Numerical solution to two-phase Stefan problems by the heat balance integral method

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ABSTRACT

Caldwell and Chiu [1, 2] have used a simple front-fixed method, namely, the Heat Balance Integral Method (HBIM), to solve one-phase solidification problems. In this paper the method is extended to two-phase Stefan problems which arise when the effect of sub-cooling is taken into consideration. Numerical results for the two-phase cylindrical problem are obtained for a range of sub-divisions (n), sub-cooling parameters (ϕ), Stefan number (α) and ratio of conductivities (κ_1/κ_2). The effects of variation of parameters on the solidification process are also studied.

INTRODUCTION

Melting and solidification problems occur in numerous important areas of science, engineering and industry. For example, freezing and thawing of foods, production of ice, ice formation on pipe surface, solidification of steel and chemical reaction all involve either a melting or solidification process. Mathematically, melting/solidification problems are special cases of moving boundary problems. Problems in which the solution of a differential equation has to satisfy certain conditions on the boundary of a prescribed domain are referred to as boundary-value problems. In the cases of melting/solidification problems, however, the boundary of the domain is not known in advance, so the solution of melting/solidification problems requires solving the diffusion or heat-conduction equation in an unknown region which has also to be determined as part of the solution.

Moving boundary problems are often called Stefan problems with reference to early work of J. Stefan, who arround 1890 was interested in the melting of the polar ice cap. There exists only few exact solutions to the melting/solidification problems, and existing closed form solutions to these significant problems are highly restrictive as to allowable initial conditions

and boundary conditions. So numerical solution becomes the main tool in the study of the moving boundary problems. Two conditions are needed in order to solve the moving boundary problems, one to determine the boundary itself and the other to complete the definition of the solution of the differential equation.

In this paper, we shall consider the numerical solutions of two-phase Stefan problems in cylindrical and spherical geometries using the Heat Balance Integral Method (HBIM) of Goodman [3] and Bell [4, 5].

PROBLEM FORMULATION

The governing equations for two-phase Stefan problems are similar to those for one-phase problems. However, we have to distinguish the temperature in the liquid phase from the temperature in the solid phase. In the following formulae, T_1 and T_2 represent the temperature in the liquid phase and solid phase respectively.

$$\frac{\partial T_1}{\partial t} = \frac{\kappa_1}{r^\beta} \frac{\partial}{\partial r} \left[r^\beta \frac{\partial T_1}{\partial r} \right], \qquad a < r < R(t), \qquad t > 0 \tag{1}$$

$$\frac{\partial T_2}{\partial t} = \frac{\kappa_2}{r^\beta} \frac{\partial}{\partial r} \left[r^\beta \frac{\partial T_2}{\partial r} \right], \qquad R(t) < r < S(t), \qquad t > 0 \tag{2}$$

$$T_1 = T_f, \quad r = R(t), \quad t > 0, \quad \text{and} \quad T_1 = T_s, \quad r = a, \quad t \geq 0$$

$$T_2 = T_0, \quad r \geq S(t), \quad t > 0, \quad \text{and} \quad T_2 = T_f, \quad r = R(t), \quad t \geq 0$$

and for the moving boundary and the solid-liquid interface, we have

$$\frac{\partial T_2(S,t)}{\partial r} = 0 \tag{3}$$

$$- K_1 \frac{\partial T_1(R,t)}{\partial r} + K_2 \frac{\partial T_2(R,t)}{\partial r} = L\rho_1 \frac{dR(t)}{dt} \tag{4}$$

where $\beta = 0$, 1 or 2 depending on the geometry of the problem, and represents solidification process in planes, infinite cylinders and spheres.

Heat balance integral method

Since we have to solve two partial differential equations in the two-phase problem, we have to employ two sets of sub-divisions, one for the solid region and one for the liquid region (in our formulation we use R and S respectively for this purpose). In each region, we sub-divide the temperature range into n equal intervals and use a linear approximation profile at each interval as in the case of one-phase problems (see Figure 1). The above set of partial differential equations can then be reduced to a system of ordinary differential equations.

