

Computer Sciences Department
University of Wisconsin
1210 West Dayton Street
Madison, Wisconsin 53706

NUMERICAL STUDIES OF FLOW BETWEEN
ROTATING COAXIAL DISKS

by

Donald Greenspan

Technical Report #110

January 1971



NUMERICAL STUDIES OF FLOW BETWEEN
ROTATING COAXIAL DISKS

by

Donald Greenspan

ABSTRACT

A new algorithm, which is exceptionally fast for certain choices of numerical parameters, is described for the study of nonlinear, incompressible flow between two rotating disks. Typical examples for Reynolds number R in the range $10 \leq R \leq 2000$ are described and discussed. Comparisons are made with the limited available results generated by other methods.



1. INTRODUCTION

The study of fluid motion between rotating disks is of both practical and theoretical interest (see, e.g., references [1]-[7], [9]-[17], [18]-[25], and the references contained therein). It appears that the first mathematical paper on the subject was that of von Karmen [24], who dealt with the steady flow above an infinitely large rotating disk under the assumption that axial velocity was radius independent. This model was extended to steady flow between two coaxial rotating disks by Batchelor [2], and then to nonsteady models by Greenspan [9], Greenspan and Howard [10], and Pearson [15]. In studying the related mathematical and physical problems, various techniques have been applied, including asymptotic analysis ([2], [6], [20]-[22], [25]), linearization [10], [11], [16], numerical analysis [5], [12], [13], [15], [16], and experimentation [13], [21]. Unfortunately, the results of these analyses often are either unreasonable or contradictory. Thus, as the Reynolds number becomes infinite, Batchelor [2] and Stewartson [21] both find unique, steady, limiting flows, but which are qualitatively different, while Tam [22] claims that the problem admits an infinite number of flows. Pearson [15] and Lance and Rogers [12], using different numerical techniques, generate qualitatively different flows for certain classes of nonsteady problems. Mellor, Chapple and Stokes [13] claim to have produced several classes of solutions for a given

problem by analytical-numerical means, but then can produce only one such class in laboratory experiments.

Our purpose in this paper is to study numerically only the steady motion of a viscous, incompressible fluid between two rotating, infinite coaxial disks. For simplicity, the first disk is positioned in (x, y, z) space in the plane $z = 0$ with its center at $(0, 0, 0)$ and is given an angular velocity Ω_1 , while the second disk is positioned in the plane $z = 1$ with its center at $(0, 0, 1)$ and is given an angular velocity Ω_2 . If the cylindrical coordinates of (x, y, z) are (r, θ, z) , and if the fluid at (x, y, z) has velocity components (u, v, w) , then the substitutions

$$(1.1) \quad u = -\frac{1}{2} r H'(z), \quad v = r G(z), \quad w = H(z)$$

enable one ([9], [15]) to transform the dimensionless, steady state Navier-Stokes equations to

$$(1.2) \quad H'' = M \quad , \quad 0 \leq z \leq 1$$

$$(1.3) \quad G'' + R(GH' - G'H) = 0 \quad , \quad 0 \leq z \leq 1$$

$$(1.4) \quad M'' - R(HM' + 4GG') = 0 \quad , \quad 0 \leq z \leq 1,$$

where differentiation is with respect to z . For the coaxial flow under consideration, the boundary conditions for nonlinear system (1.2)-(1.4) are ([9], [15])

$$(1.5) \quad G(0) = \Omega_1, \quad G(1) = \Omega_2$$

$$(1.6) \quad H(0) = 0, \quad H(1) = 0$$

$$(1.7) \quad H'(0) = 0, \quad H'(1) = 0.$$

The numerical method to be used is an extension of one developed for cavity flow problems [8], and which was convergent for all Reynolds numbers studied ($0 \leq R \leq 10^6$). Since the present work is largely experimental in nature, and since errors in computation often seem to be more the rule than the exception, the FORTRAN program used is being made accessible in a report [18], so that every aspect of the calculations can be reproduced by the reader.

2. THE NUMERICAL METHOD

In this section we give a precise description of the algorithm to be used for the numerical solution of (1.2)-(1.7).

Divide $0 \leq z \leq 1$ into n equal parts, each of length $h = \Delta z = \frac{1}{n}$. Let the points of subdivision be $0 = z_0 < z_1 < z_2 < \dots < z_n = 1$. Thus, $z_j = jh = \frac{j}{n}$, $j = 0, 1, 2, \dots, n$. Let S_h be the set of boundary grid points z_0 and z_n , while I_h is the set of interior grid points z_1, z_2, \dots, z_{n-1} . If F is any function defined on $S_h + I_h$, then a convenient notation will be

$$F(z_i) = F_i.$$

We attempt to approximate H , G , and M by generating three sequences, $H^{(k)}$, $G^{(k)}$, and $M^{(k)}$, $k = 0, 1, 2, \dots$, on $I_h + S_h$, each of which is convergent. This is done as follows. For all values of k , let

$$(2.1) \quad H_0^{(k)} = H_n^{(k)} = 0, \quad k = 0, 1, 2, \dots$$

$$(2.2) \quad G_0^{(k)} = \Omega_1, \quad G_n^{(k)} = \Omega_2, \quad k = 0, 1, 2, \dots$$

Set

$$(2.3) \quad H_i^{(0)} = 0, \quad i = 1, 2, \dots, n-1$$

$$(2.4) \quad G_i^{(0)} = (1-z)\Omega_1 + z\Omega_2, \quad i = 1, 2, \dots, n-1$$

$$(2.5) \quad M_i^{(0)} = 0, \quad i = 0, 1, 2, \dots, n.$$

By induction, $H^{(k+1)}$, $G^{(k+1)}$ and $M^{(k+1)}$ are generated from $H^{(k)}$, $G^{(k)}$ and $M^{(k)}$ as is shown next.

At the points z_1 and z_{n-1} write down the two equations

$$(2.6) \quad 4H_1 = H_2$$

$$(2.7) \quad 4H_{n-1} = H_{n-2},$$

which are difference approximations (see [4]) of (1.7). At each of the remaining points of I_h , write down the difference analogue

$$(2.8) \quad H_{i-1} - 2H_i + H_{i+1} = h^2 M_i^{(k)}, \quad i = 2, 3, \dots, n-2$$

of (1.2). Insertion of (2.1) into (2.6)-(2.8) results in a diagonally dominant linear algebraic system. Solve this system by SOR (point successive over-relaxation) with over-relaxation factor r_H and convergence tolerance α_1 , and denote the solution by $\bar{H}_i^{(k+1)}$, $i = 1, 2, \dots, n-1$. Define $H_i^{(k+1)}$, $i = 1, 2, \dots, n-1$, by the smoothing formula

$$(2.9) \quad H_i^{(k+1)} = \rho \bar{H}_i^{(k+1)} + (1 - \rho) H_i^{(k)}, \quad \begin{cases} i = 1, 2, \dots, n-1 \\ k = 0, 1, 2, \dots \\ 0 \leq \rho \leq 1. \end{cases}$$

