Numerical Solutions to Nash–Cournot Equilibria in Coupled Constraint Electricity Markets

Javier Contreras, *Member, IEEE*, Matthias Klusch, and Jacek B. Krawczyk

***Abstract*—A numerical method based on a relaxation algorithm and the Nikaido–Isoda function is presented for the calculation of Nash–Cournot equilibria in electricity markets. Nash equilibrium is attained through a relaxation procedure applied to an objective function, the Nikaido–Isoda function, which is derived from the existing profit maximization functions calculated by the generating companies. We also show how to use the relaxation algorithm to compute, and enforce, a coupled constraint equilibrium, which occurs if regulatory, generation, and distribution (and more) restrictions are placed on the companies and entire markets. Moreover, we use the relaxation algorithm to compute players' payoffs under several player configurations. This is needed for the solution of our game under cooperative game theory concepts, such as the bilateral Shapley value and the kernel. We show that the existence of both depends critically on demand price elasticity. The numerical method converges to a unique solution under rather specific but plausible concavity conditions. A case study from the IEEE 30-bus system, and a three-bus bilateral market example with a dc model of the transmission line constraints are presented and discussed.**

***Index Terms*—Bilateral Shapley value, coalition formation, coupled constraints, electricity markets, kernel, Nash–Cournot equilibrium, Nikaido–Isoda function, relaxation algorithm.**

I. Introduction

POWER system restructuring is transforming traditional vertically integrated monopolies into deregulated entities. Competition is fostered by newly created electricity markets where buyers and sellers can trade electricity in auctions or through bilateral agreements.

Cost minimization techniques used by electric utilities in the past are being replaced by efficient bidding algorithms. Currently, the objective of the electric utilities is profit maximization, where prices are determined by suppliers, consumers, transmission line owners, and other participants.

Perfect competitive markets are very difficult to attain in the electricity industry, mainly because of the small number of players that compete. On the other hand, network constraints affect the competitiveness of the market, since market bidders produce bottlenecks that may induce a large increase in prices. Thus, the assumption of the market as being an imperfect one is sensible.

Oligopolistic market models have been applied to study electricity markets since the beginning of the restructuring upheaval [1]–[5]. Our model is similar in spirit to Hobbs' [5] as both papers are concerned with Nash–Cournot equilibria [6] in electricity markets. However, there are two practical differences between our papers. While Hobbs' method in [5] depends on existence of a solution to a system of equations and inequalities, which result from mixed complementarity (Kuhn–Karesh–Tucker) conditions, ours relies on a function minimization procedure. The other difference is in the kind of equilibrium each paper is trying to establish: Hobbs in [5] endeavours to compute a Nash–Cournot equilibrium that would also satisfy a market clearing condition. We look for a *coupled constraint* equilibrium, which is a rather new solution concept to game theory problems (explained below) where the action space is jointly restricted for all players. We consider the latter kind of equilibrium an appropriate solution concept for many electricity market problems.

Using the approach of [5] or looking for an analytical solution to a particular concave game of several players with nonlinear profit functions and, possibly, constraints might be difficult. The Nikaido–Isoda function and a relaxation algorithm are combined in [7]–[9] to create a numerical method (NIRA) for solutions of infinite games. The method is attractive in that the most advanced computational routine required is minimization of a multivariate function. A sequential improvement of the Nikaido–Isoda function is obtained through a relaxation algorithm that is proved to converge to a Nash equilibrium for a wide class of problems, including nondifferential payoffs and coupled constraint games [7], [10].

The feature of handling games with a constrained strategy space is of particular importance for electricity market modeling. In a typical problem of electricity generation and distribution, the competing economic agents' strategy space is *coupled*. This is due to (mainly) capacity constraints and Kirchhoff's laws, and signifies that in the problem, there are joint constraints imposed on the combined strategy space of all agents. This means that the set of options available to an agent depends on the other agents' choices. If all agents act simultaneously, no traditional noncooperative game theory concept can be used to solve such a game. However, Rosen's normalized equilibrium, in which he introduces to solve games subject to a coupled constraint set, (called, for short, here and in [9] *coupled constraint games*) can be applied [10]. A contribution of this paper is to apply this solution concept in the context of electricity markets.

Manuscript received June 14, 2003.

J. Contreras is with the Escuela Técnica Superior de Ingenieros Industriales, Universidad de Castilla—La Mancha, Ciudad Real 13071, Spain (e-mail: Javier.Contreras@uclm.es).

M. Klusch is with the German Research Center for Artificial Intelligence, Saarbrücken 66123, Germany (e-mail: klusch@dfki.de).

J. B. Krawczyk is with the School of Economics and Finance, Victoria University of Wellington, Wellington 6000, New Zealand (e-mail: Jacek.Krawczyk@vuw.ac.nz).

Digital Object Identifier 10.1109/TPWRS.2003.820692

To illustrate how this concept works, we have solved several electricity market games. In particular, we have solved a (slightly modified) problem posed in [5] to compare the two methods.

The paper is organized as follows. Section II provides an introduction to basic definitions and concepts. Section III presents the relaxation algorithm and an illustrative example. Section IV shows several case studies where the algorithm is applied, both with and without transmission network constraints. For one of the examples, cooperative game theory solution concepts, such as bilateral Shapley value [11], [12] and the kernel [13], [14] are used to analyze coalition formation. The solutions, which might be computationally involved, are achieved by a sequential use of the relaxation algorithm. Concluding remarks are shown in Section V.

II. Definitions and Concepts

An n-person *game* is a formal representation or a mathematical model of a situation in which a number of *players* (that can be electricity companies) interact in a setting of strategic interdependence. This means that the welfare of a player depends upon his[1] own actions and on the actions of the other participants in the game. An n-person *game* (in normal form) is defined as a three-tuple $\{N, (X_i), (\phi_i), i \in N\}$, where N is the set of players; $N = \{1, 2, \ldots, \mathrm{n}\}$, X_i, is the set of strategies (or strategy space) of player i; and ϕ_i, $i \in N$ is the payoff (or welfare, utility, profit, etc.) function of player i that assigns a real number to each element of the Cartesian product of the strategy spaces $X_1 \times X_2 \times \ldots \times X_N$.

An agent plays a game through actions. An *action* is a choice that a player makes, according to his own *strategy*. Since a game sets a framework of strategic interdependence, a participant should be able to have enough information about its own and other players' past actions. This is called the *information set*. A *strategy* is a rule that tells the player which action(s) he should take, according to his own information set at any particular stage of a game. Finally, a *payoff* function expresses the utility that a player obtains given a strategy profile for all players.

More formally stated, assume that there are $i = 1, \ldots, n$ players participating in a game. Each player can take an individual action represented by a vector x_i. All players, when acting together, can take a collective action, which is a vector $\mathbf{x} = (x_1, \cdots, x_n)$. Denote by X_i an action set[2] of player i, by $\phi_i : X_i \rightarrow \Re$ his payoff function, and by X the collective action set. Then, if $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ are elements of the collective action set, an element $(y_i|\mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ of the collective action set can be seen as a set of actions where the ith player plays y_i while the remaining agents are playing $x_j, j = 1, 2, \ldots, i-1, i+1, \ldots, n$.

A point $\mathbf{x}^* = (x_1^*, \cdots, x_n^*)$ is called the *Nash equilibrium point* if, for each i

$$\phi_i(\mathbf{x}^*) = \max_{(x_i|\mathbf{x}^*) \in X} \phi_i(x_i|\mathbf{x}) \tag{1}$$

[1]Despite the political correctness trend, we need to use singular personal and possessive pronouns to address a nongender specific individual agent. We adopt the convention that the word "he" and "his" refer to a singular genderless player of a game.

[2]Which is identical to the strategy set if the information set is empty.

Notice that $\mathbf{x}^*$ solves the game $\{N, (X_i), (\phi_i), i \in N\}$ in the following sense: at $\mathbf{x}^*$ no player can improve his individual payoff by a unilateral (i.e., his own) action.

In order to compute the Nash equilibrium, we introduce the Nikaido–Isoda function [15]. This function transforms an equilibrium problem into an optimization problem. Let ϕ_i be the payoff function of player i, then the *Nikaido–Isoda function* $\Psi(\mathbf{x}, \mathbf{y})$ is defined as

$$\Psi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \left[\phi_i(y_i|\mathbf{x}) - \phi_i(\mathbf{x})\right]. \tag{2}$$

From (2), it follows that $\Psi(\mathbf{x}, \mathbf{x}) \equiv 0$. Each summand of the Nikaido–Isoda function represents the improvement in payoff that a player will receive when he changes his action from x_i to y_i, while all other players continue playing according to $\mathbf{x}$. That means that one player changes his action while the others do not. Thus, the function represents the sum of these improvements in the payoff. Note that the maximum value of this function is always nonnegative for a given $\mathbf{x}$. Also, the function is nonpositive for all feasible $\mathbf{y}$ when $\mathbf{x}^*$ is a Nash equilibrium, since no player can improve his payoff at equilibrium. In consequence, each summand can be at most zero at the Nash equilibrium.

