorem 2 below.

Theorem 2 *If $V(\tau, S) \in C^1(\Theta)$, then the value function V satisfies the Hamilton–Jacobi–Bellman equation of the following:*

$$\begin{aligned}&-V_\tau(\tau, S) + \mathcal{H}(\tau, S, D_S V(\tau, S)) = 0, \quad \forall\, (\tau, S) \in [t_1, t_2) \times \mathbb{R}^{(m+1)(n+1)}, \\ &V(t_2, S) = \|g^-(S) - W^*\|^2, \quad \forall\, S \in \mathbb{R}^{(m+1)(n+1)},\end{aligned} \tag{4.1}$$

in which the Hamiltonian $\mathcal{H}$ is given by

$$\begin{aligned}&\mathcal{H}(\tau, S, D_S V(\tau, S)) \\ &= \sup_{h \in \mathbf{E}} \left\{ \left\langle -\mathcal{F}(S)\left[\mathcal{A}S + \mathcal{B}\sigma\varepsilon((g^-(S_0))^4 - W_{\text{ext}}^4) + \mathcal{B}(g^-(S_0) - W_{\text{H}_2\text{O}})h\right], D_S V(\tau, S)\right\rangle \right. \\ &\quad \left. -\|h\|^2 - \sum_{j=1}^n \max\left(G_j(g^-(S)), 0\right)\right\}, \quad \forall\, (\tau, S, D_S V(\tau, S)) \in \Theta \times \mathbb{R}^{(m+1)(n+1)},\end{aligned} \tag{4.2}$$

where $S = (S_0, S_1, \ldots, S_n)^T$ and $D_S V$ denotes the partial Fréchet gradient of $V(\tau, \cdot)$.

Proof For any $0 < \delta < t_2 - \tau$ and any given constant control $h \in \mathbf{E}$, by the dynamic programming principle (3.7),

$$V(\tau, S) \leq \int_\tau^{\tau+\delta} \left[\|h\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(T(t))), 0\right)\right] \mathrm{d}t + V(\tau + \delta, T(\tau + \delta)),$$

where T is the state corresponding to h and initial condition $T(\tau) = S$. Therefore,

$$0 \leq \frac{1}{\delta}\int_\tau^{\tau+\delta} \left[\|h\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(T(t))), 0\right)\right] \mathrm{d}t + \frac{V(\tau + \delta, T(\tau + \delta)) - V(\tau, T(\tau))}{\delta}.$$

Letting $\delta \to 0^+$ gives

$$-V_\tau(\tau, S) + \mathcal{H}(\tau, S, D_S V(\tau, S)) \leq 0. \tag{4.3}$$

On the other hand, for any given $\eta > 0$, $0 < \delta < t_2 - \tau$ and $h \in \mathbf{E}$, by Proposition 1, one can find a smooth $\hat{h}(\cdot) \in \mathcal{U}(\tau, t_2)$ such that $\hat{h}(\tau) = h$ and

$$V(\tau, S) + \eta\delta \geq \int_{\tau}^{\tau+\delta} \left[\|\hat{h}(t)\|^2 + \sum_{j=1}^{n} \max\left(G_j(g^{-}(\hat{T}(t))), 0 \right) \right] \mathrm{d}t + V(\tau + \delta, \hat{T}(\tau + \delta)),$$

where $\hat{T}(\cdot)$ is the state corresponding to $\hat{h}(\cdot)$ and the initial condition $\hat{T}(\tau) = S$. Therefore,

$$\begin{aligned}
\eta &\geq \frac{1}{\delta} \int_{\tau}^{\tau+\delta} \left[\|\hat{h}(t)\|^2 + \sum_{j=1}^{n} \max\left(G_j(g^{-}(\hat{T}(t))), 0 \right) \right] \mathrm{d}t \\
&\quad + \frac{V(\tau + \delta, \hat{T}(\tau + \delta)) - V(\tau, S)}{\delta} \\
&= \frac{1}{\delta} \int_{\tau}^{\tau+\delta} \left[\|\hat{h}(t)\|^2 + \sum_{j=1}^{n} \max\left(G_j(g^{-}(\hat{T}(t))), 0 \right) \right] \mathrm{d}t + V_{\tau}(\tau, S) \\
&\quad + \Big\langle D_S V(\tau, S), \frac{1}{\delta} \int_{\tau}^{\tau+\delta} \mathcal{F}(\hat{T}(t)) \Big[\mathcal{A}\hat{T}(t) + \mathcal{B}\sigma\varepsilon((g^{-}(\hat{T}_0(t)))^4 - W_{\mathrm{ext}}^4) \\
&\quad + \mathcal{B}(g^{-}(\hat{T}_0(t)) - W_{\mathrm{H_2O}})\hat{h}(t) \Big] \mathrm{d}t \Big\rangle + \frac{1}{\delta} o(|\delta| + \|\hat{T}(\tau + \delta) - S\|) \\
&\geq \frac{1}{\delta} \int_{\tau}^{\tau+\delta} -\mathcal{H}(t, T(t), D_T V(t, T(t))) \mathrm{d}t + V_{\tau}(\tau, S) \\
&\quad + \frac{1}{\delta} o(|\delta| + \|\hat{T}(\tau + \delta) - S\|).
\end{aligned}$$