At each point of I_h , write down next the following forward-backward difference analogues of (1.3):

$$(2.10) \quad G_{i-1} + [-2 + \text{Rh}H_i^{(k+1)}]G_i + [1 - \text{Rh}H_i^{(k+1)}]G_{i+1} \\ = -\frac{1}{2}\text{Rh}G_i^{(k)} [H_{i+1}^{(k+1)} - H_{i-1}^{(k+1)}]; \text{ if } H_i^{(k+1)} < 0,$$

$$(2.11) \quad [1 + \text{Rh}H_i^{(k+1)}]G_{i-1} + [-2 - \text{Rh}H_i^{(k+1)}]G_i + G_{i+1} \\ = -\frac{1}{2}\text{Rh}G_i^{(k)} [H_{i+1}^{(k+1)} - H_{i-1}^{(k+1)}]; \text{ if } H_i^{(k+1)} \geq 0.$$

Solve the resulting diagonally dominant linear algebraic system by SOR with over-relaxation factor r_G and convergence tolerance α_2 , call the solution $\bar{G}^{(k+1)}$, and on I_h define $G^{(k+1)}$ by

$$(2.12) \quad G_i^{(k+1)} = \mu \bar{G}_i^{(k+1)} + (1 - \mu)G_i^{(k)}, \quad \begin{cases} i = 1, 2, \dots, n-1 \\ k = 0, 1, 2, \dots \\ 0 \leq \mu \leq 1. \end{cases}$$

Note that (2.10)-(2.11) avoid the possible eigenvalue problems inherent in viewing (1.3) as an equation in G by using the iterate $G^{(k+1)}$ to approximate G'' and G' and using the iterate $G^{(k)}$ to approximate G .

To construct $M^{(k+1)}$ on $I_h + S_h$, first set (see [4])

$$(2.13) \quad M_0^{(k+1)} = \delta_1 [2H_1^{(k+1)}/h^2] + (1 - \delta_1) M_0^{(k)}, \quad \begin{cases} k = 0, 1, 2, \dots \\ 0 \leq \delta_1 \leq 1 \end{cases}$$

$$(2.14) \quad M_n^{(k+1)} = \delta_1 [2H_{n-1}^{(k+1)}/h^2] + (1 - \delta_1) M_n^{(k)}, \quad \begin{cases} k = 0, 1, 2, \dots \\ 0 \leq \delta_1 \leq 1. \end{cases}$$

Using (2.13) and (2.14) as boundary values, write down at each point of I_h the following forward-backward difference analogues of (1.4):

$$(2.15) \quad M_{i-1} + [-2 + \text{Rh}H_i^{(k+1)}]M_i + [1 - \text{Rh}H_i^{(k+1)}]M_{i+1} \\ = 2\text{Rh}G_i^{(k+1)} [G_{i+1}^{(k+1)} - G_{i-1}^{(k+1)}], \quad \text{if } H_i^{(k+1)} < 0,$$

$$(2.16) \quad [1 + \text{Rh}H_i^{(k+1)}]M_{i-1} + [-2 - \text{Rh}H_i^{(k+1)}]M_i + M_{i+1} \\ = 2\text{Rh}G_i^{(k+1)} [G_{i+1}^{(k+1)} - G_{i-1}^{(k+1)}], \quad \text{if } H_i^{(k+1)} \geq 0.$$

Solve the resulting diagonally dominant, linear algebraic system by SOR with successive over-relaxation factor r_M and convergence tolerance α_3 , call the solution $\bar{M}^{(k+1)}$, and define $M^{(k+1)}$ on I_h by

$$(2.17) \quad M_i^{(k+1)} = \delta_2 \bar{M}_i^{(k+1)} + (1 - \delta_2) M_i^{(k)}, \quad \begin{cases} i = 1, 2, \dots, n-1 \\ k = 0, 1, 2, \dots \\ 0 \leq \delta_2 \leq 1. \end{cases}$$

For given positive tolerances $\varepsilon_1, \varepsilon_2, \varepsilon_3$, the iteration proceeds for $k = 0, 1, 2, \dots$, until for **some** value $k = K$, one has

$$(2.18) \quad |H^{(K+1)} - H^{(K)}| < \varepsilon_1, \quad \text{uniformly in } I_h$$

$$(2.19) \quad |G^{(K+1)} - G^{(K)}| < \varepsilon_2, \quad \text{uniformly on } I_h$$

$$(2.20) \quad |M^{(K+1)} - M^{(K)}| < \varepsilon_3, \quad \text{uniformly on } I_h + S_h.$$

Finally one verifies whether or not $H^{(K+1)}$, $G^{(K+1)}$, $M^{(K+1)}$ are solutions of the difference equations being solved, and, if they are, then they are taken to be the respective approximations of H , G , M .

3. EXAMPLES

From the large number of examples run on the UNIVAC 1108, several which are typical, which are of physical interest, and which display readily the changes of flow patterns with increasing Reynolds number, will be presented in this section.

For $h = \frac{1}{50}$, $\Omega_1 = 1$, and $\Omega_2 = 0$, the results for H , H' and G , with H' determined by central differences, are shown graphically for Reynolds numbers 10, 100, 1000 in Figures 1, 2, 3, respectively. The other parameter choices were: (a) for $R = 10$, $\rho = 0.9$, $\mu = 0.9$, $\delta_1 = 0.8$, $\delta_2 = 0.1$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.0$, $\alpha_1 = \alpha_2 = \alpha_3 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 0.005$, (b) for $R = 100$, $\rho = 0.9$, $\mu = 0.9$, $\delta_1 = 0.8$, $\delta_2 = 0.1$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.5$, $\alpha_1 = \alpha_2 = \epsilon_1 = \epsilon_2 = 0.001$, $\alpha_3 = \epsilon_3 = 0.05$, (c) for $R = 1000$, $\rho = 0.9$, $\mu = 0.05$, $\delta_1 = 0.9$, $\delta_2 = 0.2$, $r_H = 1.8$, $r_G = 1.1$, $r_M = 1.5$, $\alpha_1 = \epsilon_1 = 0.005$, $\alpha_2 = \epsilon_2 = 0.03$, $\alpha_3 = \epsilon_3 = 0.3$. The number of outer iterations necessary for convergence for Reynolds numbers 10 and 100 were 72 and 182, respectively. To speed convergence for $R = 1000$, $H_i^{(0)}$ and $M_i^{(0)}$ were modified to agree with

$$(3.1) \quad H^{(0)}(z) = \begin{cases} -0.6z, & 0 \leq z \leq \frac{1}{2} \\ (0.6)(z-1), & \frac{1}{2} \leq z \leq 1 \end{cases}$$

$$(3.2) \quad M^{(0)}(z) = \begin{cases} -12 + 36z, & 0 \leq z \leq \frac{1}{2} \\ -6 - 24(z-1), & \frac{1}{2} \leq z \leq 1, \end{cases}$$

respectively, at the grid points, and convergence was attained in 19 outer iterations. Choices (3.1) and (3.2) were motivated by the results for $R = 100$ shown in Figure 2. The maximum running time of all cases discussed thus far was under thirty seconds and convergence resulted for a variety of other choices of parameters.