In conclusion, when the Nikaido–Isoda function satisfies certain concavity conditions (defined in Appendix A) and cannot be made (significantly) positive for a given $\mathbf{y}$, the Nash equilibrium point is (approximately) reached. This is used to construct a termination condition for the relaxation algorithm, such that when an ε is chosen, the Nash equilibrium is obtained when $\max_{\mathbf{y} \in \Re^m} \Psi(\mathbf{x}^s, \mathbf{y}) < \varepsilon$, where s is the iterative step of the relaxation algorithm. See Appendix A and [8].

An element $\mathbf{x}^* \in X$ is referred to as a *Nash normalized equilibrium point* if

$$\max_{y \in X} \Psi(\mathbf{x}^*, \mathbf{y}) = 0. \tag{3}$$

Given the concavity conditions, a Nash normalized equilibrium is also a Nash equilibrium point [16].

Finally, the optimum response function is introduced. It is the result of maximizing the Nikaido–Isoda function, where all players try to improve their payoffs. The *optimum response function* at the point $\mathbf{x}$ is

$$Z(\mathbf{x}) = \arg\max_{\mathbf{y} \in X} \Psi(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, Z(\mathbf{x}) \in X. \tag{4}$$

This function returns the set of players' actions whereby they all try to unilaterally maximize their respective payoffs. So, by "playing" actions $Z(\mathbf{x})$ rather than $\mathbf{x}$, the players approach the equilibrium.

In the next section, an algorithm that uses the Nikaido–Isoda function to compute a Nash normalized equilibrium is presented. At each iteration of the algorithm, the players wish to move to a point that represents an improvement on the current player's point. Technical definitions that are used in the convergence theorem of the algorithm are included in Appendix A.

III. Relaxation Algorithm

A. Relaxation Algorithm

In order to find a Nash equilibrium of a game, having an initial estimate $\mathbf{x}^0$, the relaxation algorithm of the optimum response function, when $Z(\mathbf{x})$ is single-valued and the concavity conditions are satisfied, is

$$\mathbf{x}^{s+1} = (1-\alpha_s)\mathbf{x}^s + \alpha_s Z(\mathbf{x}^s), \quad \mathrm{s} = 0, 1, 2, \ldots \tag{5}$$

where $0 < \alpha_s \leq 1$ An iterative step $s+1$ is constructed as a weighted average of the improvement point $Z(\mathbf{x}^s)$ and the current point $\mathbf{x}^s$. The optimum response function $Z(\mathbf{x}^s)$ is calculated after solving an optimization problem, as seen in (4). The averaging shown in (5) ensures convergence of the algorithm, under certain conditions [7], [9]. At each stage, the optimum response of a player is chosen, assuming that the rest will play as they did in the previous period. Thus, by taking a sufficient number of iterations, the algorithm converges to the Nash equilibrium $\mathbf{x}^*$. The problem can be either considered a centralized optimization model or a calculation of the succession of actions by the players at each stage, where players choose their optimum response given the actions of the opponents in the previous period.

The theorem that ensures convergence of the relaxation algorithm is presented in full detail in Appendix A. Condition 5 of the theorem is of special importance for the solution to games, in which the strategy space of competing generation and distribution agents is coupled (e.g., due to Kirchhoff's laws). Such games are *coupled constraint games* [10] and possess equilibrium solutions under a rather technical (but likely satisfied) assumption. The assumption is that the game is *diagonally strictly concave*[3] (DST). It follows from [17] that if condition 5 is satisfied, then the underlying game *is* DSC. Therefore, if the relaxation algorithm converges to an equilibrium, then, this equilibrium is a coupled constraint game solution.

B. Duopoly Example

We will use a simple example to illustrate, quite in detail[4] how a Nash equilibrium can be computed using the Nikado–Isoda function. In this example [9], there are two identical firms that sell an identical product on the same market. Each firm x_i chooses its production such that its profit is maximized. Let α, λ, and ρ be constants (price intercept, linear cost coefficient, and inverse elasticity, respectively). Using the inverse demand equation, the market price becomes

$$p(\mathbf{x}) = \alpha - \rho(x_1 + x_2) \tag{6}$$

and the profit made by firm i is

$$\phi_i(\mathbf{x}) = p(\mathbf{x})x_i - \lambda x_i = [\alpha - \lambda - \rho(x_1 + x_2)]\, x_i. \tag{7}$$

[3]Loosely speaking, *diagonal strict concavity* means that each player has more control over his payoff than the other players have over it.

[4]A reader not interested in this level of detail can proceed to Section IV where motivating electricity market case studies are analyzed.

The Nikaido–Isoda function is

$$\begin{aligned}\Psi(\mathbf{x},\mathbf{y}) = {}& [\alpha - \lambda - \rho(y_1 + x_2)]\, y_1 - [\alpha - \lambda - \rho(x_1 + x_2)]\, x_1 \\ &+ [\alpha - \lambda - \rho(x_1 + y_2)]\, y_2 - [\alpha - \lambda - \rho(x_1 + x_2)]\, x_2\end{aligned} \tag{8}$$

leading to an optimum response function

$$Z(\mathbf{x}) = \arg\max_{\mathbf{y}\in X} \Psi(\mathbf{x},\mathbf{y}) = \frac{\alpha-\lambda}{2\rho}(1,1) - \frac{1}{2}(x_2, x_1). \tag{9}$$

The above maximization provides the "improvement" values for y_1 and y_2, given "current" x_1 and x_2 (initial or calculated in the previous iteration). Given differentiability and weak convex-concavity (see Appendix A) of (8) and because there are no constraints or production limits in this case, the improvement values are the result of just making the first derivatives of (8) w.r.t. y_1 and y_2 equal to zero, respectively. From (9), it can be seen that

$$\begin{aligned} Z(x_1, x_2) &= (y_1, y_2), \quad \text{where} \\ y_1 &= \frac{\alpha-\lambda}{2\rho} - \frac{x_2}{2}; \\ y_2 &= \frac{\alpha-\lambda}{2\rho} - \frac{x_1}{2}. \end{aligned} \tag{10}$$

Once y_1 and y_2 are known, they will, through (5), become the "current" values and, in the next iteration, new "improvement" values will be produced. The process continues until convergence is reached [i.e., no significant improvement in (8) can be achieved].

Notice that in this example, all conditions of the convergence theorem (see Appendix A) are met. In particular, the matrix (See the equation at the bottom of page.) whose positive definiteness is required for the satisfaction of condition 5 of the convergence theorem (see Appendix A), is strictly positive definite for a positive ρ. $(\Psi_{x1}(\mathbf{x},\mathbf{y}), \Psi_{x2}(\mathbf{x},\mathbf{y}))|_{y=x}$ and $(\Psi_{y1}(\mathbf{x},\mathbf{y}), \Psi_{y2}(\mathbf{x},\mathbf{y}))|_{y=x}$ are the Jacobians of the Nikaido–Isoda function evaluated at $\mathbf{y} = \mathbf{x}$, $\Psi_{xx}(\mathbf{x},\mathbf{y})|_{y=x}$ is the Hessian of the Nikaido-Isoda function w.r.t. the first argument and $\Psi_{yy}(\mathbf{x},\mathbf{y})|_{y=x}$ is the Hessian of the Nikaido–Isoda function w.r.t. the second argument, both evaluated at $\mathbf{y} = \mathbf{x}$. The result of the Nash equilibrium is $x_i^N = ((\alpha-\lambda)/(3\rho))$ with a corresponding payoff $\phi_i(x_i^N) = ((\alpha-\rho)^2/(9\rho))$, where superscript N stands for Nash equilibrium. For the particular case of $\alpha = 20$, $\lambda = 4$ and $\rho = 1$, then $x^N = ((16/3), (16/3))$. Note that the relaxation algorithm obtains the solution to the fixed point problem posed in (10): $x_1 = ((\alpha-\lambda)/(2\rho)) - (x_2/2)$; $x_2 = ((\alpha-\lambda)/(2\rho)) - (x_1/2)$, after an iterative process.

In the next section, we analyze situations with a larger number of agents. We use the relaxation algorithm to compute solutions to competitive Nash–Cournot games and also to cooperative games. For the latter, we apply the bilateral Shapley value [12] and the kernel [14] as solution concepts.

