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Numerical Study of Vortex Motion in the Two Dimensional Ginzburg-Landau Equation

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The spiral pattern can emerge in the two dimensional generalized Ginzburg-Landau equation. There exists a vortex as a topological defect in the center of the spiral. We carried out some numerical simulations to study the motion of an interacting pair of vortices and a vortex driven by the faster pacemaker.

Spiral patterns and targetlike patterns are interesting and well-known wave patterns in the Belousov-Zhabotinsky reaction.[1),2)] Such waves can emerge generally in two dimensional excitable or oscillatory media.[1)~7)] We consider only oscillatory media in the following. There is a pacemaker region in the center of such patterns and it sends out the waves. For the targetlike pattern the pacemaker arises from heterogeneous nuclei in the vicinity of which the oscillation frequency is somewhat higher than that of bulk medium. For the spiral pattern there is a singularity point in the center of the spiral, where the amplitude of oscillation becomes zero and the phase of oscillation cannot be defined. The phase singularity point is a kind of topological defects called vortex and it plays a role of the pacemaker. A three dimensional extension of rotating spiral waves is called scroll waves and there is a vortex line in the center of the scroll.[3),4)]

It is known experimentally that many spiral patterns coexist stably and they move around hardly. It is also known that if spiral patterns and targetlike patterns coexist initially, the targetlike patterns are gradually eaten away by the spiral patterns and eventually all the targetlike patterns disappear and only the spiral patterns survive. It is because the pace of oscillation of the spiral wave is generally faster than that of targetlike waves.

We carried out numerical simulations to see if the spiral patterns are really stable and cannot move. The model equation we used in this paper is the generalized Ginzburg-Landau equation:[5)]

$$\dot{W}=(1+i\omega(\boldsymbol{x}))W-(1+ic_2)|W|^2\cdot W+(1+ic_1)\nabla^2 W\,, \tag{1}$$

where $W=X+iY=R\exp(i\theta)$ is a complex variable, and c_1 and c_2 are parameters and $\omega(\boldsymbol{x})$ denotes heterogeneity of the oscillating frequency. If $\nabla^2 W=0$ and $\omega(\boldsymbol{x})=c_0$, the uniform oscillation $W(\boldsymbol{x}, t)=\exp\{i(c_0-c_2)t\}$ is observed. Spiral patterns are found in this system, when c_1 or c_2 has a nonzero value and $\omega(\boldsymbol{x})=c_0$.[5),6)] In our simulations c_1 is assumed to be 0 and $\omega(\boldsymbol{x})$ is almost constant except for a local pacemaker region. We used the simple Euler method with the time step $\Delta t=0.005$ and the space step $\Delta x=0.3$. The grid size is 200×200 and therefore the space size is 60×60. And the free boundary condition is assumed.

The center of the spiral pattern is in the state of vanishing amplitude, i.e., $R=0$ or $(X, Y)=(0, 0)$, and hence the phase θ cannot be defined there. The rotation number l is defined as

$$l=\frac{1}{2\pi}\oint \nabla\theta d\boldsymbol{x}\,, \tag{2}$$

where the integral is taken around the phase singularity point. The rotation number l corresponds to the number of arms of the spiral and $l=\pm 1$ for the usual single armed spiral. The sign of l denotes the winding direction, i.e., plus (minus) sign represents counterclockwise (clockwise). The amplitude of oscillation is small around the phase singularity point and the region is called the core region. The core radius is about 4 when $c_1=0$ and $c_2=1.0$.

We study at first about a pair of spiral patterns in a uniform system, i.e., $\omega(\boldsymbol{x})$ $=$const. Two types of pairs can be considered: One is a clockwise-counterclockwise pair and the other is a clockwise-clockwise pair. In the former case the total rotation