Letting $\delta \to 0^+$ again yields $-V(\tau, S) + \mathcal{H}(\tau, S, D_S V(\tau, S)) \geq -\eta$ and hence

$$-V_{\tau}(\tau, S) + \mathcal{H}(\tau, S, D_S V(\tau, S)) \geq 0. \tag{4.4}$$

The proof is then complete by combining (4.3) and (4.4). □

Next, we give a sufficient condition of optimality.

Theorem 3 *Let $Y \in C^1(\Theta)$ satisfy (4.1) and let V be the value function. Then*

(1) $Y(\tau, S) \leq V(\tau, S)$, $\forall (\tau, S) \in \Theta$.
(2) *If there exists a $h^*(\cdot) \in \mathcal{U}(t_1, t_2)$ such that*

$$\begin{aligned}
&\Big\langle \mathcal{F}(T^*(t)) \Big[\mathcal{A}T^*(t) + \mathcal{B}\sigma\varepsilon((g^{-}(T_0^*(t)))^4 - W_{\mathrm{ext}}^4) + \mathcal{B}(g^{-}(T_0^*(t)) - W_{\mathrm{H_2O}})h^*(t) \Big], \\
&D_S Y(t, T^*(t)) \Big\rangle + \|h^*(t)\|^2 + \sum_{j=1}^{n} \max\left(G_j(g^{-}(T^*(t))), 0 \right) \\
&= -\mathcal{H}(t, T^*(t), D_S Y(t, T^*(t))),
\end{aligned} \tag{4.5}$$

which, in usual way, one writes

$$h^*(t) \in \arg\inf_{h\in\mathbf{E}} \Big\{ \big\langle \mathcal{F}(T^*(t))\big[\mathcal{A}T^*(t) + \mathcal{B}\sigma\varepsilon((g^-(T_0^*(t)))^4 - W_{\text{ext}}^4)$$

$$+\mathcal{B}(g^-(T_0^*(t)) - W_{\text{H}_2\text{O}})h\big], D_S Y(t, T^*(t))\big\rangle + \|h\|^2 + \sum_{j=1}^{n} \max\Big(G_j(g^-(T^*(t))), 0\Big)\Big\}, \tag{4.6}$$

for almost all $t \in [t_1, t_2]$, where T^* is the state corresponding to h^* and initial condition $T^*(t_1) = S^0$, then $h^*(\cdot)$ is an optimal control.

Proof For any $h(\cdot) \in \mathcal{U}(\tau, t_2)$, using the dynamic programming Eq. 4.1, we have

$$\begin{aligned} Y(t_2, T(t_2)) &= Y(\tau, S) + \int_\tau^{t_2} \left[\frac{\partial}{\partial\tau} Y(t, T(t)) + \left\langle \frac{\mathrm{d}T(t)}{\mathrm{d}t}, D_S Y(t, T(t))\right\rangle\right] \mathrm{d}t \\ &= Y(\tau, S) + \int_\tau^{t_2} \Big[\frac{\partial}{\partial\tau} Y(t, T(t)) + \big\langle \mathcal{F}(S)\big[\mathcal{A}S + \mathcal{B}\sigma\varepsilon((g^-(S_0))^4 - W_{\text{ext}}^4) \\ &\quad +\mathcal{B}(g^-(S_0) - W_{\text{H}_2\text{O}})h\big], D_S Y(t, T(t))\big\rangle\Big]\mathrm{d}t \\ &\geq Y(\tau, S) - \int_\tau^{t_2} \left[\|h(t)\|^2 + \sum_{j=1}^{n} \max\Big(G_j(g^-(T(t))), 0\Big)\right]\mathrm{d}t. \end{aligned} \tag{4.7}$$

Hence $Y(\tau, S) \leq J(\tau, S, h)$ and (1) follows.