IV. Case Studies

Two case studies are proposed to test the relaxation algorithm. The first one considers an electricity market that uses the IEEE

TABLE I
IEEE 30-BUS SYSTEM MARKET DATA

company #	generator #	P_g^{min} [MW]	P_g^{max} [MW]	P_C^{min} [MW]	P_C^{max} [MW]
1	1	0	80	0	80
2	2	0	80	0	130
	3	0	50		
3	4	0	55	0	125
	5	0	30		
	6	0	40		

TABLE II
GENERATING UNITS COST COEFFICIENTS

generator #	c_i [\$/MW^{2}h]	d_i [\$/MWh]	e_i [\$/h]
1	0.04	2	0
2	0.035	1.75	0
3	0.125	1	0
4	0.0166	3.25	0
5	0.05	3	0
6	0.05	3	0

30-bus system [4]. The second one is taken from [5], including transmission line modeling and flow constraints.

A. Case Study 1

In this case study, it is assumed that there are three generating companies and each of them possesses several generating units, as shown in Table I; P_g and P_C are the power generation of a unit and a company, respectively.

The cost of a generating unit i is of the type $C_i(P_{gi}) = (c_i/2)P_{gi}^2 + d_iP_{gi} + e_i$, whose coefficients are reported in Table II.

Assuming that the electricity demand is a strictly decreasing function of the price p, the demand function in an interval of time during a day of study considered standard can be expressed as $P_{\text{load}}(p) = P_{\text{load}}^0(p) + ap$ where $P_{\text{load}}^0(p)$ is the total power demand level expected for a selected time interval, and a represents the elasticity of the demand w.r.t. price. In particular, the standard loading condition of the IEEE 30-bus system in a selected interval is supposed to be [4]

$$P_{\text{load}}(p) = 189.2 - 0.5p \tag{12}$$

that can be also expressed conversely as

$$p = 378.4 - 2P_{\text{load}}(p) \tag{13}$$

where

$$P_{\text{load}}(p) = \sum_{i=1}^{ng} P_{gi} - P_{\text{loss}} \tag{14}$$

ng are the total number of generators and P_{loss} represents the transmission losses throughout the system. This simple example neglects losses, but they can be easily incorporated as a function of the generation power using the B-matrix loss formula, as shown in [4].

Once these premises are established, the profit made by company j that owns ngj generating units is

$$\begin{aligned}\phi_j(P_{gj}) &= p\sum_{j=1}^{ngj} P_{gj} - \sum_{j=1}^{ngj} C_j(P_{gj}) \\ &= \left(378.4 - 2\sum_{i=1}^{ng} P_{gi}\right)\sum_{j=1}^{ngj} P_{gj} \\ &\quad - \sum_{j=1}^{ngj}\left(\frac{c_j}{2}P_{gj}^2 + d_jP_{gj} + e_j\right)\end{aligned} \tag{15}$$

subject to the constraints $P_{\min gj} \leq P_{gj} \leq P_{\max gj}$. The Nikaido–Isoda function for (15) is derived in the same way as

$$\begin{aligned}&Q(\mathbf{x},\mathbf{x}) = \Psi_{xx}(\mathbf{x},\mathbf{y})|_{y=x} - \Psi_{yy}(\mathbf{x},\mathbf{y})|_{y=x} = \\ &\left.\begin{pmatrix}\frac{\partial}{\partial x_1}\Psi_{x1}(x_1,x_2,y_1,y_2) & \frac{\partial}{\partial x_2}\Psi_{x1}(x_1,x_2,y_1,y_2)\\ \frac{\partial}{\partial x_1}\Psi_{x2}(x_1,x_2,y_1,y_2) & \frac{\partial}{\partial x_2}\Psi_{x2}(x_1,x_2,y_1,y_2)\end{pmatrix}\right|_{y=x} - \\ &\left.\begin{pmatrix}\frac{\partial}{\partial y_1}\Psi_{y1}(x_1,x_2,y_1,y_2) & \frac{\partial}{\partial y_2}\Psi_{y1}(x_1,x_2,y_1,y_2)\\ \frac{\partial}{\partial y_1}\Psi_{y2}(x_1,x_2,y_1,y_2) & \frac{\partial}{\partial y_2}\Psi_{y2}(x_1,x_2,y_1,y_2)\end{pmatrix}\right|_{y=x} = \\ &\left.\begin{pmatrix}\frac{\partial}{\partial x_1}(2\rho x_1 + 2\rho x_2 - \rho y_2) & \frac{\partial}{\partial x_2}(2\rho x_1 + 2\rho x_2 - \rho y_2)\\ \frac{\partial}{\partial x_1}(2\rho x_1 + 2\rho x_2 - \rho y_1) & \frac{\partial}{\partial x_2}(2\rho x_1 + 2\rho x_2 - \rho y_1)\end{pmatrix}\right|_{y=x} - \\ &\left.\begin{pmatrix}\frac{\partial}{\partial y_1}(-\rho x_2 - 2\rho y_1) & \frac{\partial}{\partial y_2}(-\rho x_2 - 2\rho y_1)\\ \frac{\partial}{\partial y_1}(-\rho x_1 - 2\rho y_2) & \frac{\partial}{\partial y_2}(-\rho x_1 - 2\rho y_2)\end{pmatrix}\right|_{y=x} = \\ &\rho\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} - (-\rho)\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = 2\rho\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\end{aligned} \tag{11}$$

(8) was. Below, company-by-company, the new variables y_j are replacing the values of P_{gj}

$$\begin{aligned}\Psi(P_{gj}, y_j) = \{&[378.4 - 2(y_1 + P_{g2} + \cdots + P_{g6})]\, y_1 \\ &- \left(\frac{c_1}{2}y_1^2 + d_1 y_1 + e_1\right)\Big\} \\ &- \{[378.4 - 2(P_{g1} + P_{g2} + \cdots + P_{g6})]\, P_{g1} \\ &- \left(\frac{c_1}{2}P_{g1}^2 + d_1 P_{g1} + e_1\right)\Big\} \\ &+ \{[378.4 - 2(P_{g1} + y_2 + y_3 + \cdots)]\,(y_2 + y_3) \\ &- \left(\frac{c_2}{2}y_2^2 + d_2 y_2 + e_2\right) \\ &- \left(\frac{c_3}{2}y_3^2 + d_3 y_3 + e_3\right)\Big\} \\ &- \{[378.4 - 2(P_{g1} + P_{g2} + P_{g3} + \cdots)] \\ &\times (P_{g2} + P_{g3}) - \left(\frac{c_2}{2}P_{g2}^2 + d_2 P_{g2} + e_2\right) \\ &- \left(\frac{c_3}{2}P_{g3}^2 + d_3 P_{g3} + e_3\right)\Big\} \\ &+ \{[378.4 - 2(P_{g1} + \cdots + y_4 + y_5 + y_6)] \\ &\times (y_4 + y_5 + y_6) - \left(\frac{c_4}{2}y_4^2 + d_4 y_4 + e_4\right) \\ &- \left(\frac{c_5}{2}y_5^2 + d_5 y_5 + e_5\right) \\ &- \left(\frac{c_6}{2}y_6^2 + d_6 y_6 + e_6\right)\Big\} \\ &- \{[378.4 - 2(P_{g1} + \cdots + P_{g4} + P_{g5} + P_{g6})] \\ &\times (P_{g4} + P_{g5} + P_{g6}) \\ &- \left(\frac{c_4}{2}P_{g4}^2 + d_4 P_{g4} + e_4\right) \\ &- \left(\frac{c_5}{2}P_{g5}^2 + d_5 P_{g5} + e_5\right) \\ &- \left(\frac{c_6}{2}P_{g6}^2 + d_6 P_{g6} + e_6\right)\Big\}. \end{aligned} \tag{16}$$

Thus, (16) has three terms: the first corresponds to company #1 who owns one generator, the second to company #2, with two generators, and the third to company #3, with three generators. This function is weakly convex-concave, since the there are positive square terms of P_{gj} and negative square terms of y_j. Thus, the game also qualifies as diagonally strictly concave. For this Nikaido–Isoda function, the optimum response function can be written as follows:

$$\begin{aligned} &Z(\mathbf{x}) = \arg\max_{\mathrm{y}} \Psi(P_{gj}, y_j) \\ &\text{subject to} \\ &P_{\min gj} \le y_j \le P_{\max gj}. \end{aligned} \tag{17}$$

Relationship (17) produces the values of $y_1, \cdots, y_6$, given the values for $P_{g1}, \cdots, P_{g6}$. The latter comes either from an initial estimation (only in the first iteration), or from a previous iteration of the relaxation algorithm (5). Finally, both $y_1, \cdots, y_6$ and the previous iteration values for $P_{g1}, \cdots, P_{g6}$ are plugged in the relaxation formula (5) to obtain a new value of $P_{g1}, \cdots, P_{g6}$ and the next iteration starts. Convergence conditions are met, since $\Psi_{xx}(P_{gj}, y_j)|_{yj=Pgi} - \Psi_{yy}(P_{gj}, y_j)|_{yj=Pgj}$ is positive definite. In fact, the matrix is similar to the one in (11), but there is a positive extra term from the quadratic cost function in the diagonal terms. Thus, since the inverse demand (which is equal to 2) is also positive, the convergence is guaranteed.

