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Numerical treatment of fourth-order singular boundary value problems using new quintic B-spline approximation technique

Muhammad Kashif Iqbal [1], Muhammad Waseem Iftikhar [2], Muhammad Shahid Iqbal [3], Muhammad Abbas [4,*]

[1]*Department of Mathematics, Government College University, Faisalabad, Pakistan*
[2]*Department of Mathematics, National Textile University, Faisalabad, Pakistan*
[3]*Department of Mathematics, University of Okara, Okara, Pakistan*
[4]*Department of Mathematics, University of Sargodha, Sargodha, Pakistan*

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ABSTRACT

Singular boundary value problems (SBVPs) are cropped up in mathematical modeling of many real-life phenomena such as chemical reactions, electro-hydrodynamics, aerodynamics, thermal explosions, fluid dynamics, and atomic nuclear reactions. In this work, a new quintic B-spline approximation technique has been presented for the numerical solution of fourth-order singular boundary value problems. The fifth-degree basis spline functions are brought into play together with a new approximation for fourth-order derivative. The proposed numerical technique is proved to be uniformly convergent in the entire domain. In order to corroborate this work, the proposed scheme has been implemented on some test problems. The comparison of computational outcomes advocates the superior performance of the presented algorithm over current methods on the topic.

1. Introduction

In this work, we have considered the following class of fourth-order SBVPs.

$$\alpha u^{(4)}(x)+\frac{\beta}{x}u^{(3)}(x)+v(x)u''(x)+w(x)u'(x)=f(x,u), x\in[0,1], \quad (1)$$

with one of the following the initial/boundary conditions:

$$\begin{cases} u(0)=\beta_1,\ u'(0)=\beta_2,\ u''(0)=\beta_3,\ u'''(0)=0 \\ u(0)=\beta_1,\ u'(0)=\beta_2,\ u(1)=\beta_3,\ u'(1)=\beta_4\ , \\ u(0)=\beta_1,\ u'(0)=\beta_2,\ u(1)=\beta_3,\ u''(1)=\beta_4 \end{cases} \quad (2)$$

where, $\alpha, \beta, \alpha_i's$ are constants and $v(x), w(x)$ are smooth functions and f, f_u are supposed to be continuous in the entire domain with $f_u \geq 0$. In recent years, SBVPs have attracted a considerable amount of research work. Khuri (2001) explored the numerical solution of generalized Lane-Emden type equations by means of a new decomposition method based on Adomian polynomials. Kim and Chun (2010) employed a modified Adomian decomposition method to obtain the series solution of higher-order SBVP's. Aruna and Kanth (2013) studied the series solution of higher-order SBVP's using differential transformation method. Wazwaz (2015) proposed the Variational iteration method for the numerical solution of fourth-order SBVP's. Taiwo and Hassan (2015) presented a new iterative decomposition method to solve higher-order initial and boundary value problems. The numerical solution to fourth-order Emden-Flower type equations has been discussed in Wazwaz et al. (2015) using the Adomian decomposition method. Parand and Delkhosh (2017) proposed a generalized fractional-order of Chebyshev functions for solving singular Lane-Emden type equations.

The spline interpolating functions have been employed frequently for solving initial and boundary value problems (BVP's). The third-degree spline functions were brought into use for solving second order SBVP's in Abukhaled et al. (2011), Caglar et al. (2009), and Goh et al. (2012; 2011). Khuri and Sayfy (2014) developed an adaptive cubic B-spline (CBS) collocation scheme to investigate the approximate solution of second-order Emden-Flower type equations. The fourth-degree polynomial spline functions were utilized by Akram (2011) for the numerical solution of third-order self-adjoint singularly perturbed BVP's. Akram and Amin (2012) used a quintic polynomial spline for solving fourth-

* Corresponding Author.
Email Address: muhammad.abbas@uos.edu.pk (M. Abbas)
https://doi.org/10.21833/ijaas.2020.06.007
Corresponding author's ORCID profile:
https://orcid.org/0000-0002-0491-1528

order singularly perturbed BVP's. Lodhi and Mishra (2016) employed the quintic B-spline (QnBS) collocation method for solving fourth-order singularly perturbed SBVP's.

In this paper, we have explored the approximate solution of fourth-order SBVP's by dint of QnBS functions reinforced with a new approximation for fourth-order derivative. In recent years, several numerical techniques have been proposed for the numerical solution of SBVP's, but as far as we know, this technique is novel and has not been employed for solving BVP's before.