For the second assertion, let $h^*(\cdot) \in \mathcal{U}(t_1, t_2)$ satisfy (4.5). Substitute $(h^*(\cdot), T^*(\cdot))$ into (4.7) to get an equality. In particular, $Y(t_1, S^0) = J(t_1, S^0, h^*)$. By (1), $V(t_1, S^0) = J(t_1, S^0, h^*)$, which shows that $h^*(\cdot)$ is an optimal control. The proof is complete. □

Proposition 3 (1) *The sufficient condition of optimality (4.5) is also necessary if the value function V is smooth. Therefore $(T^*(\cdot), h^*(\cdot))$ is an optimal control-trajectory pair if and only if*

$$\begin{aligned} V_t(t, T^*(t)) &+ \big\langle \mathcal{F}(T^*(t))\big[\mathcal{A}T^*(t) + \mathcal{B}\sigma\varepsilon((g^-(T_0^*(t)))^4 - W_{\text{ext}}^4) \\ &+\mathcal{B}(g^-(T_0^*(t)) - W_{\text{H}_2\text{O}})h^*(t)\big], D_S V(t, T^*(t))\big\rangle \\ +\|h^*(t)\|^2 &+ \sum_{j=1}^{n} \max\Big(G_j(g^-(T^*(t))), 0\Big) = 0 \text{ for almost all } t \in [t_1, t_2]. \end{aligned} \tag{4.8}$$

(2) *Let V be the value function. Then for any control-trajectory pair $(h^*(\cdot), T^*(\cdot))$, $T^*(t_1) = S^0$, the function*

$$t \to V(t, T^*(t)) - \int_t^{t_2} \left[\|h^*(\rho)\|^2 + \sum_{j=1}^{n} \max\Big(G_j(g^-(T^*(\rho))), 0\Big)\right] \mathrm{d}\rho \tag{4.9}$$

is nondecreasing on $[t_1, t_2]$. Moreover, $(h^*(\cdot), T^*(\cdot))$ is an optimal control-trajectory pair if and only if the above function is constant on $[t_1, t_2]$.

Proof This proof is standard and we omit the details here. □

Now we give a definition of viscosity solution to the Hamilton–Jacobi–Bellman equation (4.1).

Definition 1 Let $u(\tau, S) \in C(\Theta)$. We say that

(1) u is a viscosity supersolution to the first equation of (4.1) in Θ if

$$-\varphi_\tau(\tau, S) + \mathcal{H}(\tau, S, D_S\varphi(\tau, S)) \geq 0$$

for any $\varphi \in C^1(\Theta)$ such that $u-\varphi$ has a local minimum, relative to Θ at $S \in \mathbb{R}^{(m+1)(n+1)}$.

(2) u is a viscosity subsolution to the first equation of (4.1) in Θ if

$$-\varphi_\tau(\tau, S) + \mathcal{H}(\tau, S, D_S\varphi(\tau, S)) \leq 0$$

for any $\varphi \in C^1(\Theta)$ such that $u-\varphi$ has a local maximum, relative to Θ at $S \in \mathbb{R}^{(m+1)(n+1)}$.

(3) u is a viscosity solution to the first equation of (4.1) in Θ if it is simultaneously a viscosity subsolution and supersolution in Θ.

Theorem 4 *The value function V is a viscosity solution of the Hamilton–Jacobi–Bellman equation (4.1) in Θ.*

Proof Let $\varphi \in C^1(\Theta)$ and $(\tau, S) \in \Theta$ be a local maximum point of $V - \varphi$. Let $h \in \mathbf{E}$ be an arbitrary constant control and T be the state corresponding to h and initial condition $T(\tau) = S$. Then for any given $\delta > 0$, $(r, T(r)) \in \mathbb{B}((\tau, S), \delta)$, the ball centered at (τ, S) with radius δ in Θ, for all sufficiently small r. Hence

$$\varphi(\tau, S) - \varphi(r, T(r)) \leq V(\tau, S) - V(r, T(r))$$

for all r small enough. By the dynamic programming principle (3.7), we get

$$\varphi(\tau, S) - \varphi(r, T(r)) \leq \int_\tau^r \left[\|h\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(T(t))), 0\right)\right] \mathrm{d}t.$$