TABLE III
IEEE 30-Bus System Nash Equilibrium Results

P_{g1} = 46.66 MW	P_{C1} = 46.66 MW	Profit= $/h 4397.82
P_{g2} = 32.16 MW	P_{C2} = 47.16 MW	Profit= $/h 4479.92
P_{g3} = 15 MW		
P_{g4} = 22.13 MW	P_{C3} = 46.79 MW	Profit= $/h 4389.79
P_{g5} = 12.33 MW		
P_{g6} = 12.33 MW		

NIRA-2, a software package programmed in Matlab, has been used to solve this example [18]. Final results with an optimized size step (see Appendix A) after 18 iterations are presented in Table III, where P_{gi} is the production of generator i, and P_{Cj} is the total production of company j. The final price is 97.19 U.S.$/MWh. Note that the same results can be achieved by applying the traditional Nash–Cournot equilibrium conditions expressed as a system of equations. However, our method appears more robust in that it abstracts from the analytical form of those conditions.

The above results are obtained assuming that the three companies compete against each other. However, some of the generating companies may like to form a cartel to increase their overall profits. In other words, it is possible that the final productions of the companies are not necessarily the ones from Table III, but lower, by exerting market power withholding energy. Since the number of companies involved is small, it is feasible to study all possible companies' coalitions. After enumerating all of the combinations, the relaxation algorithm can be applied to each one, which makes up a game where the players are the coalitions, and compute a Nash equilibrium in a game between coalitions. In this way, cooperation among the players creates several scenarios that have different Nash equilibria.

However, what is not solved by the relaxation algorithm is how to split the profit that results from cooperation. Cooperative game theory concepts, such as the bilateral Shapley value (BSV) [12] and the kernel [14] are useful tools to study how the Nash equilibrium value can be split among the companies. Both represent a dynamic way to build coalitions among players and to allocate profits after the coalitions are formed, see Appendix B for details. Seeking profit maximization, each company can join other companies and become part of a new player composed of two or more firms.

Table IV presents five different scenarios in which the companies can be arranged according to all possible coalition combinations. The coalition values $v\{i,j\}$ express the profit obtained by a coalition composed of players (companies) i and j, as given by (15). Coalitions' values are always a result of a Nash equilibrium in our game. These values are calculated applying the relaxation algorithm to the Nikaido–Isoda function corresponding to each scenario, and then obtaining the final individual profits per coalition (or coalition values) after the iterative algorithm has converged. Obviously, the first scenario is as in Table III, the second scenario has two players: $\{1,2\}$ and

TABLE IV
IEEE 30-BUS SYSTEM COALITION SCENARIOS. DEMAND ELASTICITY = 0.5

Coalitions	Production (MW)	Price ($/MWh)	Coalition values v{i,j}	Total profit ($/h)
{1},	46.66	97.19	**4397.82**	13,267.53
{2},	47.16		**4479.92**	
{3}	46.79		**4389.79**	
{1, 2},	62.77	128.32	**7914.99**	15,691.01
{3}	62.27		7776.02	
{1, 3},	62.48	128.37	**7836.26**	15,713.33
{2}	62.54		7877.07	
{2, 3},	62.76	128.68	**7925.80**	15,714.65
{1}	62.1		7788.85	
{1, 2, 3}	93.81	190.78	**17,665.02**	17,665.02

3, and the last scenario is not a game but a grand coalition optimization problem.[5] For example, if there were two players, such as company $1, 2$ and company 3, then, there would be only two terms in the Nikaido–Isoda function, and not three, as in (16). The Nikaido-Isoda function for the second scenario would be as follows:

$$\begin{aligned}\Psi(P_{gj}, y_j) = &\{[378.4-2(y_1+y_2+y_3+P_{g4}+P_{g5}+P_{g6})]\\ &\times(y_1+y_2+y_3)-\left(\frac{c_1}{2}y_1^2+d_1y_1+e_1\right)\\ &-\left(\frac{c_2}{2}y_2^2+d_2y_2+e_2\right)-\left(\frac{c_3}{2}y_3^2+d_3y_3+e_3\right)\Big\}\\ &-\{[378.4-2(P_{g1}+P_{g2}+\cdots+P_{g6})]\\ &\times(P_{g1}+P_{g2}+P_{g3})-\left(\frac{c_1}{2}P_{g1}^2+d_1P_{g1}+e_1\right)\\ &-\left(\frac{c_2}{2}P_{g2}^2+d_2P_{g2}+e_2\right)\\ &-\left(\frac{c_3}{2}P_{g3}^2+d_3P_{g3}+e_3\right)\Big\}\\ &+\{[378.4-2(P_{g1}+P_{g2}+P_{g3}+y_4+y_5+y_6)]\\ &\times(y_4+y_5+y_6)-\left(\frac{c_4}{2}y_4^2+d_4y_4+e_4\right)\\ &-\left(\frac{c_5}{2}y_5^2+d_5y_5+e_5\right)-\left(\frac{c_6}{2}y_6^2+d_6y_6+e_6\right)\Big\}\\ &-\{[378.4-2(P_{g1}+\cdots+P_{g4}+P_{g5}+P_{g6})]\\ &\times(P_{g4}+P_{g5}+P_{g6})-\left(\frac{c_4}{2}P_{g4}^2+d_4P_{g4}+e_4\right)\\ &-\left(\frac{c_5}{2}P_{g5}^2+d_5P_{g5}+e_5\right)\\ &-\left(\frac{c_6}{2}P_{g6}^2+d_6P_{g6}+e_6\right)\Big\}. \qquad (18)\end{aligned}$$

In order to determine the coalitions that are actually formed, we need to extract the coalition values that each player or coalition has in the game. Note that they always correspond to the minimal values—profits—that a coalition can guarantee for itself against any other coalition. These values are highlighted in Table IV and shown in Table V, and they are the basis for the coalition formation and cost allocation algorithms. For instance, player 1 has a value of U.S.$ 4397.82 when playing against independent players 2 and 3, and a value of U.S.$ 7788.85 when playing against player $\{2, 3\}$. Therefore, the minimum that 1 can guarantee for himself in any scenario is U.S.$ 4397.82. Note that the enumeration of coalitions must be exhaustive, such that the set of all coalitions contains $[\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}]$. All coalition-related calculations are done using the COALA-IDEAS multiagent and coalition formation software [19].

[5]Note that (16) is the Nikaido–Isoda function when there are three independent companies. If two or more companies form a coalition, the Nikaido–Isoda function is different, and the number of terms changes.

TABLE V
COALITION VALUES OF THE GAME

Coalition	Coalition values v{i,j}
{1}	4397.82
{2}	4479.92
{3}	4389.79
{1, 2}	7914.99
{1, 3}	7836.26
{2, 3}	7925.80
{1, 2, 3}	**17,665.02**

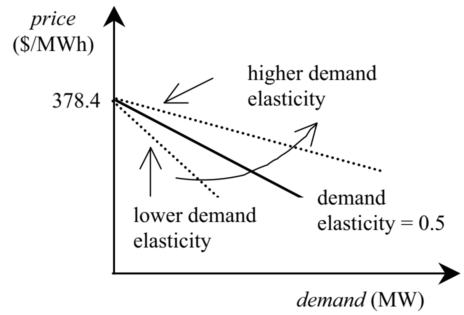

Fig. 1. Price pivoting produces changes in demand elasticity.

The game shown in Table V is subadditive, except for the grand coalition, meaning that the value of any coalition $\{i, j\}$ is always less than the sum of the values of i and j. The complementary case is called a superadditive game. From the coalition formation simulation, it is observed that this game has neither BSV nor kernel solutions. The same situation occurs when the demand becomes more inelastic, such that $p = 378.4 - 4P_{\text{load}}(p)$.

However, when the demand becomes more elastic (i.e., $p = 378.4 - P_{\text{load}}(p)$), BSV and kernel solutions do exist. Fig. 1 depicts the changes in elasticity when pivoting around the price intercept.

For the new elasticity coefficient, the game now is neither subadditive nor superadditive, since sometimes it is better to be alone and sometimes to join someone, depending on the coalition values or joint profits.

In this new game shown in Table VI, both the BSV and kernel solutions follow the same coalition building sequence $[\{1\}, \{2\}, \{3\}] \rightarrow [\{1\}, \{2, 3\}] \rightarrow [\{1, 2, 3\}]$. Table VII shows all of the coalitions and values for the new game.