$$\frac{d}{dt} \left[\frac{R_{i+1}^{\beta+2} - R_i^{\beta+2}}{R_{i+1} - R_i} \right] = (\beta+1)(\beta+2) \left[\frac{\kappa_1 R_i^\beta}{R_{i+1} - R_i} - \frac{\kappa_1 R_{i+1}^\beta}{R_{i+2} - R_{i+1}} \right], \quad i = 0, 1, \ldots, n-2$$

$$\frac{d}{dt}\left[\frac{R_n^{\beta+2} - R_{n-1}^{\beta+2}}{R_n - R_{n-1}} + \alpha n(\beta+2)R_n^{\beta+1}\right] = (\beta+1)(\beta+2)\left[\frac{\kappa_1 R_{n-1}^{\beta}}{R_n - R_{n-1}} - \frac{\kappa_2 \phi S_0^{\beta}}{S_1 - S_0}\right]$$

$$\frac{d}{dt}\left[\frac{S_{j+1}^{\beta+2} - S_j^{\beta+2}}{S_{j+1} - S_j}\right] = (\beta+1)(\beta+2)\left[\frac{\kappa_2 S_j^{\beta}}{S_{j+1} - S_j} - \frac{\kappa_2 S_{j+1}^{\beta}}{S_{j+2} - S_{j+1}}\right], \quad j = 0, 1, \ldots, n-2$$

$$\frac{d}{dt}\left[\frac{S_n^{\beta+2} - S_{n-1}^{\beta+2}}{S_n - S_{n-1}}\right] = (\beta+1)(\beta+2)\frac{\kappa_2 S_{n-1}^{\beta}}{S_n - S_{n-1}}$$

Note that we have two very important parameters in this system of ODEs; where $\alpha = L/c(T_f - T_s)$ is the Stefan number and $\phi = (T_0 - T_f)/(T_f - T_s)$ is the sub-cooling parameter. The two-phase Stefan problem reduces to a one-phase problem when we set $\phi = 0$ (the system of ODEs decouples to two systems of ODEs). When solving this system of ODEs, we are mainly interested in the position of the phase change front $R(t) = R_n$ and the heat front $S(t) = S_n$.

Cylindrical solidification

For the cylindrical case (see Figure 2), we set $\beta = 1$ and the system of ordinary differential equations becomes,

$$(2R_{i+1} + R_i)\dot{R}_{i+1} + (R_{i+1} + 2R_i)\dot{R}_i = \frac{6\kappa_1 R_i}{R_{i+1} - R_i} - \frac{6\kappa_1 R_{i+1}}{R_{i+2} - R_{i+1}}$$

$$(2(1+3\alpha n)R_n + R_{n-1})\dot{R}_n + (R_n + 2R_{n-1})\dot{R}_{n-1} = \frac{6\kappa_1 R_{n-1}}{R_n - R_{n-1}} - \frac{6\phi\kappa_2 R_n}{S_1 - S_0}$$

$$(2S_{j+1} + S_j)\dot{S}_{j+1} + (S_{j+1} + 2S_j)\dot{S}_j = \frac{6\kappa_2 S_j}{S_{j+1} - S_j} - \frac{6\kappa_2 S_{j+1}}{S_{j+2} - S_{j+1}}$$

$$(2S_n + S_{n-1})\dot{S}_n + (S_n + 2S_{n-1})\dot{S}_{n-1} = \frac{6\kappa_2 S_{n-1}}{S_n - S_{n-1}}$$

We need a special starting solution for the initial conditions in order to solve the system of ODEs. The method of obtaining such a solution is to assume in the initial small time period $R_i(t)$ and $S_i(t)$ have the following forms,

$$R_i(t) = a + r_{i,1}t^{1/2} + r_{i,2}t + r_{i,3}t^{3/2} + \cdots$$

$$S_j(t) = a + s_{j,1}t^{1/2} + s_{j,2}t + s_{j,3}t^{3/2} + \cdots$$

and then substitute the above series into the system of ODEs and equate the coefficents of like powers of t (see Poots [6]). This results in a system of non-linear algebraic equations for which we can solve to determine the values of R_i and S_j.