This work is composed as follows: We shall review some key ideas of QnBS interpolation in section 2. The new QnBS approximation for the fourth-order derivative has been formulated in section 3. The numerical method is described in section 4. The derivation of uniform convergence is given in section 5. The numerical results and discussions are presented in section 6.

1.1. Quintic B-spline functions

We uniformly partition the interval $[a,b]$ by $n+1$ equidistant knots $x_i = x_0 + ih,\ i = 0(1)n,$ where, $n \in \mathbb{Z}^+, a = x_0, b = x_n$ and $h = \frac{b-a}{n}$. Let us extended $[a,b]$ to $[a+5h, b+5h]$ with equidistant knots $x_i = x_0 + ih, (\ i = -5,-4,-3,\dots,n+3,n+4,n+5)$ and define the typical fifth-degree basis spline functions as (Lodhi and Mishra, 2016; Xu and Lang, 2014):

$$B_j(x) = \frac{1}{120h^5}\begin{cases} (x-x_{j-3})^5, & x \in [x_{j-3}, x_{j-2}] \\ h^5 + 5h^4(x-x_{j-2}) + 10h^3(x-x_{j-2})^2 + 10h^2(x-x_{j-2})^3 & \\ +5h(x-x_{j-2})^4 - 5(x-x_{j-2})^5, & x \in [x_{j-2}, x_{j-1}] \\ 26h^5 + 50h^4(x-x_{j-1}) + 20h^3(x-x_{j-1})^2 - 20h^2(x-x_{j-1})^3 & \\ -20h(x-x_{j-1})^4 + 10(x-x_{j-1})^5, & x \in [x_{j-1}, x_j] \\ 26h^5 + 50h^4(x_{j+1}-x) + 20h^3(x_{j+1}-x)^2 - 20h^2(x_{j+1}-x)^3 & \\ -20h(x_{j+1}-x)^4 + 10(x_{j+1}-x)^5, & x \in [x_j, x_{j+1}] \\ h^5 + 5h^4(x_{j+2}-x) + 10h^3(x_{j+2}-x)^2 + 10h^2(x_{j+2}-x)^3 & \\ +5h(x_{j+2}-x)^4 - 5(x_{j+2}-x)^5, & x \in [x_{j+1}, x_{j+2}] \\ (x_{j+3}-x)^5, & x \in [x_{j+2}, x_{j+3}] \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

For a sufficiently differentiable function $u(x)$, there corresponds a unique QnBS, $U(x)$, s.t.

$$U(x) = \sum_{j=-2}^{n+2} c_j B_j(x), \quad (4)$$

where, c_j's are, constants, yet to be calculated. Let U_i, m_i, M_i, T_i and F_i denote the quintic B-spline approximations for $u(x)$ and its first four derivatives at the i^{th} knot respectively.

Using Eqs. 3 and 4, we have:

$$U_i = U(x_i) = \sum_{j=i-2}^{i+2} c_j B_j(x_i) = \frac{1}{120}(c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}), \quad (5)$$

$$m_i = U'(x_i) = \sum_{j=i-2}^{i+2} c_j B_j'(x_i) = \frac{1}{24h}(-c_{i-2} - 10c_{i-1} + 10c_{i+1} + c_{i+2}), \quad (6)$$

$$M_i = U''(x_i) = \sum_{j=i-2}^{i+2} c_j B_j''(x_i) = \frac{1}{6h^2}(c_{i-2} + 2c_{i-1} - 6c_i + 2c_{i+1} + c_{i+2}), \quad (7)$$

$$T_i = U'''(x_i) = \sum_{j=i-2}^{i+2} c_j B_j'''(x_i) = \frac{1}{2h^3}(-c_{i-2} + 2c_{i-1} - 2c_{i+1} + c_{i+2}), \quad (8)$$

$$F_i = U^{(4)}(x_i) = \sum_{j=i-2}^{i+2} c_j B_j^{(4)}(x_i) = \frac{1}{h^4}(c_{i-2} - 4c_{i-1} + 6c_i - 4c_{i+1} + c_{i+2}). \quad (9)$$