Final results in terms of profit allocation for companies 1, 2, and 3 are as follows:

- BSV profit allocation (U.S.$ 10 658.84, U.S.$ 12 316.99, U.S.$ 12 259.22);
- Kernel profit allocation (U.S.$ 10 634.80, U.S.$ 12 179.37, U.S.$ 12 420.88).

Final results show that the total profit of U.S.$ 35 235.05 can be divided among the three companies according to these ratios. It represents a final agreement on splitting profits if acting

TABLE VI
IEEE 30-BUS SYSTEM COALITION SCENARIOS. DEMAND ELASTICITY = 1

Coalitions	Production (MW)	Price ($/MWh)	Coalition values v{i, j}	Total profit ($/h)
{1},	80	102.26	**7893.23**	27,309.61
{2},	98		**9737.07**	
{3}	98.13		**9679.31**	
{1, 2},	125.26	129.05	**15,820.57**	31,369.37
{3}	124.09		15,548.80	
{1, 3},	125.65	129.57	**15,865.07**	31,413.62
{2}	123.17		15,548.55	
{2, 3},	147.3	151.1	**21,810.59**	33,610.94
{1}	80		11,800.35	
{1, 2, 3}	187.31	191.09	**35,235.05**	35,235.05

TABLE VII
COALITION VALUES OF THE GAME

Coalition	**Coalition values v{i, j}**
{1}	**7893.23**
{2}	**9737.07**
{3}	**9679.31**
{1, 2}	**15,820.57**
{1, 3}	**15,865.07**
{2, 3}	**21,810.59**
{1, 2, 3}	**35,235.05**

as a cartel. Thus, it can be observed that price-demand elasticity is the key to increase the chance of cooperation among oligopolistic firms. If the demand becomes more elastic, higher prices will be paid (for a given demand) and the "cake" to share will be bigger. This will make a final agreement more plausible (than when the cake is smaller) and all players will most likely end up in a grand coalition, increasing their profits.[6]

B. Case Study 2

The second example is taken from [5] although slightly modified. It is just one of the many case studies presented in that paper. Our modified example assumes no arbitrage (existence of marketers that can sell and buy megawatts from producers and consumers) and a linear dc network. In the example selected, indexes i and j indicate nodes. Each company f owns several generating units distributed throughout the network. $C(P_{fgi})$ is the cost per megawatt-hour of generating unit g that belongs to company f and is placed at node i; its production is P_{fgi} MW. The maximum capacity of a generator is $P_{\max fgi}$ MW. Consumers at node i consume q_i MW. At each node, linear demand functions are assumed to be of the form $p_i(q_i) = P_{io} - (P_{io}/Q_{io})q_i$ U.S.$/MWh, where P_{io} and Q_{io} are the price and quantity intercepts, respectively. It is also assumed that the market is bilateral, and s_{fj} MW are sold by the company f to consumers at node j. Market clearing is such that $\sum_f s_{fj} = q_j$. Also, an energy balance is imposed on each company $\sum_{i,g} P_{fgi} = \sum_j s_{fj}$. Given that each company f chooses generation P_{fgi} and sales s_{fi} to maximize profit U.S.$/h, which is equal to revenue minus generation costs

$$\max \sum_j \left[P_{jo} - \left(\frac{P_{jo}}{Q_{jo}}\right) \left(s_{fj} + \sum_{k \neq f} s_{kj} \right) \right] s_{fj} - \sum_{i,g} C(P_{fgi}) P_{fgi}$$

$$\text{subject to :}$$

$$\begin{aligned} & P_{fgi} \leq P_{\max fgi}, \quad \forall \text{ nodes } i, \text{ generators } g \\ & \sum_j s_{fj} = \sum_{i,g} P_{fgi} \\ & \sum_j s_{fj} = q_j \\ & \forall s_{fj}, P_{fgi} \geq 0. \end{aligned} \tag{19}$$

We are interested in a noncooperative Nash–Cournot solution to the game at hand. This means that we are looking for a distribution of generation and the corresponding payoffs such that no player can improve his own payoff by a unilateral action.

Numerical data for the general formulation of problem (19) are as follows. There are three buses $i = 1$, 2, and 3, each of which has customers. Generation only occurs at buses 1 and 2 and each pair of buses is connected by a single transmission line. The demand functions are $p_i(q_i) = 40 - 0.08q_i$ for buses $i = 1$, 2, and $p_3(q_3) = 32 - 0.0516q_3$ U.S.$/MWh. Thus, the demand is more elastic at the demand-only node 3 (bus 3). Firm's 1 generator is placed at $i = 1$ and firm's 2 at $i = 2$. Both generators have unlimited capacity and constant marginal costs are U.S.$ 15/MWh for firm 1 and U.S.$ 20/MWh for firm 2.

Considering these data, two cases are run. The first case assumes that there are no limits on the line flows and, therefore, there is no congestion. As a result, both firms solve the following optimization problems subject to linear constraints:

Firm 1:

$$\max \{[40 - 0.08(s_{11} + s_{21})] s_{11} + [40 - 0.08(s_{12} + s_{22})] s_{12} + [32 - 0.0516(s_{13} + s_{23})] s_{13} - 15P_{1,1,1}\}.$$

Firm 2:

$$\max \{[40 - 0.08(s_{11} + s_{21})] s_{21} + [40 - 0.08(s_{12} + s_{22})] s_{22} + [32 - 0.0516(s_{13} + s_{23})] s_{23} - 20P_{2,2,2}\}$$

subject to

$$\begin{aligned} & P_{1,1,1} = s_{11} + s_{12} + s_{13}, \\ & P_{2,2,2} = s_{21} + s_{22} + s_{23}, \\ & q_1 = s_{11} + s_{21}, \\ & q_2 = s_{12} + s_{22}, \\ & q_3 = s_{13} + s_{23}, \\ & P_{1,1,1} + S_{\text{base}} \frac{(\theta_2 - \theta_1)}{x_{12}} + S_{\text{base}} \frac{(\theta_3 - \theta_1)}{x_{13}} = q_1, \\ & P_{2,2,2} + S_{\text{base}} \frac{(\theta_1 - \theta_2)}{x_{12}} + S_{\text{base}} \frac{(\theta_3 - \theta_2)}{x_{23}} = q_2, \\ & S_{\text{base}} \frac{(\theta_1 - \theta_3)}{x_{13}} + S_{\text{base}} \frac{(\theta_2 - \theta_3)}{x_{23}} = q_3, \\ & s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23} \geq 0, \\ & \theta_{\min} \geq \theta_1, \theta_2, \theta_3 \geq \theta_{\max} \end{aligned} \tag{20}$$

[6]Note that the conclusion about a bigger cake and the coalition formation is also valid for when we pivot the price schedule around the demand intercept. However, in that case, prices (and the cake) will increase with the schedules becoming steeper. This is the conclusion that can be found in [20] where the system operates with a price-responsive demand.

where θ_i, $i = 1, 2, 3$ are the bus angles in radians, the bus reference angle is at node 3 ($\theta_3 = 0$), and x_{ij} is the reactance value in per unit. The three lines have equal impedances of 0.2 p.u., S_{base} is the base power: 100 MVA, and the angle limits are set to ± 0.35 radians. The first five constraints in (20) can be dropped, since all variables other than s_{ij} are a linear combination of them. The last three linear constraints in (20) can be converted into equivalent inequalities. Every equality is equivalent to two inequalities: "greater than or equal" and "less than or equal," to be satisfied simultaneously. Thus, written in this convention, we would have had six inequality constraints. However, we drop the three constraints, which are "greater than or equal." This is because the Lagrange multipliers associated with those constraints are zero. Thus, the resulting problem has eight decision variables s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23}, θ_1, θ_2.