Spherical solidification

For the spherical case, we set $\beta = 2$ and the system of ordinary differential equations becomes,

$$(3R_{i+1}^2 + 2R_{i+1}R_i + R_i^2)\dot{R}_{i+1} + (R_{i+1}^2 + 2R_{i+1}R_i + 3R_i^2)\dot{R}_i =$$

$$12\kappa_1 R_i^2 \Big/ (R_{i+1} - R_i) - 12\kappa_1 R_{i+1}^2 \Big/ (R_{i+2} - R_{i+1})$$

$$\left(3(1+3\alpha n)R_n^2 + 2R_nR_{n-1} + R_{n-1}^2\right)\dot{R}_n + \left(R_n^2 + R_nR_{n-1} + 3R_{n-1}^2\right)\dot{R}_{n-1} =$$

$$12\kappa_1 R_{n-1}^2 \Big/ (R_n - R_{n-1}) - 12\phi\kappa_2 R_n^2 \Big/ (S_1 - S_0)$$

$$\left(3S_{j+1}^2 + 2S_{j+1}S_j + S_j^2\right)\dot{S}_{j+1} + \left(S_{j+1}^2 + 2S_{j+1}S_j + 3S_j^2\right)\dot{S}_j = \frac{12\kappa_2 S_j^2}{S_{j+1} - S_j} - \frac{12\kappa_2 S_{j+1}^2}{S_{j+2} - S_{j+1}}$$

$$\left(3S_n^2 + 2S_nS_{n-1} + S_{n-1}^2\right)\dot{S}_n + \left(S_n^2 + 2S_nS_{n-1} + 3S_{n-1}^2\right)\dot{S}_{n-1} = \frac{12\kappa_2 S_{n-1}^2}{S_n - S_{n-1}}$$

The procedure for solving this problem is the same as for the two-phase cylindrical problem described in the last section.

NUMERICAL RESULTS

Numerical results for the two-phase cylindrical problem are presented in Figures 3–10 for a range of sub-divisions (n), sub-cooling parameters (ϕ), Stefan numbers (α) and the ratio of conductivities κ_1/κ_2. By changing one of the parameters while keeping the others constant we can study the effect of varying the parameters on the solidification process. For each parameter which we consider, we present two graphical results. One of them shows the effect of varying parameters on the position of phase change front $R(t)$ and the other shows the effect on the heat front $S(t)$.

Number of sub-divisions (n)

The number of sub-divisions which we use can have a great effect on the accuracy of the results. However, Figure 3 shows that for the phase change position $R(t)$, the results for $n = 4$ are almost indistinguishable from the results for $n = 16$. On the other hand, from Figure 4 we can see that the convergence of the heat front $S(t)$ is much slower. The curve for $n = 16$ is not very close to the curve for $n = 8$.

Sub-cooling parameter (ϕ)

Figures 5 and 6 show the effect of different sub-cooling parameters (ϕ) on the movement of phase change position and heat front. It is very clear that the sub-cooling parameter (ϕ) has a great effect on the phase change position. In Figure 5, the curve $\phi = 0$ corresponds to the case of zero sub-cooling (i.e. one-phase problem). From the graphs we can see that large values of ϕ can slow the solidification process substantially (the curves are flatter). On the other hand, the effect on the heat front is much smaller as the curves are much closer in Figure 6.

Stefan number (α)

Figures 7 and 8 show that the effect of the Stefan number (α) on the solidification process is similar to that of the sub-cooling parameter (ϕ). Large values of Stefan number will slow down the movement of the phase change position $R(t)$. On the other hand, the heat front position $S(t)$ is almost unaffected by the Stefan number.

Ratio of conductivities (κ_1/κ_2)

Figures 9 and 10 demonstrate the effect of the ratio of conductivities (κ_1/κ_2). A large value of this ratio will speed up the movement of the phase change position while the effect on the heat front position is small.

SUMMARY AND COMMENTS

Our numerical experiments for two-phase Stefan problem show that the solidification process can be affected by the physical parameters. The effects mainly appear in the movement of the solid-liquid interface, whereas the effect on the heat front position is relatively small. We have to mention that the results for the position of heat front may not be the same order of accuracy as those for the phase change position. This is because we have used the same number of sub-divisions for both regions. Further improvments on the method can be made by using different numbers of sub-divisions for two different regions.

The simple method described in this paper can also be used in real life applications. For example, in the design of water/oil pipes and tanks, cylindrical or spherical geometries are the most common shapes used.