Divide by $r - \tau > 0$ on both sides above and let $r \to \tau$, to obtain

$$-\varphi_\tau(\tau, S) - \left\langle \mathcal{F}(S)\left[\mathcal{A}S + \mathcal{B}\sigma\varepsilon((g^-(S_0))^4 - W_{\mathrm{ext}}^4) + \mathcal{B}(g^-(S_0) - W_{\mathrm{H_2O}})h\right], D_S\varphi(\tau, S)\right\rangle$$
$$\leq \|h\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(S)), 0\right).$$

This shows that

$$-\varphi_\tau(\tau, S) + \mathcal{H}(\tau, S, D_S V(\tau, S)) \leq 0$$

for any $h \in \mathbf{E}$. Therefore, V is a viscosity subsolution to (4.1) in $(\tau, S) \in \Theta$.

Next assume that $(\tau, S) \in \Theta$ is a local minimum point of $V - \varphi$, that is, for a given $\delta > 0$, $V(\tau, S) - V(r, T(r)) \leq \varphi(\tau, S) - \varphi(r, T(r))$ for all $(r, T(r)) \in \mathbb{B}((\tau, S), \delta)$. For any given $\rho > 0$ and $h \in \mathbf{E}$, by the dynamic programming principle (3.7) and Proposition 1, there exists a smooth $\overline{h} \in \mathcal{U}(\tau, t_2)$ depending on ρ and r such that $\overline{h}(\tau) = h$ and

$$V(\tau, S) \geq \int_\tau^r \left[\|\overline{h}(t)\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(\overline{T}(t))), 0\right)\right] \mathrm{d}t + V(r, \overline{T}(r)) - r\rho,$$

where $\overline{T}$ is the state corresponding to $\overline{h}$ and initial condition $\overline{T}(\tau) = S$. Therefore,

$$\varphi(\tau, S) - \varphi(r, \overline{T}(r)) \geq \int_\tau^r \left[\|\overline{h}\|^2 + \sum_{j=1}^n \max\left(G_j(g^-(\overline{T}(t))), 0\right)\right] \mathrm{d}t - r\rho.$$

Divide by $r - \tau > 0$ and let $r \to \tau$ to get

$$-\varphi_\tau(\tau, S) - \big\langle \mathcal{F}(S)\big[\mathcal{A}S + \mathcal{B}\sigma\varepsilon((g^-(S_0))^4 - W_{\rm ext}^4) + \mathcal{B}(g^-(S_0) - W_{\rm H_2O})\overline{h}\big], D_S\varphi(\tau, S)\big\rangle$$
$$\geq \|h\|^2 + \sum_{j=1}^{n} \max\left(G_j(g^-(\overline{T}(\tau))), 0\right) - \rho.$$

Since ρ is arbitrary, the last inequality reads

$$-\varphi_\tau(\tau, S) + \mathcal{H}(\tau, S, D_S V(\tau, S)) \geq 0,$$

that is, V is a viscosity supersolution to (4.1). Therefore, V is a viscosity solution to (4.1) in Θ. The proof is complete. □

To end this section, let us go back to the argumentation of optimality conditions. It is well-known that the assertion (1) of Proposition 3 will lead to the classical verification theorem, which plays an important role in testing the optimality for a given control-trajectory pair, and more importantly, in constructing the optimal feedback control [22]. In fact, suppose the value function $V(\tau, S)$ is smooth, then $V(\tau, S)$ is a classical solution to the Hamilton–Jaconbi–Bellman equation claimed by Theorem 2:

$$-V_\tau(\tau, S) + \mathcal{H}(\tau, S, D_S V(\tau, S)) = 0.$$

Hence the optimality condition (4.8) is equivalent to (4.5). It is noted that (4.5) says that $h^*(\cdot) \in \mathcal{U}(t_1, t_2)$ is an optimal control with the the initial state S^0 if and only if (4.6) holds, and (4.6) is the formula of finding the optimal feedback control.

However, (4.6) is true only when the value function is differentiable. This is not true usually. Nevertheless, it is our basis of numerical solution to the optimal feedback control discussed in the next section.

5 Finite difference scheme for the numerical solution of optimal feedback control

In this section, we use the viscosity solution approach to get numerical solutions of the optimal feedback control, one of the main tasks of this paper.