For (20), the Nikaido–Isoda function becomes

$$\begin{aligned}\Psi(s_{ij}, y_{ij}) = &\{[40 - 0.08(y_{11} + s_{21})]\, y_{11} \\ &+ [40 - 0.08(y_{12} + s_{22})]\, y_{12} \\ &+ [32 - 0.0516(y_{13} + s_{23})]\, y_{13} \\ &-15(y_{11} + y_{12} + y_{13})\} \\ &- \{[40 - 0.08(s_{11} + s_{21})]\, s_{11} \\ &+ [40 - 0.08(s_{12} + s_{22})]\, s_{12} \\ &+ [32 - 0.0516(s_{13} + s_{23})]\, s_{13} \\ &- 15(s_{11} + s_{12} + s_{13})\} \\ &+ \{[40 - 0.08(s_{11} + y_{21})]\, y_{21} \\ &+ [40 - 0.08(s_{12} + y_{22})]\, y_{22} \\ &+ [32 - 0.0516(s_{13} + y_{23})]\, y_{23} \\ &- 20(y_{21} + y_{22} + y_{23})\} \\ &- \{[40 - 0.08(s_{11} + s_{21})]\, s_{21} \\ &+ [40 - 0.08(s_{12} + s_{22})]\, s_{22} \\ &+ [32 - 0.0516(s_{13} + s_{23})]\, s_{23} \\ &- 20(s_{21} + s_{22} + s_{23})\}. \end{aligned} \tag{21}$$

And, from (21), the optimum response function is

$$Z(\mathbf{x}) = \arg\max_{\mathrm{y}} \Psi(s_{ij}, y_{ij})$$

subject to

$$\begin{aligned} &(y_{11} + y_{12} + y_{13}) + S_{\text{base}}\frac{(\theta_2 - \theta_1)}{x_{12}} + S_{\text{base}}\frac{(\theta_3 - \theta_1)}{x_{13}} \le y_{11} + y_{21}, \\ &(y_{21} + y_{21} + y_{23}) + S_{\text{base}}\frac{(\theta_1 - \theta_2)}{x_{12}} + S_{\text{base}}\frac{(\theta_3 - \theta_2)}{x_{23}} \le y_{12} + y_{22}, \\ &S_{\text{base}}\frac{(\theta_1 - \theta_3)}{x_{13}} + S_{\text{base}}\frac{(\theta_2 - \theta_3)}{x_{23}} \le y_{13} + y_{23}, \\ &y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23} \ge 0, \\ &\theta_{\min} \ge \theta_1, \theta_2, \theta_3 \ge \theta_{\max}. \end{aligned} \tag{22}$$

From (22), the values of $y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, \theta_1, \theta_2, \theta_3$ are obtained, and, with the previous iteration values of s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23}, all of them are plugged in the relaxation formula. This procedure is repeated until convergence is reached. Note that there is no need for an extra term in the Nikaido–Isoda function in regards to the angles θ_i, because they only appear as part of the constraints. Testing the Nikaido–Isoda function (21) for weak concavity-convexity offers similar results to case study 1, so we can assert convergence also in this situation. Note that this is a game with coupled constraints.[7] The results after 18 iterations using an optimized step-size are the same as in [5], as shown in Table VIII.

TABLE VIII
GENERATION AND SALES FOR THE THREE-BUS EXAMPLE

SALES BY FIRM 1 (MWh)			SALES BY FIRM 2 (MWh)			GENERATION BY FIRM F	
S_{11}	S_{12}	S_{13}	S_{21}	S_{22}	S_{23}	P_{11}	P_{22}
125	125	142.1	62.5	62.5	45.2	392.1	170.2

And, therefore, the quantities demanded, according to (20) are $q_1 = 187.5$, $q_2 = 187.5$, and $q_3 = 187.3$. Prices at nodes are, according to the linear demand functions: 25, 25, and 22.3 U.S.$/MWh, respectively. Angles at nodes 1 and 2 are 0.2613 radians (14.97°) and 0.1134 radians (6.49°), for nodes 1 and 2, respectively. The flows through the lines are 73.95, 130.65, and 56.7 MW for lines 1-2, 1-3, and 2-3, respectively. Profits for firms 1 and 2 are 3542.1 and 730.6 U.S.$/h, respectively.

The second case considers a limit of 25 MW in the transmission capacity of the line that connects buses 1 and 2. The line flow limit of 25 MW on line 1-2 can be formulated as another linear constraint added to the ones in (22)

$$\left|\frac{\theta_1 - \theta_2}{x_{12}}\right| \le \frac{25}{S_{\text{base}}} \tag{23}$$

where x_{12} is the reactance of line 1-2 in p.u., and S_{base} is the base power in megavolt amperes. Constraint (23) can be also depicted as a two-in-one set of constraints

$$\frac{\theta_1 - \theta_2}{x_{12}} \le \frac{25}{S_{\text{base}}} \tag{23-a}$$

$$\frac{\theta_2 - \theta_1}{x_{12}} \le \frac{25}{S_{\text{base}}}. \tag{23-b}$$

This is a game[8] with coupled constraints, and the relaxation algorithm can find a unique equilibrium solution.[9] This will be a combination of the decision variables such that the constraints will be satisfied *and* no player will be able to improve his payoff by a unilateral move. Part of the solution will constitute the Lagrange multipliers that a regional regulator will be able to use to enforce the equilibrium, presumably desired.

The overall results of the relaxation algorithm with a constant size step of 0.5 are shown in Tables IX and X.

The quantities demanded, according to (21), (22), and (23) are $q_1 = 199.1$, $q_2 = 175.9$, and $q_3 = 187.3$. Prices at nodes are, according to the linear demand functions: 24.1, 25.9, and 22.3 U.S.$/MWh, respectively. Angles at nodes 1 and 2 are 0.2123 radians (12.16°) and 0.1623 radians (9.3°), for nodes 1 and

[7]Case 1 is set as a coupled constraint game, since S_{11}, S_{12}, S_{13}, and θ_1 belong to player 1 and S_{21}, S_{22}, S_{23}, and θ_2 belong to player 2, and they are coupled in (20). However, in this example (but not in general), the constraint set in (22) is nonactive and the corresponding Lagrange multipliers are zero, because the angles' limits are not reached. In addition, any demand change in the nodes will not alter the productions in Table VIII, just the angles' values.

[8]In this case, both (22) and (23) are active.

[9]Numerical experimentation indicates that the game is *diagonally strictly concave*, which is enough to guarantee this unique solution (see [21] and [10] for details).

TABLE IX
GENERATION AND SALES FOR THE THREE-BUS EXAMPLE AND LINE FLOW LIMIT OF 25 MW IN LINE T_{12}

SALES BY FIRM 1 (MWh)			SALES BY FIRM 2 (MWh)			GENERATION BY FIRM F	
s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}	P_{11}	P_{22}
113.4	101.8	115.1	85.7	74.1	72.2	330.3	232

TABLE X
CONVERGENCE IN THE THREE-BUS EXAMPLE AND LINE FLOW LIMIT OF 25 MW IN LINE T_{12}

Iteration	s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}
0	0	0	0	0	0	0
1	78.1	74.9	79.9	65.7	62.5	60.6
2	100.7	94.4	102.9	81.3	75	72.8
3	108.2	99.7	110.2	84.9	76.4	74.2
4	111	101.1	113	85.6	75.8	73.6
5	112.2	101.6	114.1	85.7	75.1	73.1
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	113.3	101.8	115.1	85.7	74.1	72.2
⋮	⋮	⋮	⋮	⋮	⋮	⋮
16	113.4	101.8	115.1	85.7	74.1	72.2

2, respectively. The flows through the lines are: 25 (line flow limit), 106.15, and 81.15 MW for lines 1-2, 1-3, and 2-3, respectively. Profits for firms 1 and 2 are 2985 and 956.9 U.S.$/h, respectively. The following Lagrange multipliers are computed $\lambda_1 = 278.66$, $\lambda_2 = 0$, $\lambda_3 = 139.33$, $\lambda_4 = 417.99$, $\lambda_5 = 0$. The first three correspond to the set of linear inequality constraints in (22), the fourth to constraint (23-a), and the fifth to constraint (23-b). The termination condition for the algorithm is $\varepsilon = 0.000\,01$.

Suppose that the coupled constraint game that we have solved above was in fact a local electricity authority's problem to establish generation levels that satisfy (23) as well as the other necessary transmission restrictions. If the authority is empowered to charge the agents for some deviations from the desired levels, it can easily compel them to implement the desired Nash equilibrium solution. This can be achieved by using the above Lagrange multipliers.