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6. Poots, G. 'On the application of integral methods to the solution of problems involving the solidification of liquids initially at fusion temperature' *Int. J. Heat Mass Transfer* **5**, pp525–531, 1962.
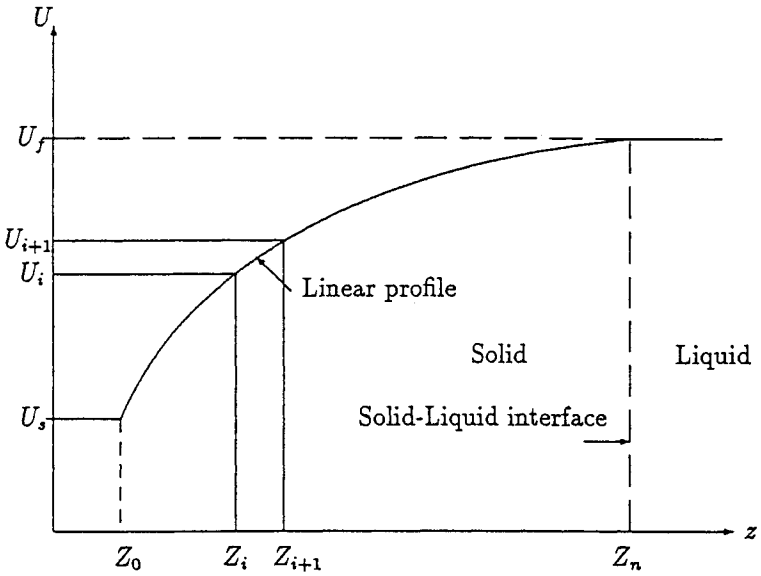
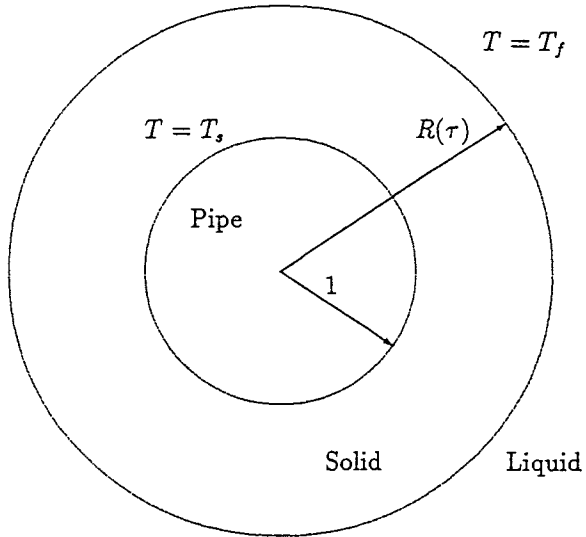Figure 1: Sub-division of the region for HBIM
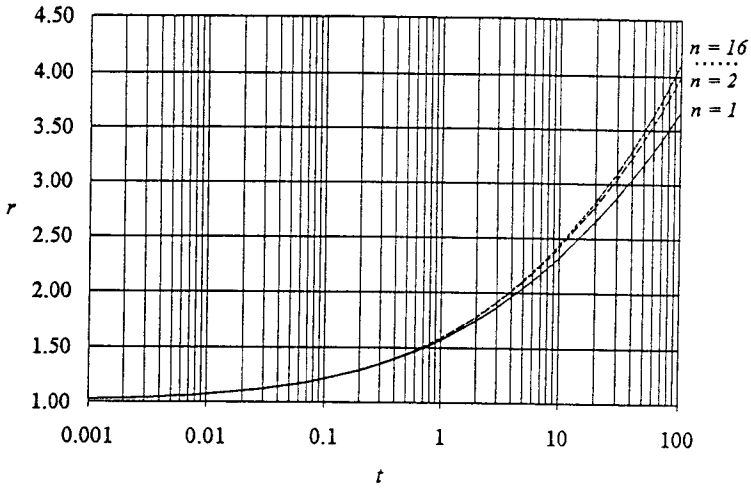