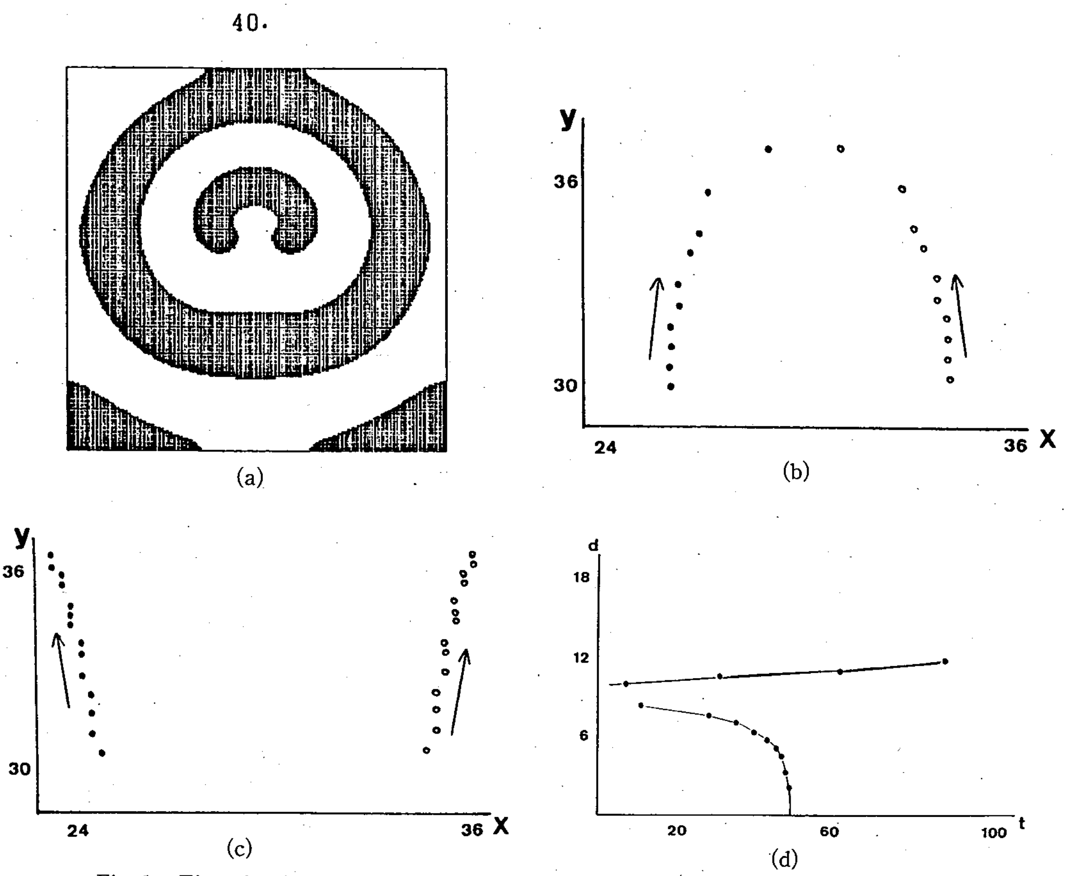
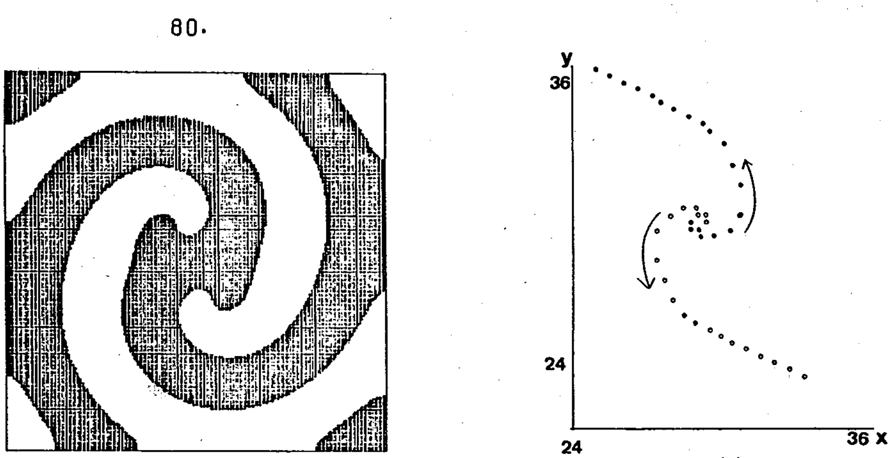


Fig. 1. Time developement of a clockwise-counterclockwise pair of spiral patterns.
(a) Phase pattern at $t=40$ when the initial distance r_0 between the two vortices is 8.1.
(b) Trajectories of the vortices for $r_0=8.1$ in the two dimensional space.
(c) Trajectories of the vortices for $r_0=9.9$ in the two dimensional space.
(d) Time development of the distance between the vortices for $r_0=8.1$ and 9.9.

number is 0 and it is possible that the pair of spirals are merged and disappear. In the latter case the total rotation number is 2, and if the two spirals are merged, a two-armed spiral may emerge. We carried out numerical simulations for $c_2=1.0$ and $\omega(\boldsymbol{x})=0$. The two phase singularity was arranged apart from each other as an initial condition, and the initial distance between the vortices was changed in several ways. Time development of the pattern and the positions of the vortices were investigated.