The Lagrange multipliers of the *active* constraints, which are computed by the relaxation algorithm, can be used to enforce the production levels of the above quantities as follows. The authority announces that for a unit of constraint violation each player will be charged

$$\begin{aligned}&\lambda_1 \max\left(0, P_{1,1,1} + S_{\text{base}}\frac{(\theta_2-\theta_1)}{x_{12}} + S_{\text{base}}\frac{(\theta_3-\theta_1)}{x_{13}} - q_1\right)\\&+ \lambda_3 \max\left(0, S_{\text{base}}\frac{(\theta_1-\theta_3)}{x_{13}} + S_{\text{base}}\frac{(\theta_2-\theta_3)}{x_{23}} - q_3\right)\\&+ \lambda_4 \max\left(0, S_{\text{base}}\left(\frac{\theta_1-\theta_2}{x_{12}}\right) - 25\right). \qquad (24)\end{aligned}$$

This is a threat that all players have to incorporate in their payoff functions. The modified payoff functions of firms 1 and 2 thus become

Firm 1:

$$\begin{aligned}\max\{&[40-0.08(s_{11}+s_{21})]\,s_{11}+[40-0.08(s_{12}+s_{22})]\,s_{12}\\&+[32-0.0516(s_{13}+s_{23})]\,s_{13}-15P_{1,1,1}\\&-\lambda_1 \max\left(0, P_{1,1,1}+S_{\text{base}}\frac{(\theta_2-\theta_1)}{x_{12}}\right.\\&\qquad\left.+S_{\text{base}}\frac{(\theta_3-\theta_1)}{x_{13}}-q_1\right)\\&-\lambda_3 \max\left(0, S_{\text{base}}\frac{(\theta_1-\theta_3)}{x_{13}}+S_{\text{base}}\frac{(\theta_2-\theta_3)}{x_{23}}-q_3\right)\\&-\lambda_4 \max\left(0, S_{\text{base}}\left(\frac{\theta_1-\theta_2}{x_{12}}\right)-25\right)\Bigg\}.\end{aligned}$$

Firm 2:

$$\begin{aligned}\max\{&[40-0.08(s_{11}+s_{21})]\,s_{21}+[40-0.08(s_{12}+s_{22})]\,s_{22}\\&+[32-0.0516(s_{13}+s_{23})]\,s_{23}-20P_{2,2,2}\\&-\lambda_1 \max\left(0, P_{1,1,1}+S_{base}\frac{(\theta_2-\theta_1)}{x_{12}}\right.\\&\qquad\left.+S_{base}\frac{(\theta_3-\theta_1)}{x_{13}}-q_1\right)\\&-\lambda_3 \max\left(0, S_{\text{base}}\frac{(\theta_1-\theta_3)}{x_{13}}+S_{\text{base}}\frac{(\theta_2-\theta_3)}{x_{23}}-q_3\right)\\&-\lambda_4 \max\left(0, S_{\text{base}}\left(\frac{\theta_1-\theta_2}{x_{12}}\right)-25\right)\Bigg\}. \qquad (25)\end{aligned}$$

The problem set in (25) can be solved again by the NIRA approach by modifying the payoff functions when constructing the Nikaido–Isoda function, and also by removing the constraint set. The result of including the threat in the payoff functions is that the firms solve now their individual optimization problems, *decoupled* through the use of λ_1, λ_3, and λ_4 and stick to the solutions in *their own best interest*. The *decoupled* Nash equilibrium numerical results using the NIRA approach are the *same* as in Table IX. This corroborates the correctness of the Lagrange multipliers computation and compliance implementation.

Finally, note that in difference to [5], no congestion-based wheeling fees are considered in our paper. Consequently, our profits for the second case are different than the ones in [5].

V. CONCLUSION

This paper presents a new approach to find Nash equilibria in electricity markets. It is based on the Nikaido–Isoda function and a relaxation algorithm (NIRA). The Nikaido–Isoda function indicates when the Nash equilibrium has been reached (i.e., the players cannot improve their profits). The relaxation algorithm is the way to converge to the Nash equilibrium by iterating with a weighted average of the players' improvements. Thus, the method can be seen either as centralized optimization or as distributed optimization, where the generating companies solve their own profit maximization subproblems.

Several case studies of electricity markets that use the relaxation algorithm to achieve Nash equilibrium are presented. The first case study shows the importance of the price-demand elasticity. Changes in elasticity can produce different sets of coalitions among generating companies that can be studied with concepts such as the bilateral Shapley value or the kernel. The second case models transmission network constraints and introduces flow limits and shows how the NIRA approach can be used for enforcement of results that are satisfying local authority's constraints.

APPENDIX A

We present several definitions and remarks from mathematical literature to help the reader understand the concept of the Nikaido–Isoda function and its usage in electricity economics.

For the relaxation algorithm to converge to a unique equilibrium, the Nikaido–Isoda function needs to be *weakly convex-concave*. The notions of weak convexity and weak concavity weaken the concept of strict convexity and concavity. The family of weakly convex-concave functions includes smooth functions (derivatives of all orders are continuous) as well as many nondifferentiable functions. Many "real-life" payoff functions satisfy are *weakly convex-concave*.

Definition 1a: Let X be a convex subset of the Euclidean space $\Re^{m_i}$. A continuous function $f : X \to \Re$ is called *weakly convex* on X if for all $\mathbf{x} \in X, \mathbf{y} \in X, 0 \le \alpha \le 1$ the following inequality holds:

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \ge f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) r(\mathbf{x},\mathbf{y}),$$
$$\text{and } \frac{r(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \text{ as } \|\mathbf{x}-\mathbf{y}\| \to 0 \; \forall \mathbf{x} \in X.$$

Definition 2a: A function $f(\mathbf{x})$ is called *weakly concave* on X if the function $-f(\mathbf{x})$ is weakly convex on X. This means that for all $\mathbf{x} \in X, \mathbf{y} \in X, 0 \le \alpha \le 1$ the following inequality holds:

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \le f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) \mu(\mathbf{x},\mathbf{y}),$$
$$\text{and } \frac{\mu(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \text{ as } \|\mathbf{x}-\mathbf{y}\| \to 0 \; \forall \mathbf{x} \in X.$$

Definition 3a: A function of two vector arguments $f(\mathbf{x},\mathbf{y})$ is referred to as *weakly convex-concave* if it satisfies weak convexity w.r.t. its first argument and weak concavity w.r.t. its second. That is, for a fixed $\mathbf{z} \in X$

$$\alpha f(\mathbf{x},\mathbf{z}) + (1-\alpha) f(\mathbf{y},\mathbf{z}) \ge f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y},\mathbf{z}) + \alpha(1-\alpha) r(\mathbf{x},\mathbf{y};\mathbf{z})$$
$$0 \le \alpha \le 1, \text{ and } \frac{r(\mathbf{x},\mathbf{y};\mathbf{z})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \text{ as } \|\mathbf{x}-\mathbf{y}\| \to 0 \; \forall \mathbf{x},\mathbf{y} \in X$$

and

$$\alpha f(\mathbf{z},\mathbf{x}) + (1-\alpha) f(\mathbf{z},\mathbf{y}) \le f(\mathbf{z},\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) \mu(\mathbf{x},\mathbf{y};\mathbf{z})$$
$$0 \le \alpha \le 1, \text{ and } \frac{\mu(\mathbf{x},\mathbf{y};\mathbf{z})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \text{ as } \|\mathbf{x}-\mathbf{y}\| \to 0 \; \forall \mathbf{x},\mathbf{y} \in X$$

where $r(\mathbf{x},\mathbf{y};\mathbf{z})$ and $\mu(\mathbf{x},\mathbf{y};\mathbf{z})$ are called the *residual terms*.

Obviously, a Nikaido–Isoda function is of two arguments, at least. If it is twice continuously differentiable with respect to both arguments, the residual terms satisfy [22]:

$$r(\mathbf{x},\mathbf{y};\mathbf{y}) = \frac{1}{2}\langle A(\mathbf{x},\mathbf{x})(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + o_1\left(\|\mathbf{x}-\mathbf{y}\|^2\right)$$

and

$$\mu(\mathbf{y},\mathbf{x};\mathbf{x}) = \frac{1}{2}\langle B(\mathbf{x},\mathbf{x})(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + o_2\left(\|\mathbf{x}-\mathbf{y}\|^2\right)$$

where $A(\mathbf{x},\mathbf{x}) = \Psi_{xx}(\mathbf{x},\mathbf{y})|_{y=x}$ is the Hessian of the Nikaido–Isoda function w.r.t. the first argument and $B(\mathbf{x},\mathbf{x}) = \Psi_{yy}(\mathbf{x},\mathbf{y})|_{y=x}$ is the Hessian of the Nikaido–Isoda function w.r.t. the second argument, both evaluated at $\mathbf{y} = \mathbf{x}$. Note that if the function $\Psi(\mathbf{x},\mathbf{y})$ is convex w.r.t. $\mathbf{x}$ and concave w.r.t. $\mathbf{y}$, then $o_1(\|\mathbf{x}-\mathbf{y}\|^2) = o_2(\|\mathbf{x}-\mathbf{y}\|^2) = 0$. Thus, the remainder terms necessary for checking convergence conditions of the relaxation algorithm are simplified. Also note that to prove the last convergence condition of the relaxation algorithm shown in Section III, assuming that $\Psi(\mathbf{x},\mathbf{y})$ is twice continuously differentiable, it will suffice to show that $A(\mathbf{x},\mathbf{x}) - B(\mathbf{x},\mathbf{x})$ is strictly positive (see [9] and [21] for details). This is a relatively straightforward algebraic exercise, see (11).