Figure 2: Cylindrical solidification

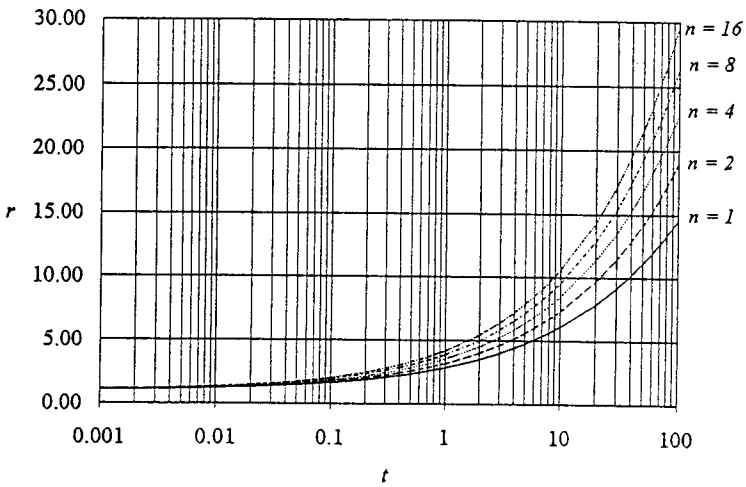Figure 3: Effect of number of sub-divisions (n) on $R(t)$.

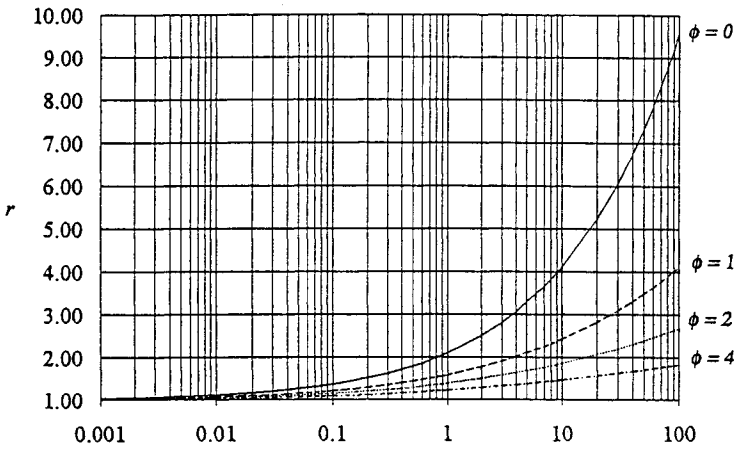Figure 4: Effect of number of sub-divisions (n) on $S(t)$.

Heat Transfer

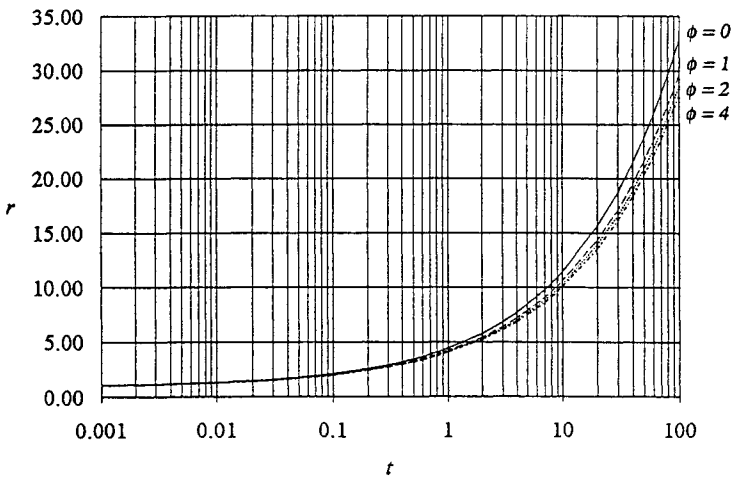Figure 5: Effect of sub-cooling parameter (ϕ) on $R(t)$.
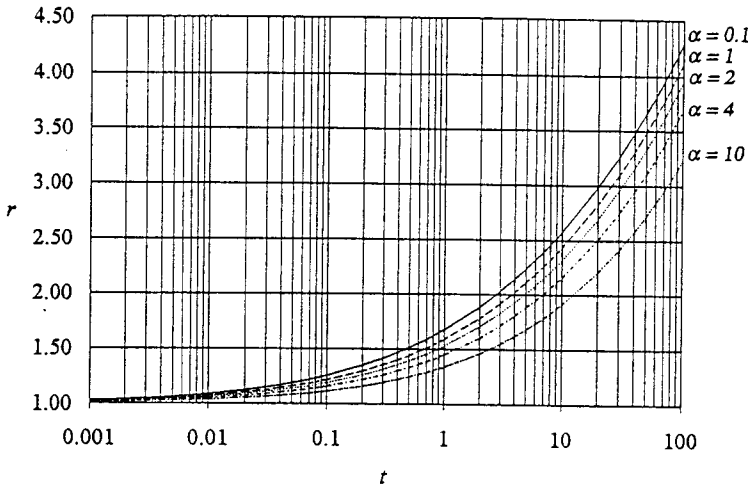