In $h = \frac{1}{50}$, $\Omega_1 = 1$, and $\Omega_2 = -1$, the results for H , H' and G are shown graphically for Reynolds numbers 10, 100, 1000 in Figures 4, 5, 6, respectively. The other parameter choices were: (a) for $R = 10$, $\rho = 0.9$, $\mu = 0.9$, $\delta_1 = 0.8$, $\delta_2 = 0.1$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.0$, $\alpha_1 = \alpha_2 = \alpha_3 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 0.005$, (b) for $R = 100$, $\rho = 0.9$, $\mu = 0.2$, $\delta_1 = 0.8$, $\delta_2 = 0.1$, $r_H = 1.8$, $r_G = 1.0$, $r_M = 1.5$, $\alpha_1 = \alpha_2 = \epsilon_1 = \epsilon_2 = 0.001$, $\alpha_3 = \epsilon_3 = 0.05$, (c) for $R = 1000$, $\rho = 0.1$, $\mu = 0.1$, $\delta_1 = 0.05$, $\delta_2 = 0.9$, $r_H = 1.8$, $r_G = 0.9$, $r_M = 1.1$, $\alpha_1 = \epsilon_1 = 0.005$, $\alpha_2 = \epsilon_2 = 0.03$, $\alpha_3 = \epsilon_3 = 0.3$. The number of outer iterations necessary for convergence for Reynolds numbers 10 and 100 were 67 and 31. Convergence for Reynolds number 1000 was achieved in 22 outer iterations by modifying $H_i^{(0)}$ and $M_i^{(0)}$ to agree with

$$(3.3) \quad H^0(z) = \begin{cases} (-1.2)z & , \quad 0 \leq z \leq \frac{1}{4} \\ -0.3 + (1.2)(z - \frac{1}{4}) & , \quad \frac{1}{4} \leq z \leq \frac{3}{4} \\ (-1.2)(z - 1) & , \quad \frac{3}{4} \leq z \leq 1 \end{cases}$$

$$(3.4) \quad M^{(0)}(z) = \begin{cases} -10 + 52z & , \quad 0 \leq z \leq \frac{1}{4} \\ 3 - 12(z - \frac{1}{4}) & , \quad \frac{1}{4} \leq z \leq \frac{3}{4} \\ 10 + 52(z - 1) & , \quad \frac{3}{4} \leq z \leq 1 \end{cases}$$

respectively, at the grid points.

Finally, because of the broad interest in large Reynolds numbers and because the contradictory conclusions reached in certain cases of counter rotation, attention was turned to refined calculations for the case $R = 2000$, $\Omega_1 = 1$, and $\Omega_2 = -1$. In Figures 7 and 8 are shown the results for H , H' , G , and M for the parameter choices $h = 1/400$, $\rho = 0.05$, $\mu = 0.05$, $\delta_1 = 0.1$, $\delta_2 = 0.925$, $r_H = 1.8$, $r_G = r_M = 0.8$, $\alpha_1 = 0.0003$, $\alpha_2 = 0.001$, $\alpha_3 = 0.002$, $\epsilon_1 = \epsilon_2 = 0.001$, $\epsilon_3 = 0.01$ and for initial functions (3.3) and (3.4). Convergence was achieved in 128 outer iteration, which required seven minutes of running time. The increase in precision is readily apparent by comparing Figures 6 and 7. Moreover, since Figure 7 shows clearly that the fluid has separated into two distinct parts which rotate with relatively large, but with opposite, angular velocities, it is concluded the numerical solution supports Batchelor in the Batchelor-Stewartson controversy ([2], [20], [22]).

With regard to the computations by other methods, only the work of Pearson and that of Lance and Rogers appear to be numerically

rigorous and moderately successful. For $R \leq 100$, their results and ours for the case of a single rotating disk are completely comparable. Thereafter, various results differ widely. For example, for $R = 1000$, $\Omega_1 = 1$ and $\Omega_2 = -1$, Pearson produced two, distinct solutions but failed to produce the symmetric one. By relaxing the convergence tolerances, we too were able to produce more than one solution. However, sharpening these tolerances always resulted in one and only one solution. It is also rather interesting to observe that if, for Pearson's results (see [15], p. 632, Figure 9), the points where G and H cross the z axis are relocated to $z = \frac{1}{2}$, while the points where H_z crosses the z axis are relocated symmetrically about $z = \frac{1}{2}$, then the resulting configurations for G , H and H_z are qualitatively analogous to those shown in our Figure 7. One can only surmise, then, that Pearson's time dependent calculations are stable, but relatively inaccurate due to an accumulation of roundoff error. Such types of calculations are very common in the study of nonlinear problems.

In the study of $R = 1023$, $\Omega_1 = 1$, $\Omega_2 = -1$, Lance and Rogers assumed symmetry and reformulated the problem on the half interval $0 \leq z \leq \frac{1}{2}$. All their previous calculations were limited to $R \leq 529$, but the use of symmetry allowed for a decrease in grid size and a corresponding increase in Reynolds number. Their results (see [12],

pp. 119-120) show that the main body of the fluid is only slightly disturbed, thus contradicting the flow shown in our Figure 7, and thereby supporting Stewartson in the Batchelor-Stewartson controversy. However, Lance and Rogers failed to demonstrate that the problem they study on the interval $0 \leq z \leq \frac{1}{2}$ is, in fact, equivalent to the given problem (1.2)-(1.7) of counter rotating disks. Indeed, numerically, they should have required that the differential equations be satisfied on $z = \frac{1}{2}$, which they failed to do ([12], p. 119, eq. 5.8). Indeed, since the solution shown in our Figure 7 also satisfies the conditions (5.8) of [12], the Lance and Rogers formulation has at least two solutions, namely theirs and ours, and appears to be a weaker problem than the one originally posed. The question as to whether or not the Lance and Rogers solution satisfies the differential equations on $z = \frac{1}{2}$ does not, however, appear to be trivial.