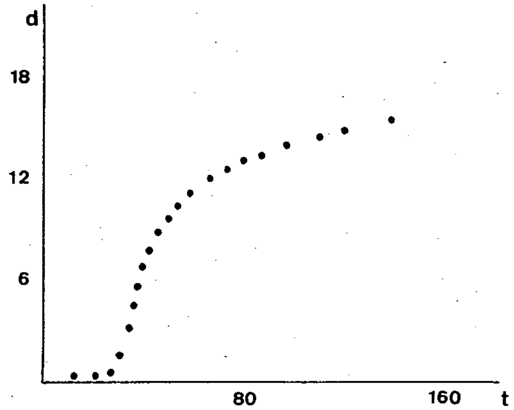
When the initial distance is large, the vortices seem to stay still and the spirals rotate steadily in both cases. But when the initial distance is not so large, the vortices can move. We show a result of the simulation in Fig. 1 for a clockwise-counterclockwise pair. Figure 1(a) is a phase pattern when the initial distance r_0 is 8.1. A small cross is marked at each point where $\mathrm{Im}\, W=Y>0$. A pair of spirals is seen and drifts upward. Figures 1(b) and (c) show the trajectories of vortices for $r_0=8.1$ and 9.9. For both initial distances the vortices drift upward. For $r_0=8.1$ the two vortices approach gradually and are merged and disappear. But for $r_0=9.9$ they go away from each other. This result suggests that there is a critical distance beyond which two vortices are repulsive. Figure 1(d) shows the distance of the two vortices as a function of time. As the vortices approach enough, the velocity of the approach and the upward-drift becomes fast. When $c_1=c_2=0$, Eq. (1) becomes the usual Ginzburg-Landau equation and the system has a potential density function: $-1/2|W|^2+1/4|W|^4+1/2|\nabla W|^2$. In the usual Ginzburg-Landau equation a vortex can exist but it cannot work as a pacemaker and therefore cannot send out a spiral waves. When two vortices are arranged initially, they only attract each other, as the integrated potential function is decreased.[8),9)] They cannot drift toward the vertical direction to the line which connects the two vortices. A similar vertical drift is known for the vortex motion in the fluid.

Figure 2 shows a result for a clockwise-clockwise pair of spirals. Initially they are set very closely, and the phase pattern is almost a two-armed spiral pattern. The vortices are repulsive and go away from each other as shown in Figs. 2(b) and (c). The velocity is slow when the distance between the vortices is short. It is probably because the two-armed spiral pattern is an unstable but stationary pattern and plays



(a) (b)

Fig. 2. (continued)



(c)

Fig. 2. Time development of a clockwise-clockwise pair of spiral patterns when the initial distance r_0 is very small.
(a) Phase pattern at $t=80$.
(b) Trajectories of the two vortices in the two dimensional space.
(c) Time development of the distance between the vortices.

a part of a saddle point in the dynamical system. The drift motion is also seen in this case, but the directions are opposite for the two vortices. Thus the trajectory of the vortex takes a shape of spiral and the two trajectories are point-symmetric with each other. The velocity becomes slow as the two vortices become far away.

One vortex can stay still anywhere because of the translational symmetry of the original equation (1), if the system-size is infinite. The existence of the vortex breaks the translational symmetry. It is therefore impossible that another vortex stays still everywhere apart from the former vortex. The experimental fact that many spirals coexist and move hardly may suggest that the repulsive or attractive force becomes rapidly weaker as the distance between the spirals is long.

Next we study a case when one vortex and a pacemaker region due to the heterogeneity of the oscillating frequency coexist. A circular region whose radius is 6 has higher natural frequency by ω_0 than the surrounding, and a spiral pattern is arranged initially apart from the pacemaker region. When ω_0 is small, the spiral pattern is hardly affected by the heterogeneity, i.e., the vortex is almost stationary and the spiral rotates steadily. This corresponds to the experimental fact that a spiral pattern overcomes a targetlike pattern and only the spiral pattern survives. But when ω_0 is large, the circular region can emit targetlike waves faster than the core of the spiral. Then the targetlike pattern becomes preferable. But the topological defect or the vortex cannot disappear by itself. What happens then? Figure 3 shows the result of a simulation for $\omega_0=0.6$. The spiral pattern is eaten away gradually but the vortex does not move until the targetlike waves surge upon the core of the spiral. Then the core of the spiral pattern is deformed and starts to move. The vortex moves around the circular region and the spiral is wound off. Finally it will go away from the boundary, and the targetlike pattern overcomes the spiral pattern. There is a shockwave-like structure where the targetlike waves collide with the spiral waves. It separates the spiral region from the targetlike region. It is suggested from our simulations that the distance between the vortex and the shocklike boundary is important for the vortex motion. When the distance is long, the spiral occupies a large region and it winds up many times and then the core of the spiral is stable. But when the targetlike waves surge upon the core region or the shocklike boundary approaches the core region, the core becomes unstable and starts to move.