Figure 6: Effect of sub-cooling parameter (ϕ) on $S(t)$.
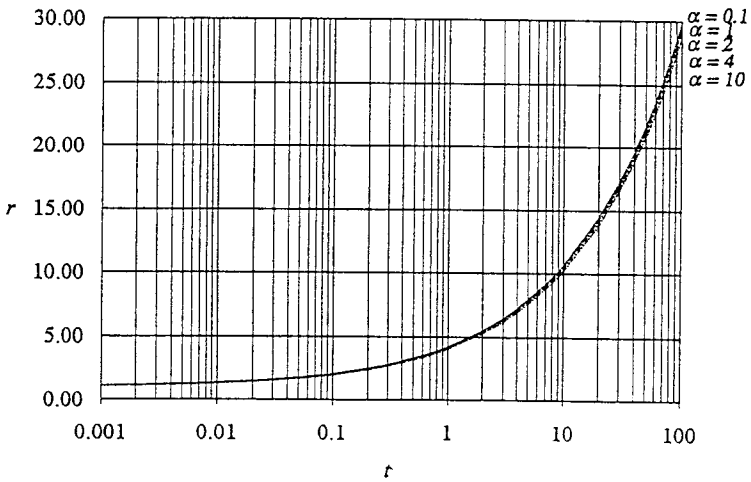
Figure 7: Effect of Stefan number (α) on $R(t)$.
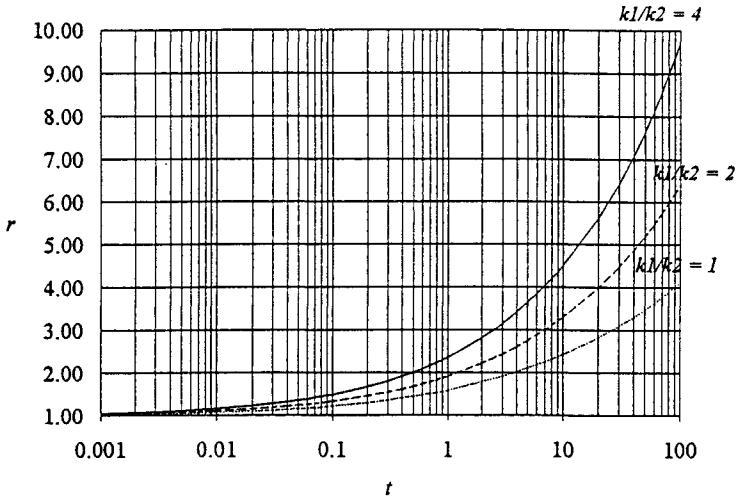

Figure 8: Effect of Stefan number (α) on $S(t)$.
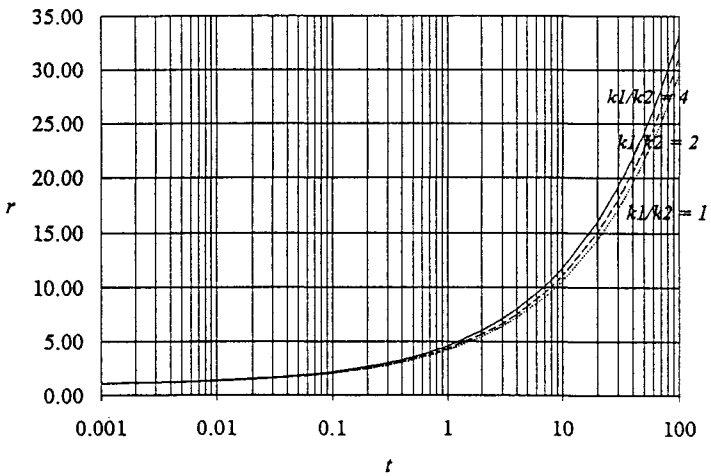
Figure 9: Effect of ratio of conductivities (κ_1/κ_2) on $R(t)$.

Figure 10: Effect of ratio of conductivities (κ_1/κ_2) on $S(t)$.