The first step is to discretize the Hamilton–Jacobi–Bellman equation (4.1). To do this, let $\tau_j = t_2 + j\Delta\tau, j = 0, 1, \ldots, N$ where $\Delta\tau = (t_1 - t_2)/N$ and N is an integer. For given $\eta > 0$, we approximate the Fréchet partial derivative as the following:

$$\langle D_S V(\tau, S), f\rangle = \left\langle D_S V(\tau, S), \eta\frac{f}{\|f\|}\right\rangle \frac{\|f\|}{\eta} \approx \left[V\left(\tau, S + \eta\frac{f}{\|f\|}\right) - V(\tau, S)\right]\frac{\|f\|}{\eta}, \quad (5.1)$$

where $f = f(h, S) = -\mathcal{F}(S)\big[\mathcal{A}S + \mathcal{B}\sigma\varepsilon((g^-(S_0))^4 - W_{\rm ext}^4) + \mathcal{B}(g^-(S_0) - W_{\rm H_2O})h\big]$. Let $f_i^j = f(h_i^j, S^{(i)})$ and $f_i = f(h, S^{(i)})$. For the initial state $S^{(0)} = S^0$ set

$$S^{(i)} = S^{(i-1)} + \frac{f_{i-1}}{\|f_{i-1}\|}\eta, \quad i = 1, 2, \ldots, M. \quad (5.2)$$

Approximate the Hamilton–Jacobi–Bellman (4.1) by the difference scheme, to obtain [8]

$$\begin{cases} \dfrac{V_i^{j+1}-V_i^j}{\Delta\tau}+\dfrac{V_i^j-V_{i-1}^j}{\eta}\|f_i^j\|+\|h_i^j\|^2+\sum\limits_{p=1}^{n}\max\Big(G_p(g^-(S^{(i)})),0\Big)=0, \\ h_i^{j+1}=\arg\inf\limits_{h\in\mathbf{E}}\left\{\dfrac{V_i^{j+1}-V_{i-1}^{j+1}}{\eta}\|f_i\|+\|h\|^2+\sum\limits_{p=1}^{n}\max\Big(G_p(g^-(S^{(i)})),0\Big)\right\} \end{cases} \tag{5.3}$$

for all $i=1,2,\cdots,M$ and $j=0,1,\cdots,N$, where $V_i^j\approx V(\tau_j,S^{(i)})$. Moreover, we assume the following condition that is a sufficient condition for the stability of the above difference scheme (5.3) [13]

$$\frac{|\Delta\tau|}{\eta}\max_{1\le i\le M}\|f_i\|\le 1. \tag{5.4}$$

Summarizing, we have the following difference algorithm of solving the Hamilton–Jacobi–Bellman equation.

Algorithm of solving the Hamilton–Jacobi–Bellman equation.
Step 1 initialization. Set

$$\begin{cases} V_i^0=V(t_2,S^{(i)})=\|g^-(S^{(i)})-W^*\|^2,\ S^{(i)}=S^{(i-1)}+\dfrac{f_{i-1}}{\|f_{i-1}\|}\eta, \\ h_i^0\in\arg\inf\limits_{h\in\mathbf{E}}\left\{\dfrac{\|g^-(S^{(i)})-W^*\|^2-\|g^-(S^{(i-1)})-W^*\|^2}{\eta}\|f_i\|\right. \\ \qquad\left.+\|h\|^2+\sum\limits_{p=1}^{n}\max\Big(G_p(g^-(S^{(i)})),0\Big)\right\},\quad i=1,2,\ldots,M. \end{cases} \tag{5.5}$$

Here the formula for h_i^0 comes from

$$h(t)\in\arg\inf_{h\in\mathbf{E}}\left\{\langle f(h,T(t)),D_SV(t,T(t))\rangle+\|h\|^2+\sum_{p=1}^{n}\max\Big(G_p(g^-(T(t))),0\Big)\right\}. \tag{5.6}$$

Step 2 iteration By (5.3),

$$\begin{cases} V_i^{j+1}=(1-\dfrac{\Delta\tau}{\eta}\|f^j{}_i\|)V^j{}_i+\dfrac{\Delta\tau}{\eta}\|f^j{}_i\|V^j{}_{i-1}-\Delta\tau\|h^j{}_i\|^2 \\ \qquad -\Delta\tau\sum\limits_{p=1}^{n}\max\Big(G_p(g^-(S^{(i)})),0\Big), \\ h_i^{j+1}=\arg\inf\limits_{h\in\mathbf{E}}\left\{\dfrac{V_i^{j+1}-V_{i-1}^{j+1}}{\eta}\|f_i\|+\|h\|^2+\sum\limits_{p=1}^{n}\max\Big(G_p(g^-(S^{(i)})),0\Big)\right\} \end{cases} \tag{5.7}$$

for all $i=1,2,\ldots,M$ and $j=0,1,\ldots,N-1$.