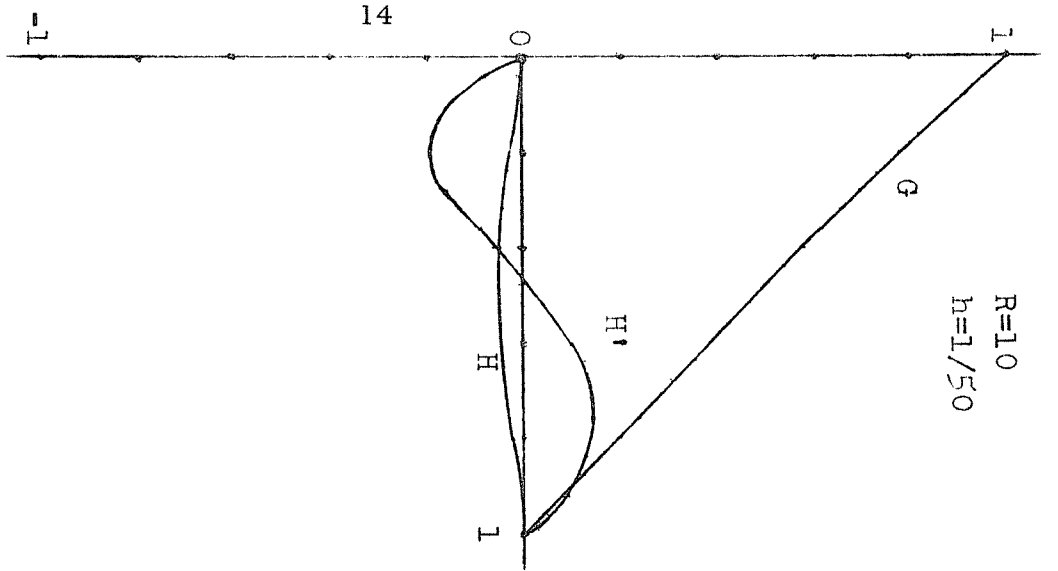


FIGURE 1

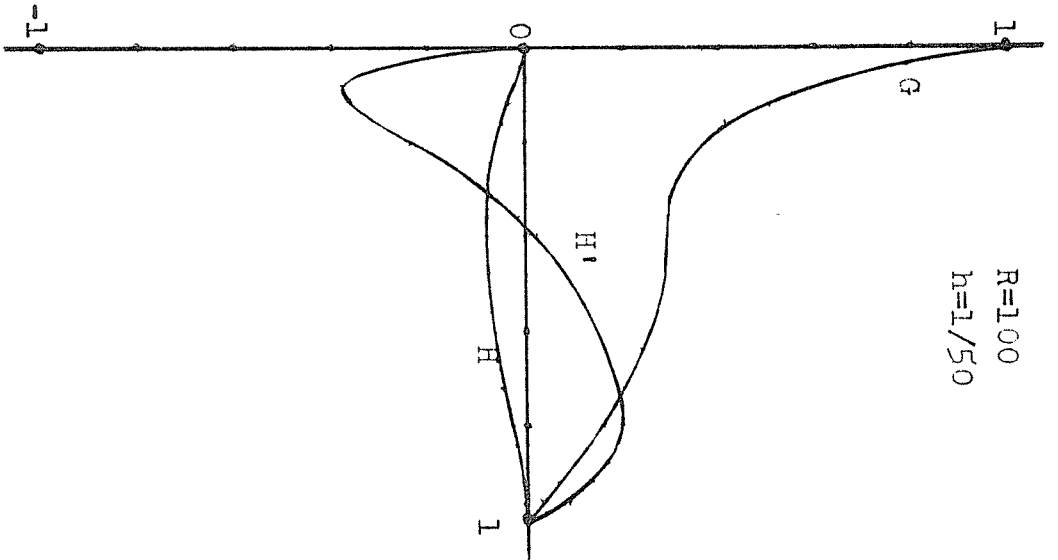


FIGURE 2

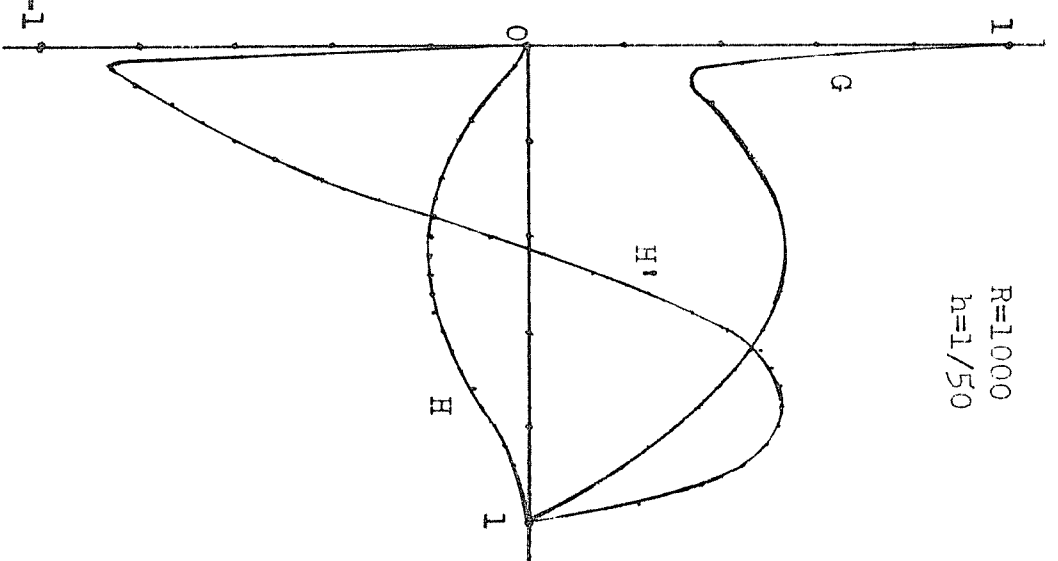


FIGURE 3

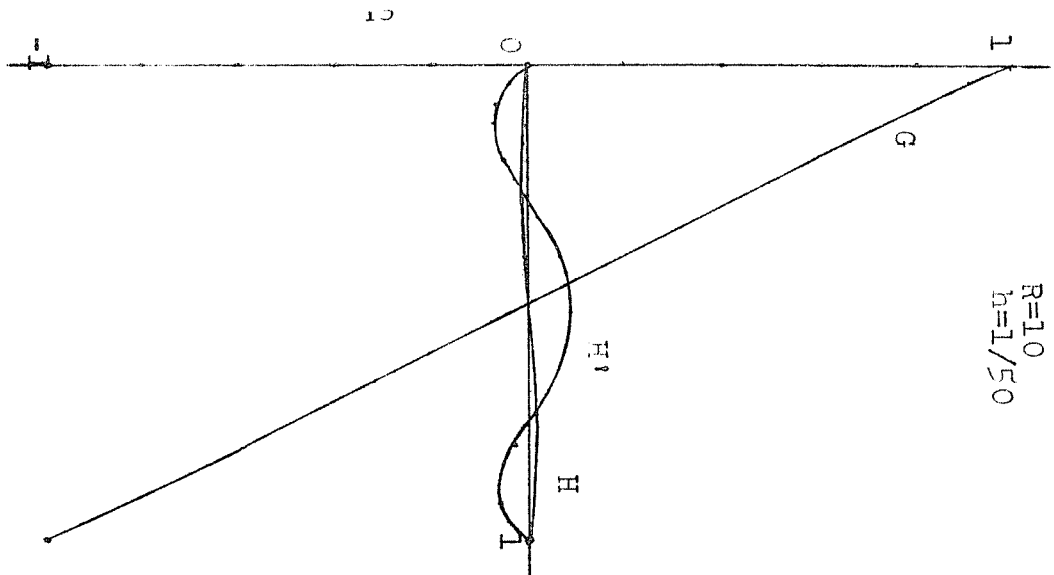


FIGURE 4

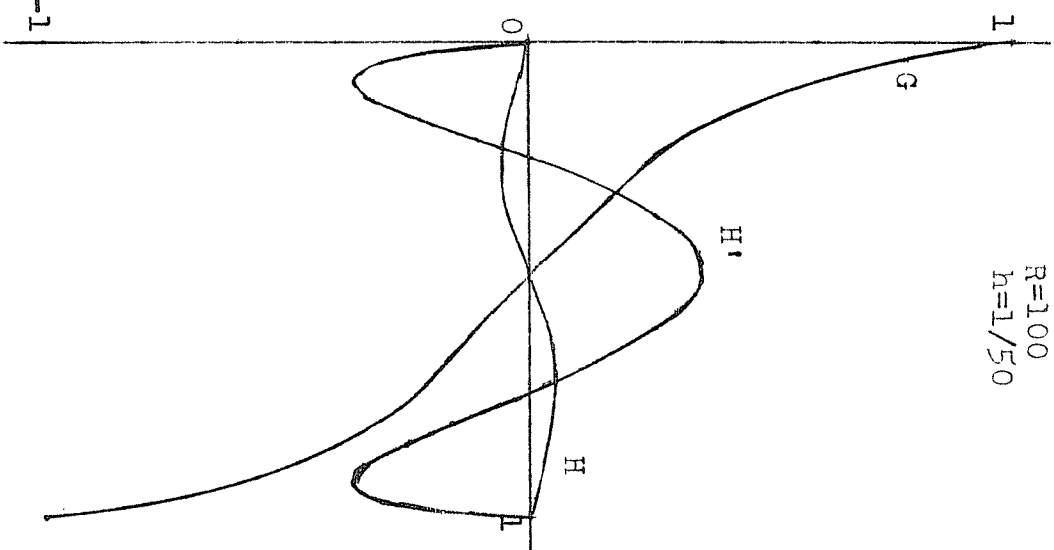


FIGURE 5

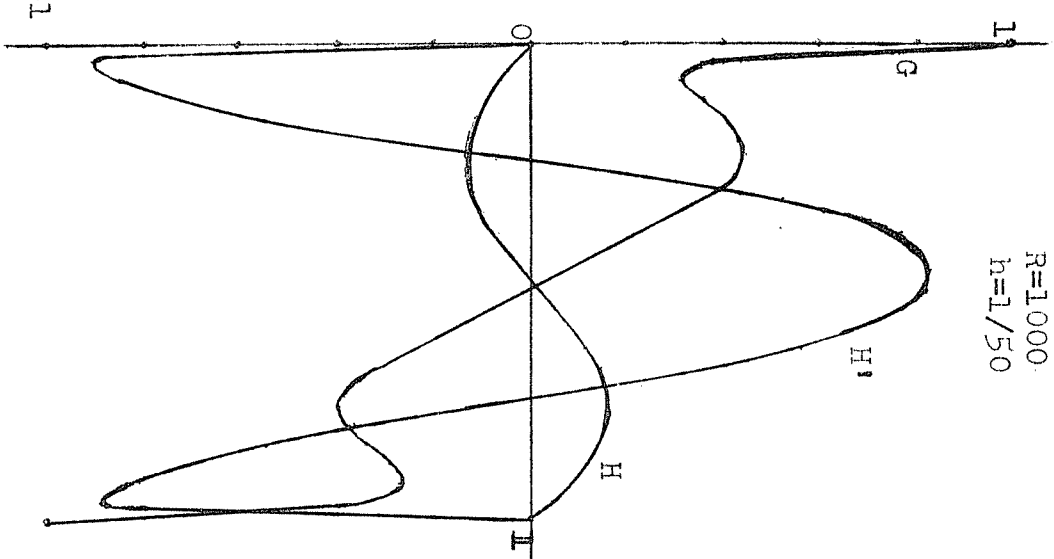


FIGURE 6

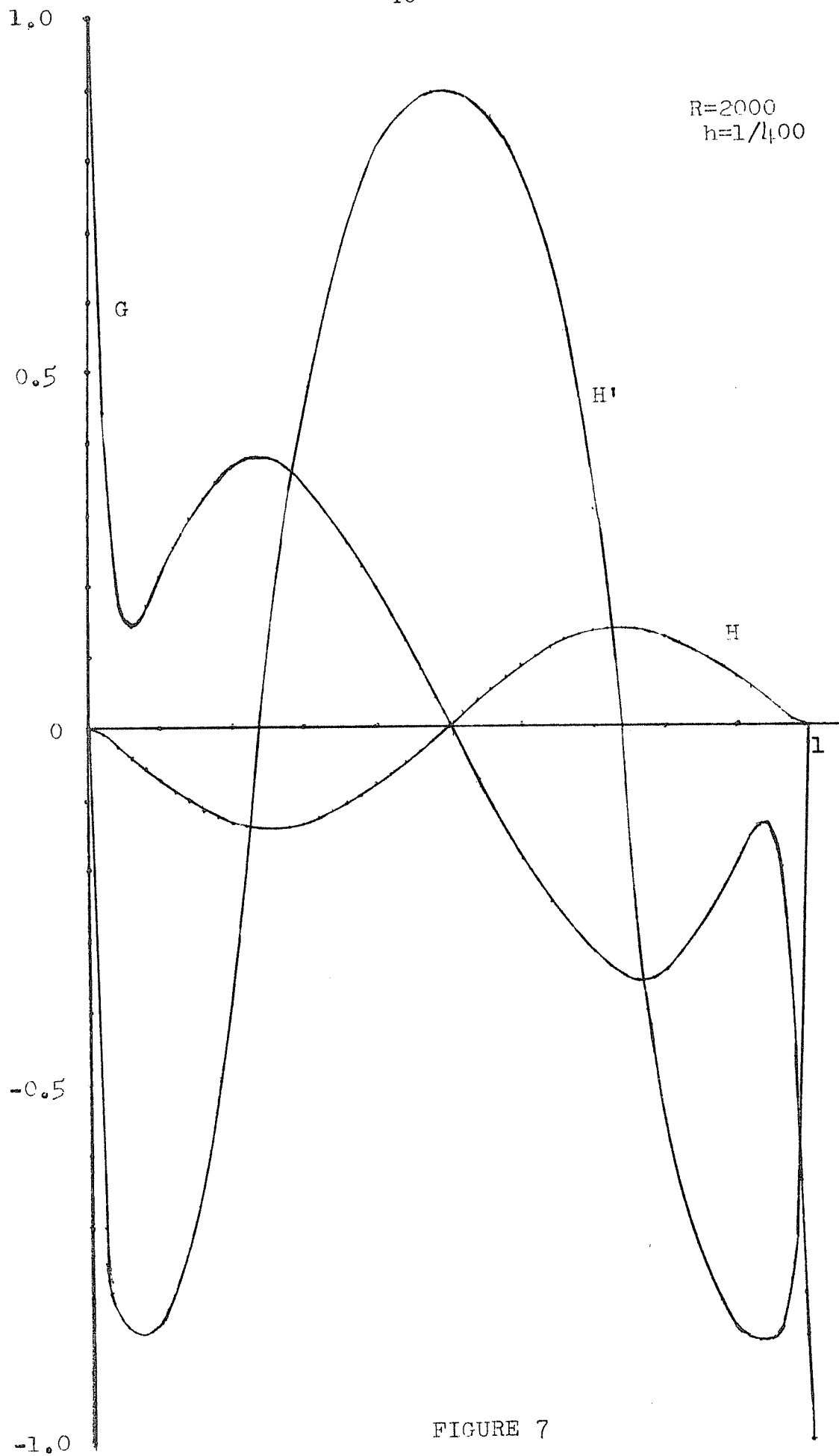


FIGURE 7

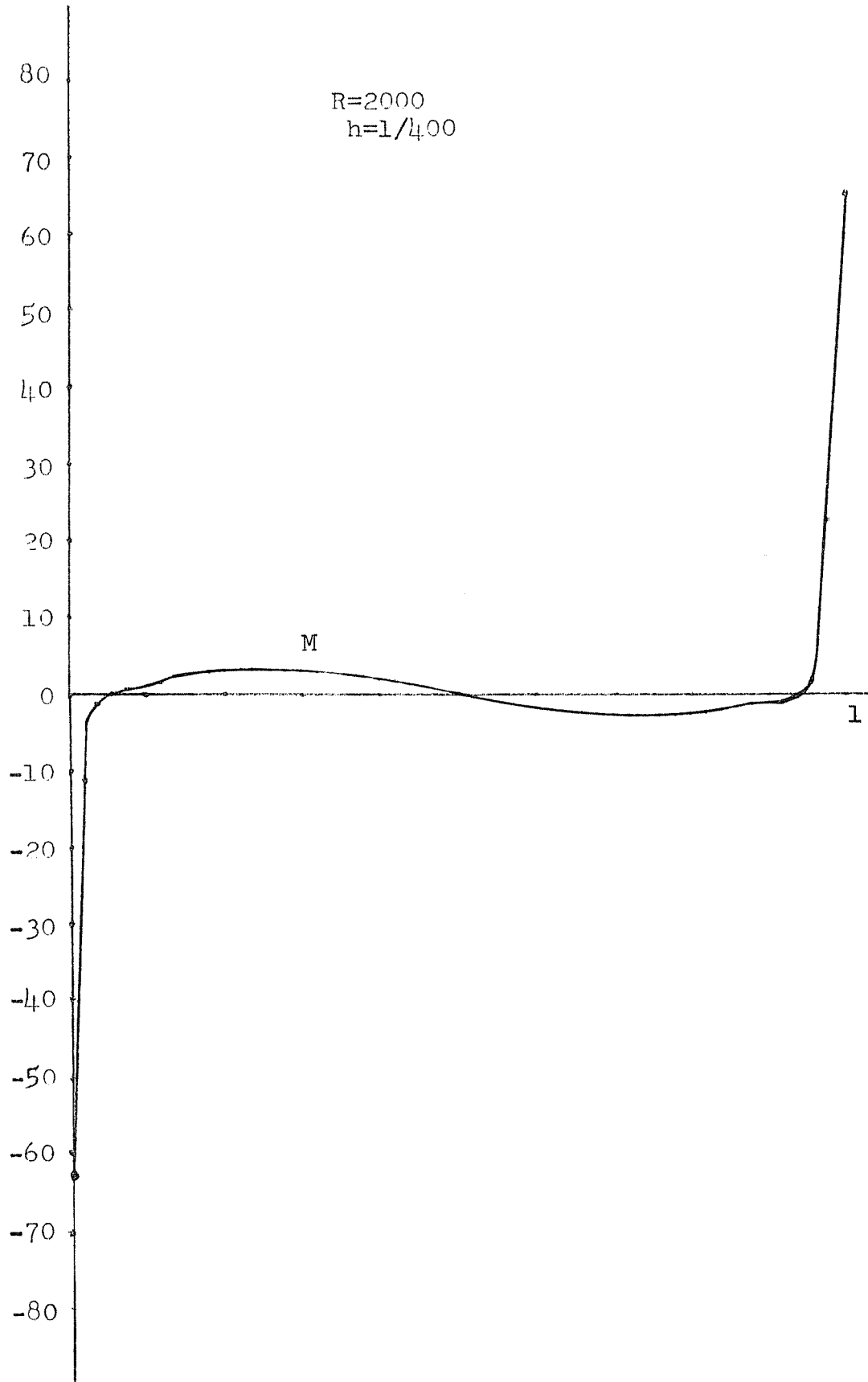


FIGURE 8

REFERENCES

1. A. Acrivos, M. J. Shah, and E. E. Peterson, "On the flow of a non-Newtonian liquid on a rotating disk," *J. Appl. Phys.*, 31, 1960, p. 963.
2. G. K. Batchelor, "Note on a class of solutions of the Navier-Stokes equations representing steady rotationally symmetric flow," *QJMAM*, 4, 1951, p. 29.
3. E. R. Benton, "On the flow due to a rotating disk," *J. Fluid Mech.*, 24, 1966, p. 781.
4. R. K. Bhatnagar, "Flow of non-Newtonian fluid between two infinite parallel discs for large values of Reynolds number—one rotating and the other at rest," *Proc. Indian Acad. Sci., Sec. A*, 58, 1963, p. 279.
5. W. G. Cochran, "The flow due to a rotating disk," *Proc. Camb. Phil. Soc.*, 30, 1934, p. 365.
6. E. M. Dobrysmann and V. P. Sadokov, "Unsteady motion of a viscous incompressible fluid near a rotating disk," *Izdat Akad. Nauk SSR, Tashkent*, 1960, p. 86.