Moreover, from Eqs. 5–9, we can establish the following relations (Fyfe, 1969; Lodhi and Mishra, 2016; Xu and Lang, 2014):

$$m_i = u'(x_i) + \frac{h^6}{5040}u^{(7)}(x_i) - \frac{h^8}{21600}u^{(9)}(x_i) + \cdots, \quad (10)$$

$$M_i = u''(x_i) + \frac{h^4}{720}u^{(6)}(x_i) - \frac{h^6}{3360}u^{(8)}(x_i) + \cdots, \quad (11)$$

$$T_i = u'''(x_i) - \frac{h^4}{240}u^{(7)}(x_i) + \frac{11h^6}{30240}u^{(9)}(x_i) + \cdots, \quad (12)$$

$$F_i = u^{(4)}(x_i) - \frac{h^2}{12}u^{(6)}(x_i) + \frac{h^4}{240}u^{(8)}(x_i) + \cdots. \quad (13)$$

From Eqs. 10–13, we can write:

$$\|m_i - u'(x_i)\|_\infty = O(h^6), \quad (14)$$
$$\|M_i - u''(x_i)\|_\infty = O(h^4), \quad (15)$$
$$\|T_i - u'''(x_i)\|_\infty = O(h^4), \quad (16)$$
$$\left\|F_i - u^{(4)}(x_i)\right\|_\infty = O(h^2). \quad (17)$$

The truncation error in F_i is $O(h^2)$ which provides a solid reason to construct a new approximation for fourth-order derivative.

2. The new approximation for $U^4(x)$

Using Eq. 3, the following expression can be established for F_{i-2} at the knots x_i, $(i = i, 2, 3, \dots, n-2)$ (Iqbal et al., 2018):

$$F_{i-2} = u^{(4)}(x_{i-2}) - \frac{h^2}{12}u^{(6)}(x_{i-2}) + \frac{h^4}{240}u^{(8)}(x_{i-2}) + \cdots$$
$$= u^{(4)}(x_i) - 2hu^{(5)}(x_i) + \frac{23h^2}{12}u^{(6)}(x_i) - \frac{7h^3}{6}u^{(7)}(x_i) + \cdots.$$

We can derive similar relations for F_{i-1}, F_{i+1}, F_{i+2} at the i^{th} knot as:

$$F_{i-1} = u^{(4)}(x_i) - h^2u^{(5)}(x_i) + \frac{5h^2}{12}u^{(6)}(x_i) - \frac{h^3}{12}u^{(7)}(x_i) + \cdots,$$
$$F_{i+1} = u^{(4)}(x_i) - hu^{(5)}(x_i) + \frac{5h^2}{12}u^{(6)}(x_i) + \frac{h^3}{12}u^{(7)}(x_i) + \cdots,$$
$$F_{i+2} = u^{(4)}(x_i) + 2hu^{(5)}(x_i) + \frac{23h^2}{12}u^{(6)}(x_i) + \frac{7h^3}{6}u^{(7)}(x_i) + \cdots.$$

Let $\tilde{F}_i$ be the new approximation to $u^{(4)}(x_i)$ *s.t.*

$$\tilde{F}_i = a_1F_{i-2} + a_2F_{i-1} + a_3F_i + a_4F_{i+1} + a_5F_{i+2}. \quad (18)$$

The above expression returns five equations involving a_i's as:

$a_1 + a_2 + a_3 + a_4 + a_5 = 1, -2a_1 - a_2 + a_4 + 2a_5 = 0,$
$23a_1 + 5a_2 - a_3 + 5a_4 + 23a_5 = 0,$
$-14a_1 - a_2 + a_4 + 14a_5 = 0, 121a_1 + a_2 + a_3 + a_4 + 121a_5 = 0,$

hence, $a_1 = -\frac{1}{240}$, $a_2 = \frac{1}{10}$, $a_3 = \frac{97}{120}$, $a_4 = \frac{1}{10}$ and $a_5 = -\frac{1}{240}$.

Substituting a_i's back into Eq. 18, we obtain,

$$\tilde{F}_i = \frac{1}{240h^4}(-c_{i-4} + 28c_{i-3} + 92c_{i-2} - 60c_{i-1} + 970c_i - 604c_{i+1} + 92c_{i+2} + 28c_{i+3} - c_{i+4}). \quad (19)$$

Now we approximate $u^{(4)}(x)$ at x_0 using four neighboring values, as:

$$\tilde{F}_0 = a_1F_0 + a_2F_1 + a_3F_2 + a_4F_3, \quad (20)$$

where,

$$F_0 = u^{(4)}(x_0) - \frac{h^2}{12}u^{(6)}(x_0) - \frac{h^4}{240}u^{(8)}(x_0) + \cdots,$$
$$F_1 = u^{(4)}(x_0) - hu^{(5)}(x_0) + \frac{5h^2}{12}u^{(6)}(x_0) + \frac{h^3}{12}u^{(7)}(x_0) + \cdots,$$
$$F_2 = u^{(4)}(x_0) + 2hu^{(5)}(x_0) + \frac{23h^2}{12}u^{(6)}(x_0) + \frac{7h^3}{6}u^{(7)}(x_0) + \cdots,$$
$$F_3 = u^{(4)}(x_0) + 3hu^{(5)}(x_0) + \frac{53h^2}{12}u^{(6)}(x_0) + \frac{17h^3}{4}u^{(7)}(x_0) + \cdots.$$