From (5.6), the optimal feedback control is

$$h^*_{S^0}(t) \in \arg\inf_{h\in\mathbf{E}} \Big\{ \langle f(h, T^*(t)), D_S V(t, T^*(t))\rangle + \|h\|^2 + \sum_{p=1}^{n} \max\Big(G_p(g^-(T^*(t))), 0\Big)\Big\}. \tag{5.8}$$

where $T^*(\cdot)$ is the optimal trajectory of the system with initial condition $T^*(t_1) = S^0$. Since (5.8) involves T^*, finding the solution of (2.10) is necessary. In this paper, the solution of (2.10) is found numerically by the classical Runge–Kutta method.

The steps below give the procedure of finding the optimal feedback control.

Steps of finding the optimal feedback control

Step 1 Call the algorithm of solving the Hamilton–Jacobi–Bellman equation to get the feedback control function $h(t_1)$. Substitute $(h(t_1), S^{(0)})$ into (2.10) to get the optimal trajectory $T^*(\ell_1)$ where $\ell_1 = t_1 + \Delta\ell$, $\Delta\ell = (t_2 - t_1)/J$ for the given integer J.

Step 2 Replace $T^*(\ell_1)$ as the initial data $S^{(0)}$ in the first step, and call the algorithm of solving the Hamilton–Jaconbi–Bellman equation again to get the feedback control function $h(\ell_1)$. Substitute $(h(\ell_1), T^*(\ell_1))$ into (2.10) to get the optimal trajectory $T^*(\ell_2)$, $\ell_2 = t_1 + 2\Delta\ell$.

Step 3 Repeat the above process until we get all feedback control functions $h(\ell_k)$ and corresponding optimal trajectory $T^*(\ell_k)$, $\ell_k = t_1 + k\Delta\ell$, $k = 0, 1, \ldots, J$, that is to say,

$$h^*_{S^0}(t) = \Big\{ h(t_1),\ h(\ell_1),\ h(\ell_2),\ \ldots,\ h(t_2)\Big\} \tag{5.9}$$

which is the optimal feedback control.

Now we are in a position to find numerical solutions of the optimal control problem (3.1), (2.10), and (3.2) based on the scheme (5.5), (5.7) and the classical Runge–Kutta method. All parameters needed in the computation are listed in the Table 1. In addition, let

$$\begin{aligned} &c_1 = 4.427\times 10^5, && c_2 = 3.0829\times 10^5,\\ &c_3 = 1.2743\times 10^5, && c_4 = 6.6941\times 10^4,\\ &c_5 = 1.4287\times 10^4 \end{aligned} \tag{5.10}$$

and d_{1i}, $\mathrm{d}_{2j} = 700$, the maximal value of control given by (2.15). The temperature profile at $t_2 = 171s$ under the control (2.15) is considered as the ideal temperature profile W^*. As in Sect. 2, $m = n = 5$.

The computation is performed in Visual C++ 6.0 and numerical results are plotted by MATLAB 6.1. Figure 8 shows the obtained optimal feedback control at total 11 grid points: $(h^*_{1i}(t), h^*_{2j}(t)), i,j = 0,1,2,3,4,5,\ 21 \le t \le 171$ (notice that h_{10} and h_{20} are corresponding to the same grid point). Under this optimal control, we get the corresponding optimal temperature profile $W^{**} = g^-(T^*)$ of the SCZ that is plotted as Fig. 9. Figure 10 presents these differences of temperature between connected different layers. From these two figures, we can see that the reheating on the forepart