7. D. R. Davies, "On the calculation of eddy viscosity and heat transfer in a turbulent boundary layer near a rapidly rotating disk," *QJMAM*, 12, 1959, p. 211.
8. D. Greenspan, "Numerical studies of prototype cavity flow problems," *The Comp. J.*, 12, 1969, p. 89.
9. H. P. Greenspan, *The Theory of Rotating Fluids*, Camb. Univ. Press, Cambridge, 1968.
10. H. P. Greenspan and L. N. Howard, "On a time-dependent motion of a rotating fluid," *J. Fluid Mech.*, 1963, p. 385.
11. R. P. Kanwal, "Impulsive rotatory motion of a circular disk in a viscous fluid," *ZAMP*, 10, 1959, p. 552.
12. G. N. Lance and M. H. Rogers, "The axially symmetric flow of a viscous fluid between two infinite rotating disks," *Proc. Roy. Soc.*, A266, 1962, p. 109.

13. G. L. Mellor, P. J. Chapple and V. K. Stokes, "On the flow between a rotating and a stationary disk," *J. Fluid Mech.*, 31, 1968, p. 95.
14. S. D. Nigam, "Rotation of an infinite plane lamina: Boundary layer growth: Motion started impulsively from rest," *Quart. Appl. Math.*, 9, 1951, p. 89.
15. C. E. Pearson, "Numerical solutions for the time-dependent viscous flow between two rotating coaxial disks," *J. Fluid Mech.*, 21, 1965, p. 623.