Eq. 20 returns the following four equations,

$a_1 + a_2 + a_3 + a_4 = 1, a_2 + 2a_3 + 3a_4 = 0, -a_1 + 5a_2 + 23a_3 + 53a_4 = 0,\ a_2 + 14a_3 + 51a_4 = 0,$

hence, $a_1 = \frac{7}{6}$, $a_2 = -\frac{5}{12}$, $a_3 = \frac{1}{3}$, $a_4 = -\frac{1}{12}$.

Using these values in Eq. 20, we get:

$$\tilde{F}_0 = \frac{1}{12h^4}(14c_{-2} - 61c_{-1} + 108c_0 - 103c_1 + 62c_2 - 27c_3 + 8c_4 - c_5). \quad (21)$$

Similarly, involving four neighboring knots at x_1, we suppose,

$$\tilde{F}_1 = a_1F_0 + a_2F_1 + a_3F_2 + a_4F_3, \quad (22)$$

where,

$$F_0 = u^{(4)}(x_1) - hu^{(5)}(x_1) + \frac{5h^2}{12}u^{(6)}(x_1) - \frac{h^3}{12}u^{(7)}(x_1) + \cdots,$$
$$F_1 = u^{(4)}(x_1) - \frac{h^2}{12}u^{(6)}(x_1) + \frac{h^4}{240}u^{(8)}(x_1) + \cdots,$$
$$F_2 = u^{(4)}(x_1) + hu^{(5)}(x_1) + \frac{5h^2}{12}u^{(6)}(x_1) + \frac{h^3}{12}u^{(7)}(x_1) + \cdots,$$
$$F_3 = u^{(4)}(x_1) + 2hu^{(5)}(x_1) + \frac{23h^2}{12}u^{(6)}(x_1) + \frac{7h^3}{6}u^{(7)}(x_0) + \cdots.$$

Eq. 22 gives the following equations involving a_i's:

$a_1 + a_2 + a_3 + a_4 = 1, \quad -a_1 + a_3 + 2a_4 = 0, \quad 5a_1 - a_2 + 5a_3 + 23a_4 = 0,\ -a_1 + a_3 + 14a_4 = 0.$

Solving the above system, we get $a_1 = \frac{1}{12}$, $a_2 = \frac{5}{6}$, $a_3 = \frac{1}{12}$, $a_4 = 0$. Substituting a_i's back into Eq. 22, we have:

$$\tilde{F}_1 = \frac{1}{12h^4}(c_{-2} + 6c_{-1} - 33c_0 + 52c_1 - 33c_2 + 6c_3 + c_4). \quad (23)$$

Working on the same lines, following approximations at the knots x_{n-1} and x_n are obtained:

$$\tilde{F}_{n-1} = \frac{1}{12h^4}(c_{n-4} + 6c_{n-3} - 33c_{n-2} + 52c_{n-1} - 33c_n + 6c_{n+1} + c_{n+2}), \quad (24)$$
$$\tilde{F}_n = \frac{1}{12h^4}(-c_{n-5} + 8c_{n-4} - 27c_{n-3} + 62c_{n-2} - 103c_{n-1} + 108c_n - 61c_{n+1} - 14c_{n+2}). \quad (25)$$

3. Description of the numerical method

Employing Quasi-linearization technique, Eq. 1 is transformed as

$$\alpha u_{m+1}{}^{(4)}(x) + \frac{\beta}{x}u_{m+1}{}^{(3)}(x) + v(x)u_{m+1}{}''(x) + w(x)u_{m+1}{}'(x) + Y_m(x)u_{m+1}(x) = Z_m(x), x \in [0,1], \quad (26)$$

where, $Y_m(x) = -(\frac{\partial f}{\partial u})_{(x,u_m)}$ and $Y_m(x) = -(\frac{\partial f}{\partial u})_{(x,u_m)}$, $m = 0, 1, 2, \ldots$.