Another similar simulation was carried out. In this simulation only the right side has higher natural frequency by ω_0 than the bulk medium and the heterogeneity gives rise to one dimensional waves. A result of the simulation for $\omega_0=0.9$, $c_2=1.0$ is shown in Fig. 4. Similar to Fig. 3 the one dimensional waves are preferable to the spiral waves. The spiral pattern is deformed into a straight long tail. The vortex drifts toward the left and downward. The velocity of the vortex is nearly constant after an initial stage. The left is the direction against the faster pacemaker region, and the tail pattern shrinks due to the downward drift. The shrinking direction seems to be characteristic of the drift motion of vortex in our system. The direction of the drift motion with respect to the characteristic hook-like pattern of the core is common as is seen in Figs. 1~4.

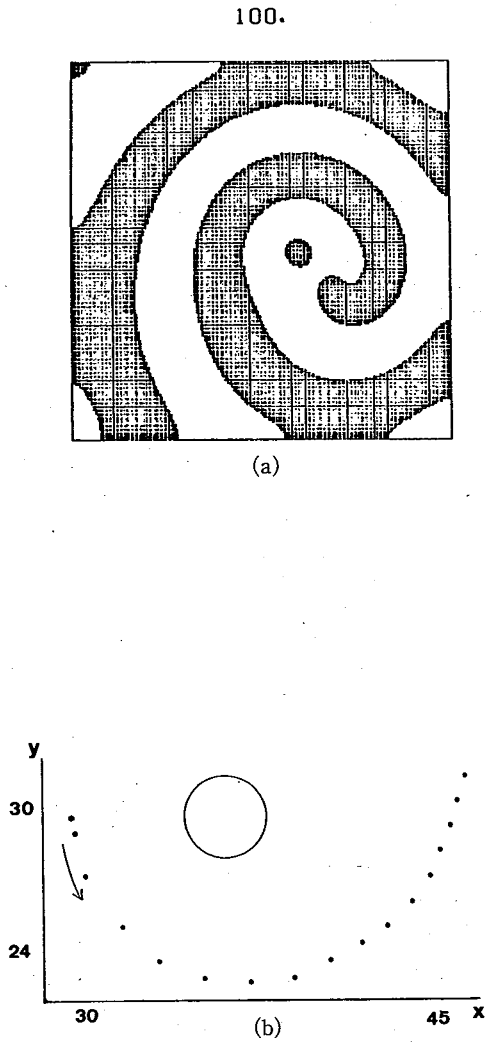


Fig. 3. Time development of a spiral pattern driven by the circular pacemaker where the natural frequency is higher by 0.6 than the surrounding region.
(a) Phase pattern at $t=100$.
(b) Trajectory of the vortex in the two dimensional space.

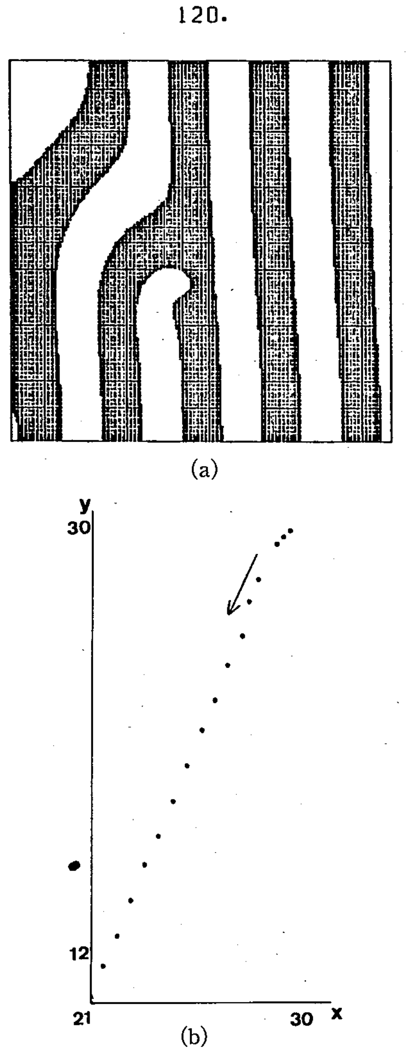


Fig. 4. Time development of a spiral pattern driven by the linear pacemaker on the right side where the natural frequency is higher by 0.9 than the bulk region.
(a) Phase pattern at $t=120$.
(b) Trajectory of the vortex in the two dimensional space.

We carried out several simulations and found that vortices can move when two vortices are closely situated or the vortex is driven by the faster pacemaker. The drift motion is observed besides the simple attractive or repulsive motion. The motion seems to have common features but we need further numerical and theoretical studies to understand the vortex motion.

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