16. H. Rasmussen, "Numerical solutions for steady, viscous linearized flow between two infinite rotating disks," *ZAMP*, 21, 1970, p. 611.
17. M. H. Rogers and G. N. Lance, "The rotationally symmetric flow of a viscous fluid in the presence of an infinite rotating disk," *J. Fluid Mech.*, 7, 1960, p. 617.
18. A Schubert, "FORTRAN program for flow between disks," Appendix, Tech. Report. 110, Department of Computer Sciences, Univ. of Wisconsin, 1971.
19. V. P. Sidlovskii, "Laminar boundary layer on an infinite rotating disk rotating in a gas," *Prikl. Mat. Meh.*, 24, 1960, p. 161.
20. A. C. Srivastava, "Flow of non-Newtonian fluids at small Reynolds number between two infinite disks: one rotating and the other at rest," *QJMAM*, 14, 1961, p. 353.
21. K. Stewartson, "On the flow between two rotating coaxial disks," *Proc. Camb. Phil. Soc.*, 49, 1953, p. 333.
22. K. K. Tam, "A note on the asymptotic solution of the flow between two oppositely rotating infinite plane disks," *SIAM J. Appl. Math.*, 17, 1969, p. 1305.
23. H. K. Theriot, "Über die laminare Anlaufströmung einer Flüssigkeit einem rotierenden Boden bei plötzlicher Änderung des drehungszustandes," *ZAMM*, 20, 1940, p. 1.