A. Relaxation Algorithm Convergence Theorem

There exists a (normalized) Nash equilibrium point to which the relaxation algorithm converges if [7]

1) X is a convex compact subset of $\Re^m$;
2) the Nikaido–Isoda function $\Psi : X \times X \to \Re$ is a weakly convex-concave function and $\Psi(\mathbf{x},\mathbf{x}) = 0$ for $\mathbf{x} \in X$;
3) the optimum response function $Z(\mathbf{x})$ is single-valued and continuous on X;
4) the residual term $r(\mathbf{x},\mathbf{y};\mathbf{z})$ is uniformly continuous on X w.r.t. $\mathbf{z}$ for all $\mathbf{x},\mathbf{y} \in X$;
5) the residual terms satisfy
$$r(\mathbf{x},\mathbf{y};\mathbf{y}) - \mu(\mathbf{y},\mathbf{x};\mathbf{x}) \ge \beta(\|\mathbf{x}-\mathbf{y}\|), \quad \mathbf{x},\mathbf{y} \in X$$
where β is an strictly increasing function;
6) the relaxation parameters α_s satisfy
   (a) $\alpha_s > 0$;
   (b) $\sum_{s=0}^{\infty} \alpha_s = \infty$;
   (c) $\alpha_s \to 0$ as $s \to \infty$.

Note that the usual linear electricity generation and capacity *coupled* constraints will naturally define the convex set X. Observe, moreover, that nondifferentiable Nikaido–Isoda functions are also possible. However, deciding on their weak convexity-concavity will be less of an easy exercise than checking strict positive definiteness of $A(\mathbf{x},\mathbf{x}) - B(\mathbf{x},\mathbf{x})$ (as in (11)).

In order for the algorithm to converge, any sequence of step-sizes α_s that satisfies condition 6 (above) will suffice, although a constant step of $\alpha_s \equiv 0.5$ leads to a quick convergence [18]. However, the last condition of the theorem will have to be replaced by the one-step optimal step-size [18], such that it minimizes the optimum response function at $\mathbf{x}^{s+1} : \alpha_s^* = \underset{\alpha\in[0,1)}{\text{argmin}} \{\max_{\mathbf{y}\in\mathbf{x}} \Psi(\mathbf{x}^{s+1}(\alpha),\mathbf{y})\}$.

Note that $\mathbf{x}^{s+1}$ depends on α. By optimizing the step-size the number of iterations decreases, but each step is longer [9], [18]. In that case, a convergence proof based on Kakutani's fixed point theorem can be found in [9].

APPENDIX B

A. BSV Method

The Shapley Value is a solution concept for a n-person cooperative game. It calculates a fair division of the utility, based on individuals' contributions, among the members of a coalition.

It is a solution concept for a n-person cooperative game. The Shapley Value can be considered as a weighted average of marginal contributions of a member to all of the possible coalitions in which it may participate. It assumes that the game is superadditive and the grand coalition is likely to be formed. The mathematical expression of the Shapley Value, is given by

$$\phi_i = \sum_{S, i \in S \subset N} \frac{(|S|-1)!\,(n-|S|)!}{n!}\left[v(S) - v\left(S - \{i\}\right)\right]$$

where i is a player, S is a coalition of players, $|S|$ is the number of players in coalition S, n is the total number of players, N is the set of all players, and $v(S)$ is the characteristic function associated with coalition S. Looking for a negotiation framework based on the Shapley value, the bilateral Shapley value (BSV) is introduced for a completely decentralized and bilateral negotiation process among rational agents.

Let $S \subseteq P(A)$ be a coalition structure on a given set of agents $A = \{a_l, \cdots, a_m\}$ where $C = C_i \cup C_j \subseteq A$, and $C_i \cap C_j = \phi$. Therefore, C is a (bilateral) coalition of disjoint (n-agent) coalitions of C_i and C_j $(n \geq 0)$. The BSV for coalition C_i in the bilateral coalition C is defined by

$$\varphi_C(C_i) = \frac{1}{2}v(C_i) + \frac{1}{2}\left(v(C) - v(C_j)\right).$$

Both coalitions C_i and C_j are willing to form coalition C, if $v(C_i) \leq \varphi_C(C_i)$ and $v(C_j) \leq \varphi_C(C_j)$.

In fact, a superadditive cooperative game is played between C_i and C_j. From the equations above, it can be seen that the founders will get half of their local contributions, and the other half obtained from cooperative work with the other entity. The second term of the BSV expression reflects the strength of each agent based on its contribution. Thus, two players will form a coalition if both obtain more value than acting alone. The coalition formation process continues if the newly formed players that are recently allied wish to team with other players to increase their value. If the process continues until the end, the grand coalition (all players) form a single team, since it is beneficial for all. More details about the method can be found in [11] and [12].

B. Kernel Method

The kernel is another solution concept for cooperative games. The kernel coalitional configurations are stable in the sense that there is an equilibrium between pairs of individual agents which are in the same coalition. Two agents A, B in a coalition C are in equilibrium if they cannot outweigh one another from C, their common coalition. Agent A can outweigh B if A is stronger than B, where strength refers to the potential of agent A to successfully claim a part of the payoff of agent B.

In each stage of the coalition formation process, the agents are in a coalitional configuration. That is, the agents are arranged in a set of coalitions $C = \{C_i\}$. During the coalition formation, agents can use the kernel solution concept to object to the payoff distribution that is attached to their coalitional configuration. The objections that agents can make are based on the excess concept. The relevant definitions are recalled now.

Excess: The excess of a coalition C with respect to a coalitional configuration C is defined as

$$e(C) = V(C) - \sum_{A_i \in C} u^i$$

where u^i is the payoff of agent A_i and $V(C)$ is the coalitional value of coalition C. The number of excesses is an important property of the kernel solution concept. Agents use the excesses as a measure of their relative strengths. Since a higher excess correlates with more strength, rational agents must search for the highest excess they have. The maximum is defined by the surplus.

Surplus and Outweight: The maximum surplus S_{AB} of agent A over agent B with respect to a coalitional configuration is defined by

$$S_{AB} = \max_{C|A \in C, B \notin C} e(C)$$

where $e(C)$ are the excesses of all the coalitions that include A and exclude B, and the coalitions C are not in the current coalitional configuration. Agent A **outweighs** agent B if $S_{AB} > S_{BA}$ and $u^B > V(B)$, where $V(B)$ is the coalitional value of agent B in a single agent coalition. The agents compare their maximum surpluses, and the one with the larger maximum surplus is stronger. The stronger agent can claim a part of the weaker agent's payoff, but this claim is limited by the individual rationality: $u^B > V(B)$. Therefore, agent A cannot claim an amount that would leave agent B with $V(B)$ or less. If two agents cannot outweigh one another, they are in **equilibrium**: A and B are in equilibrium if one of the following conditions is satisfied: 1. $S_{AB} = S_{BA}$; 2. $S_{AB} > S_{BA}$ and $u^B = V(B)$; 3. $S_{AB} < S_{BA}$ and $u^A = V(A)$. Note that equilibrium is defined only for pairs of distinct agents who are members of the same coalition. Using the concept of equilibrium, the kernel can be defined as the set of all coalitional configurations (and its associated payoffs) such that every pair of agents within the same coalition are in equilibrium. A coalitional configuration (and payoff distribution) of this type is also called kernel stable (**K-stable**). Furthermore, the kernel always exists for any coalitional configuration. However, checking the stability does not direct the agents to a specific coalitional configuration. More details about the method can be found in [13] and [14].

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Javier Contreras (M’98) was born in Zaragoza, Spain, in 1965. He received the B.S. degree in electrical engineering from the University of Zaragoza, Zaragoza, Spain, in 1989, the M.Sc. degree from the University of Southern California, Los Angeles, in 1992, and the Ph.D. degree from the University of California, Berkeley, in 1997.

Currently, he is Associate Professor at the Universidad de Castilla—La Mancha, Ciudad Real, Spain. His research interests include power systems planning, operations and economics, and electricity markets.

Matthias Klusch is a Senior Researcher at the German Research Center for Artificial Intelligence, Saarbrücken, Germany. He was Assistant Professor at the Department for Artificial Intelligence of the Free University of Amsterdam, The Netherlands. He was also Assistant Professor in the Department for Data Management at the Technical University, Chemnitz, Germany, and visiting research scientist at the Robotics Institute of Carnegie Mellon University, Pittsburgh, PA.

He is a co/editor of many books and has been widely published in the area of agent technology and intelligent information systems.

Jacek B. Krawczyk is a Professor at the School of Economics and Finance at Victoria University of Wellington, Wellington, New Zealand. He was with the Polish Academy of Sciences, Warsaw, Poland, and Mexican National Institute of Technology, Mexico City, Mexico. His research interests include computational economics as well as optimal control and dynamic games theory.

He is Associate Editor of *Environmental Modeling and Assessment*, and Editor-in-Chief of an e-journal *Information Technology for Economics and Management*. He has been published in several journals.