After removal of singularity, Eq. 26 takes the following form

$$p(x)u_{m+1}{}^{(4)}(x) + q(x)u_{m+1}{}^{(3)}(x) + v(x)u_{m+1}{}''(x) + w(x)u_{m+1}{}'(x) + Y_m(x)u_{m+1}(x) = Z_m(x), x \in [0,1], \quad (27)$$

where, $p(x) = \begin{cases} \alpha + \beta, & if\ x = 0 \\ \alpha, & if\ x \neq 0 \end{cases}$ and $q(x) = \begin{cases} 0, & if\ x = 0 \\ \frac{\beta}{x}, & if\ x \neq 0 \end{cases}.$

Similarly, transforming the boundary conditions (Eq. 2), we get:

$$\begin{cases} u_{m+1}(0) = \beta_1 \quad u_{m+1}'(0) = \beta_2 \\ u_{m+1}(1) = \beta_3 \quad u_{m+1}'(1) = \beta_4 \end{cases}. \tag{28}$$

Let us consider that the QnBS solution for Eq. 27 is given by:

$$U(x) = \sum_{j=-2}^{n+2} c_j B_j(x). \tag{29}$$

Discretizing Eq. 27 at the i^{th} knot, we obtain:

$$p(x_i)U_{m+1}^{(4)}(x_i) + q(x_i)U_{m+1}^{(3)}(x_i) + v(x_i)U_{m+1}''(x_i) + w(x_i)U_{m+1}'(x_i) + Y_m(x)u_{m+1}(x) = Z_m(x). \tag{30}$$

For $i = 2, 3, 4, \dots, n-2$, using Eqs. 5-8 and 19 in 30, we obtain:

$$\frac{p(x_i)}{240h^4}(-c_{i-4} + 28c_{i-3} + 92c_{i-2} - 60c_{i-1} + 970c_i - 604c_{i+1} + 92c_{i+2} + 28c_{i+3} - c_{i+4}) + \frac{q(x_i)}{2h^3}(-c_{i-2} + 2c_{i-1} - 2c_{i+1} + c_{i+2}) + \frac{v(x_i)}{6h^2}(c_{i-2} + 2c_{i-1} - 6c_i + 2c_{i+1} + c_{i+2}) + \frac{w(x_i)}{24h}(-c_{i-2} - 10c_{i-1} + c_{i+1} + c_{i+2}) + \frac{Y_m(x_i)}{120}(c_{i-2} + 26c_{i-1} + 66c_i + 66c_{i+1} + c_{i+2}) = Z_m(x_i). \tag{31}$$

Similarly, at the knots (x_0, x_1, x_{n-1}) and x_n, Eq. 30 produces the following equations respectively,

$$\frac{p(x_0)}{12h^4}(12c_{-2} - 61c_{-1} + 108c_0 - 103c_1 + 62c_2 - 27c_3 + 8c_4 - c_5) + \frac{q(x_0)}{2h^3}(c_{-2} + 2c_{-1} - 2c_1 + c_2) + \frac{v(x_0)}{6h^2}(c_{-2} + 2c_{-1} - 6c_0 + 2c_1 + c_2) + \frac{w(x_0)}{24h}(-c_{-2} - 10c_{-1} + 10c_1 + c_2) + \frac{Y_m(x_0)}{120}(c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2) = Z_m(x_0), \tag{32}$$

$$\frac{p(x_1)}{12h^4}(c_{-2} + 6c_{-1} - 33c_0 + 52c_1 - 33c_2 + 6c_3 + c_4) + \frac{q(x_1)}{2h^3}(-c_{-1} + 2c_0 - 2c_2 + c_3) + \frac{v(x_1)}{6h^2}(c_{-1} + 2c_0 - 6c_1 + 2c_2 + c_3) + \frac{w(x_1)}{24h}(-c_{-1} - 10c_0 + 10c_2 + c_3) + \frac{Y_m(x_1)}{120}(c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2) = Z_m(x_1), \tag{33}$$

$$\frac{p(x_{n-1})}{12h^4}(c_{n-4} + 6c_{n-3} - 33c_{n-2} + 52c_{n-1} - 33c_n + 6c_{n+1} + c_{n+2}) + \frac{q(x_{n-1})}{2h^3}(-c_{n-3} + 2c_{n-2} - 2c_n + c_{n+1}) + \frac{v(x_{n-1})}{6h^2}(c_{n-3} + 2c_{n-2} - 6c_{n-1} + 2c_n + c_{n+1}) + \frac{w(x_{n-1})}{24h}(-c_{n-3} - 10c_{n-2} + 10c_n + c_{n+1}) + \frac{Y_m(x_{n-1})}{120}(c_{n-3} + 26c_{n-2} + 66c_{n-1} + 26c_n + c_{n+1}) = Z_m(x_{n-1}), \tag{34}$$