24. T. von Karmen, "Laminare und turbulente Reibung," ZAMM, 1, 1921, p. 233.
25. J. Watson, "On the existence of solutions for a class of rotating disk flows and the convergence of a successive approximation scheme," J. Inst. Math. Appl., 1, 1965, p. 348.

A. Schubert's FORTRAN Program for flow between disks (Figure 7 case)

```

1*      PARAMETER NMAX=401
2*      REAL MU,M(NMAX,3),MO(NMAX)
3*      DIMENSION OMEGA( 2),H(NMAX,3),G(NMAX,3),DXSQ(NMAX),BO(NMAX),
4*      *   B1(NMAX),B2(NMAX),CG(NMAX),CM(NMAX),DELTA(2),HPR(NMAX)
5*      DIMENSION HO(NMAX),GO(NMAX)
6*      COMMON H,G,M
7*      COMMON/ITOUT/ITOUT/NP1/NP1/NIT/NTH,NTG,NTM/INCPR/INCPR
8*      COMMON/CON/BO,B1,B2,CG,CM/DXSQ/DXSQ/N/N/NM1/NM1/R/R
9*      DATA MAXIT,MAXOUT/800,200/,IBUG,INCOUT/1,1/,INCSOR,INCRES/801,200/
10*     DATA INSW/O/,INCPR/4/
11*     99 FORMAT(11E5.5)
12*     98 FORMAT('OSOR ITERATION FOR H FAILED IN OUTER ITERATION' IS,
13*     *   ' GO TO NEXT PARAMETER CASE.')
14*     97 FORMAT('OSOR ITERATION FOR G FAILED IN OUTER ITERATION' IS,
15*     *   ' GO TO NEXT PARAMETER CASE.')
16*     96 FORMAT('OSOR ITERATION FOR M FAILED IN OUTER ITERATION' IS,
17*     *   ' GO TO NEXT PARAMETER CASE.')
18*     95 FORMAT('OUTER ITERATION FAILED AFTER MAX. NO. OF ITERATIONS. PROC
19*     *EED TO NEXT PARAMETER CASE.')
20*     94 FORMAT(1H1 'DX =' F7.5,5X 'OMEGA(1) =' F5.2,5X 'OMEGA(2) =' F5.2,
21*     * 5X 'REYNOLDS NO. =' F7.0/1H03X 'SMOOTHING' 5X 'OVER-RELAX.' 5X
22*     * 'CONVERGENCE' 5X 'FACTOR' 9X 'FACTOR' 9X 'TOLERANCE'
23*     * 6X 'SOR TOLERANCE' / 1H0 1X 1H
24*     * F6.3,F16.3,2F17.4/2X1HGF6.3,F16.3,2F17.4/2X1HM2F6.3,F10.3,2F17.4)
25*     93 FORMAT('CONVERGENCE ATTAINED. TOTAL SOR ITERATIONS FOR H,G,M ='
26*     * 317)
27*     92 FORMAT('TOTAL SOR ITERATIONS FOR H,G,M =' 317)
28*     90 FORMAT(1X F9.3,3F8.3,14F7.3)
29*     89 FORMAT(1H )
30*     88 FORMAT('OH AT SOR ITERATION' 16)
31*     188 FORMAT('OG AT SOR ITERATION' 16)
32*     288 FORMAT('OM AT SOR ITERATION' 16)
33*     87 FORMAT('OH=PRIME')
34*     84 FORMAT(13F6.3)
35*     83 FORMAT('OH,G,M INITIALLY'/)
36*     C   READ INPUT PARAMETERS
37*     IF(INSW.NE.1) GO TO 4
38*     READ(5,84) (HO(I),I=1,NMAX)
39*     READ(5,84) (GO(I),I=1,NMAX)
40*     READ(5,84) (MO(I),I=1,NMAX)
41*     4   READ(5,99) ESH,ESG,ESM,FIN2
42*     5   READ(5,99) DX,OMEGA,R,EH,EG,EM,FIN
43*     7   READ(5,99) RHO,MU,DELTA,RH,RG,RM,END
44*     WRITE(6,94) DX,OMEGA,R,RHO,RH,EH,ESH,MU,RG,EG,ESG,DELTA,RM,EM,ESM
45*     N=1./DX+.005
46*     NP1=N+1
47*     NM1=N-1
48*     C   INITIALIZE FUNCTIONS FOR OUTER ITERATION
49*     C
50*     IF(INSW.EQ.1) GO TO 11
51*     DO 10 I=1,NP1
52*     X=FLOAT(I-1)*DX
53*     G(I,3)=(OMEGA(2)-OMEGA(1))*X+OMEGA(1)

```

```

54*      IF(X.GT. .25) GO TO 8
55*      H(I,3)=-1.2*X
56*      M(I,3)=52.0*X-10.
57*      GO TO 10
58*      8 IF(X.GT. .75) GO TO 9
59*      H(I,3)=1.2*X-.6
60*      M(I,3)=-12.0*X+6.
61*      GO TO 10
62*      9 H(I,3)=-1.2*X+1.2
63*      M(I,3)=52.0*X-42.
64*      10 CONTINUE
65*      GO TO 211
66*      11 DO 111 I=1,NP1
67*          H(I,3)=HO(I)
68*          G(I,3)=GO(I)
69*      111 M(I,3)=MO(I)
70*      211 DO 12 J=1,2
71*          H(I,J)=H(I,3)
72*          H(NP1,J)=H(NP1,3)
73*          M(I,J)=M(I,3)
74*          M(NP1,J)=M(NP1,3)
75*          G(I,J)=G(I,3)
76*      12 G(NP1,J)=G(NP1,3)
77*          WRITE(6,83)
78*          WRITE(6,90) (H(I,3),I=1,NP1,INCPR)
79*          WRITE(6,90) (G(I,3),I=1,NP1,INCPR)
80*          WRITE(6,90) (M(I,3),I=1,NP1,INCPR)
81*          DXSQ=DX**2
82*          AH1=1.-RH
83*          AH2=RH/2.
84*          RDX=R*DX
85*          AG1=1.-RG
86*          AM1=1.-RM
87*          HSQ2=2./DXSQ
88*          NTOTH=0
89*          NTOTG=0
90*          NTOTM=0
91*          ITOUT=0
92*      16 ITOUT=ITOUT+1
93*          DO 17 I=2,N
94*              H(I,1)=H(I,3)
95*              G(I,1)=G(I,3)
96*      17 M(I,1)=M(I,3)
97*          M(1,1)=M(1,3)
98*          M(NP1,1)=M(NP1,3)
99*      C      SOLVE FOR H-BAR BY SOR ITERATION
100*          NTH=0
101*          DO 18 I=2,N
102*      18 DXSQM(I)=DXSQ*M(I,1)
103*      20 NTH=NTH+1
104*          DO 22 I=2,N
105*      22 H(I,2)=H(I,3)
106*          H(2,3)=AH1*H(2,2)+RH/4.0*H(3,2)
107*          DO 25 I=3,NM1
108*      25 H(I,3)=AH1*H(I,2)+AH2*(H(I-1,3)+H(I+1,2)+DXSQM(I))
109*          H(N,3)=AH1*H(N,2)+RH/4.0*H(NM1,3)
110*      C      TEST FOR CONVERGENCE