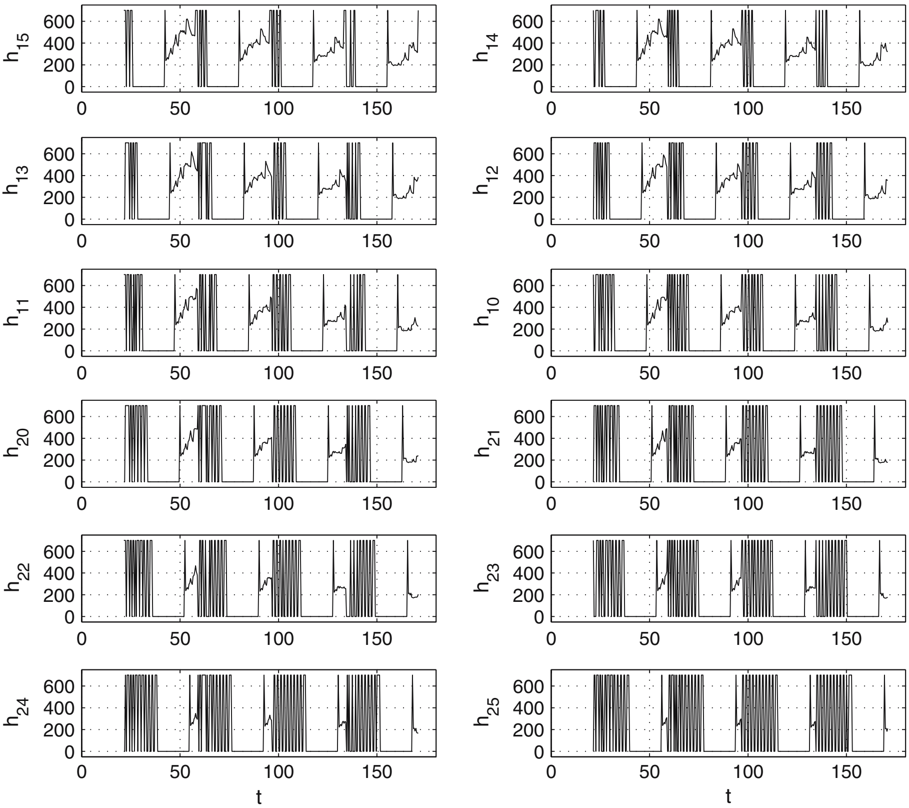

Fig. 8 Optimal feedback control h^*

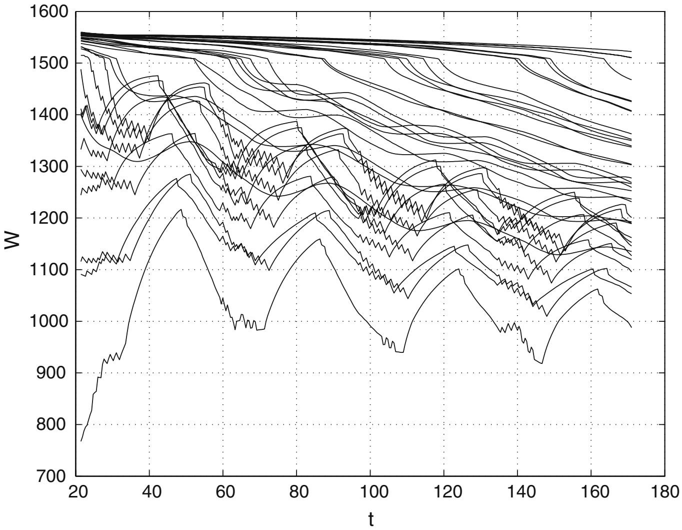

Fig. 9 Optimal temperature profile $W^{**} = g^{-}(T^*)$

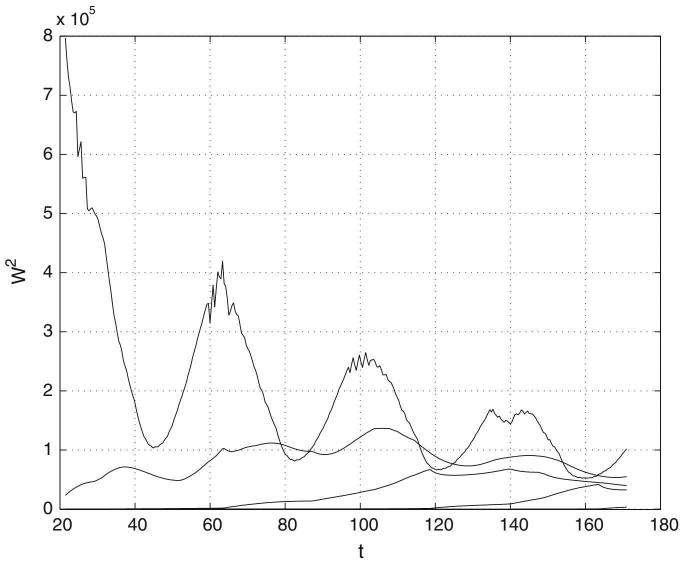

Fig. 10 $\|W_j^{**} - W_{j-1}^{**}\|^2, j = 1, ..., 5$

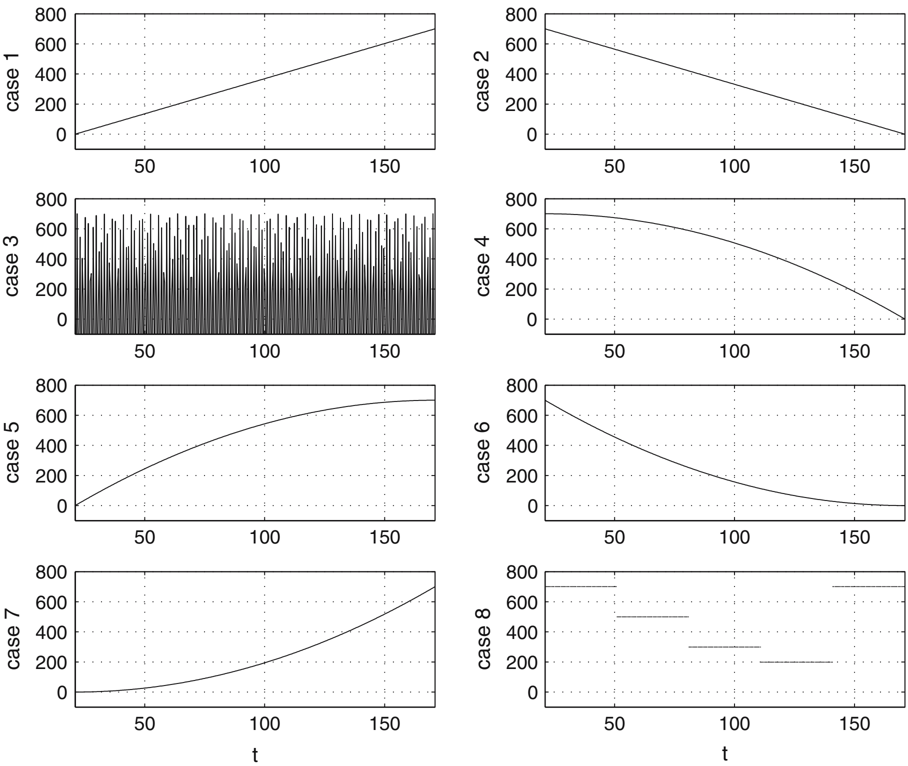

Fig. 11 Several different arbitrarily chosen admissible controls

Table 3 Different admissible controls h and their corresponding costs J

Case of h	Corresponding J
Case 1	25032343.051020
Case 2	34980529.161010
Case 3	25179601.149153
Case 4	56672703.455952
Case 5	40723454.557675
Case 6	24165907.148881
Case 7	25318719.513016
Case 8	53515593.114259
Case 9	23830013.958020

of the SCZ is effectively restrained and these temperature differences of different layers remain at most in a scope as Fig. 6 where no control is imposed.

The remaining part is the numerical experiment of checking the optimality of numerical solutions of the obtained optimal feedback control. This is done by comparing the cost functional $J(t_1, S^0, h^*)$ of the obtained optimal control-trajectory pair with $J(t_1, S^0, h)$ corresponding to the arbitrarily chosen admissible control h and its associated trajectory, that is to say, we want to know if the following inequality holds true

$$J(t_1, S^0, h^*) \leq J(t_1, S^0, h). \tag{5.11}$$

Figure 11 lists different admissible controls including that in (2.15), which is represented as case 8. We compute all corresponding cost functionals $J(t_1, S^0, h)$. The computed results are listed in the Table 3. It is seen from Table 3 that for the optimal feedback control h^*, $J(t_1, S^0, h^*) = 23830013.958020$ (labeled as case 9), which is evidently less than other cost functionals $J(t_1, S^0, h)$. From these comparisons, it seems that we do get numerical solutions of the optimal feedback control for the continuous casting of steel.

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