```

```

111*      DO 27 I=2,N
112*      IF(.NOT.(ABS(H(I,2)-H(I,3)).LT.ESH)) GO TO 33
113*  27 CONTINUE
114*  C      CONVERGENCE ATTAINED== SMOOTH SOLUTION
115*      DO 30 I=2,N
116*  30 H(I,3)=RHO*H(I,3)+(1.-RHO)*H(I,1)
117*      GO TO 34
118*  C      CONVERGENCE UNATTAINED==UPDATE FOR NEXT ITERATION UNLESS ITERATION
119*  C      LIMIT HAS BEEN EXCEEDED.
120*  33 IF(MOD(NTH,INCSOR).NE.0) GO TO 133
121*      WRITE(6,88) NTH
122*      WRITE(6,90) (H(I,3),I=1,NP1,INCPR)
123*  133 IF(NTH.LT.MAXIT) GO TO 20
124*      WRITE(6,98) ITOUT
125*      GO TO 62
126*  C      SOLVE FOR G=BAR BY SOR ITERATION
127*  34 DO 35 I=2,N
128*      BO(I)=2.-RDX*ABS(H(I,3))
129*      B1(I)=(1.+RDX*AMAX1(0.,H(I,3)))
130*      B2(I)=(1.-RDX*AMIN1(0.,H(I,3)))
131*  35 CG(I)=0.5*RDX*(G(I,1)*(H(I+1,3)-H(I-1,3)))
132*      NTG=0
133*  36 NTG=NTG+1
134*      DO 37 I=2,N
135*  37 G(I,2)=G(I,3)
136*      DO 38 I=2,N
137*  38 G(I,3)=AG1*G(I,2)+RG*(B1(I)*G(I-1,3)+B2(I)*G(I+1,2)-CG(I))/BO(I)
138*  C      TEST FOR CONVERGENCE
139*      DO 40 I=2,N
140*      IF(.NOT.(ABS(G(I,2)-G(I,3)).LT.ESG)) GO TO 44
141*  40 CONTINUE
142*  C      CONVERGENCE ATTAINED==SMOOTH SOLUTION
143*      DO 42 I=2,N
144*  42 G(I,3)=MU*G(I,3)+(1.-MU)*G(I,1)
145*      GO TO 46
146*  C      CONVERGENCE UNATTAINED==UPDATE FOR NEXT SOR ITERATION UNLESS
147*  C      MAX. ITERATION LIMIT HAS BEEN EXCEEDED.
148*  44 IF(MOD(NTG,INCSOR).NE.0) GO TO 144
149*      WRITE(6,188) NTG
150*      WRITE(6,90) (G(I,3),I=1,NP1,INCPR)
151*  144 IF(NTG.LT.MAXIT) GO TO 36
152*      WRITE(6,97) ITOUT
153*      GO TO 62
154*  C      COMPUTE M=BAR AT BOUNDARY POINTS
155*  46 M(1,3)=DELTA(1)*HSQ2*H(2,3)+(1.-DELTA(1))*M(1,1)
156*      M(NP1,3)=DELTA(1)*HSQ2*H(N,3)+(1.-DELTA(1))*M(NP1,1)
157*  C      SOLVE FOR M=BAR AT INTERIOR POINTS
158*      DO 48 I=2,N
159*  48 CM(I)=2.*RDX*(G(I,3)*(G(I+1,3)-G(I-1,3)))
160*      M(NP1,2)=M(NP1,3)
161*  148 NTM=0
162*  49 NTM=NTM+1
163*      DO 50 I=2,N
164*  50 M(I,2)=M(I,3)
165*      DO 52 I=2,N
166*  52 M(I,3)=AM1*M(I,2)-RM*(B1(I)*M(I-1,3)+B2(I)*M(I+1,2)-CM(I))/BO(I)
167*  C      TEST FOR CONVERGENCE

```

```

168*      DO 54 I=2,N
169*      IF(.NOT.(ABS(M(I,2)-M(I,3)).LT.ESM)) GO TO 56
170*      54 CONTINUE
171*      C      CONVERGENCE ATTAINED==SMOOTH SOLUTION
172*      DO 55 I=2,N
173*      55 M(I,3)=DELTA(2)*M(I,3)+(1.-DELTA(2))*M(I,1)
174*      GO TO 58
175*      C      CONVERGENCE UNATTAINED==UPDATE FOR NEXT SOR ITERATION UNLESS
176*      C      ITERATION LIMIT HAS BEEN EXCEEDED.
177*      56 IF(MOD(NTM,INCSOR).NE.0) GO TO 57
178*      WRITE(6,208) NTM
179*      WRITE(6,90) (M(I,3),I=1,NP1,INCPR)
180*      57 IF(NTM.LT.MAXIT) GO TO 49
181*      WRITE(6,96) ITOUT
182*      GO TO 62
183*      58 IF(MOD(ITOUT,INCOUT).EQ.0) CALL OUTPUT
184*      IF(MOD(ITOUT,INCRES).EQ.0) CALL TEST
185*      NTOTH=NTOTH+NTH
186*      NTOTG=NTOTG+NTG
187*      NTOTM=NTOTM+NTM
188*      C      TEST FOR CONVERGENCE OF OUTER ITERATION
189*      DO 59 I=2,N
190*      IF(.NOT.(ABS(H(I,3)-H(I,1)).LT.EH))GO TO 60
191*      IF(.NOT.(ABS(G(I,3)-G(I,1)).LT.EG))GO TO 60
192*      IF(.NOT.(ABS(M(I,3)-M(I,1)).LT.EM))GO TO 60
193*      59 CONTINUE
194*      IF(.NOT.(ABS(M(I,3)-M(I,1)).LT.EM))GO TO 60
195*      IF(.NOT.(ABS(M(NP1,3)-M(NP1,1)).LT.EM))GO TO 60
196*      C      CONVERGENCE ATTAINED==OUTPUT SOLUTION
197*      IF(MOD(ITOUT,INCOUT).NE.0) CALL OUTPUT
198*      C      COMPUTE H=PRIME
199*      DO 159 I=2,N
200*      159 HPR(I)=(H(I+1,3)-H(I-1,3))/(2.*DX)
201*      WRITE(6,87)
202*      WRITE(6,90) (HPR(I),I=1,NP1,INCPR)
203*      IF(MOD(ITOUT,INCRES).NE.0) CALL TEST
204*      WRITE(6,93) NTOTH,NTOTG,NTOTM
205*      GO TO 62
206*      C      CONVERGENCE UNATTAINED==OUTPUT VALUES FOR OUTER ITERATION AND
207*      C      UPDATE FOR NEXT OUTER ITERATION UNLESS OUTER ITERATION LIMIT
208*      C      HAS BEEN EXCEEDED.
209*      60 IF(ITOUT.LT.MAXOUT) GO TO 16
210*      WRITE(6,92) NTOTH,NTOTG,NTOTM
211*      WRITE(6,95)
212*      62 IF(ABS(END).GT.0.) GO TO 63
213*      GO TO 7
214*      63 IF(ABS(FIN).GT.0.) GO TO 64
215*      GO TO 5
216*      64 IF(ABS(FIN2).GT.0.) STOP
217*      GO TO 4
218*      END

```