$$\frac{p(x_n)}{12h^4}(c_{n-5} + 8c_{n-4} - 27c_{n-3} + 52c_{n-2} - 103c_{n-1} + 108c_n - 61c_{n+1} - 14c_{n+2}) + \frac{q(x_n)}{2h^3}(-c_{n-2} + 2c_{n-1} - 2c_{n+1} + c_{n+2}) + \frac{v(x_n)}{6h^2}(c_{n-2} + 2c_{n-1} - 6c_n + 2c_{n+1} + c_{n+2}) + \frac{w(x_n)}{24h}(-c_{n-2} - 10c_{n-1} + 10c_{n+1} + c_{n+2}) + \frac{Y_m(x_n)}{120}(c_{n-2} + 26c_{n-1} + 66c_n + 26c_{n+1} + c_{n+2}) = Z_m(x_n). \tag{35}$$

The set of boundary conditions (Eq. 28) as well give the following four equations:

$$c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2 = 120\beta_1, \tag{36}$$

$$-c_{-2} - 10c_{-1} + 10c_1 + c_2 = 24h\beta_2, \tag{37}$$

$$c_{n-2} + 26c_{n-1} + 66c_n + 26c_{n+1} + c_{n+2} = 120\beta_3, \tag{38}$$

$$-c_{n-2} - 10c_{n-1} + 10c_n + c_{n+2} = 24h\beta_4. \tag{39}$$

The system of Eqs. 31-39, with unknowns c_i's, $i = -2, -1, 0, \dots, n+2$ can be expressed in matrix notation as:

$$Ac - b = 0, \tag{40}$$

where, A represents the coefficient matrix of order $n+5$, b is a column vector with $n+5$ entities and $c = [c_{-2}c_{-1}c_0 \dots c_{n+2}]^T$. We start from $m = 0$ with an initial guess $U_0(x)$ and solve for c using a modified form of well-known Thomas algorithm. The values of c_i's are plugged into Eq. 29 to obtain $U_1(x)$. This process is continued for $m = 1, 2, 3, \dots$ until we get the desired accuracy.

4. Error analysis

Using the QnBS approximations, we can establish the following relations (Lodhi and Mishra, 2016; Xu and Lang, 2014):

$$h[U'(x_{i-2}) + 26U'(x_{i-1}) + 66U'(x_i) + 26U'(x_{i+1}) + U'(x_{i+2})] = 5[-U(x_{i-2}) - 10U(x_{i-1}) + 10U(x_{i+1}) + U(x_{i+2})], \tag{41}$$

$$h^2[U''(x_{i-2}) + 26U''(x_{i-1}) + 66U''(x_i) + 26U''(x_{i+1}) + U''(x_{i+2})] = 20[U(x_{i-2}) + 2U(x_{i-1}) - 6U(x_i) + 2U(x_{i+1}) + U(x_{i+2})], \tag{42}$$

$$h^3[U'''(x_{i-2}) + 26U'''(x_{i-1}) + 66U'''(x_i) + 26U'''(x_{i+1}) + U'''(x_{i+2})] = 60[-U(x_{i-2}) + 2U(x_{i-1}) - 2U(x_{i+1}) + U(x_{i+2})]. \tag{43}$$

Similarly, using Eqs. 7, 8, and 19, we have:

$$h^4U^{(4)}(x_i) = \frac{h^2}{40}[-U''(x_{i-2}) + 114U''(x_{i-1}) - 142U''(x_i) + 30U''(x_{i+1}) - U''(x_{i+2})] + \frac{7h^3}{10}[U'''(x_{i-1}) + 2U'''(x_i)]. \tag{44}$$

Employing the operator notation, $E^\lambda(U'(x_i)) = U'(x_{i+\lambda})$, $\lambda \in \mathbb{Z}$, Eqs. 41-43 are written as (Fyfe, 1969):

$$h[E^{-2} + 26E^{-1} + 66 + 26E^1 + E^2]U'(x_i) = 5[-E^{-2} - 10E^{-1} + 10E^1 + E^2]u(x_i), \tag{45}$$

$$h^2[E^{-2} + 26E^{-1} + 66 + 26E^1 + E^2]U''(x_i) = 20[E^{-2} - 2E^{-1} - 6 + 10E^1 + E^2]u(x_i), \tag{46}$$

$$h^3[E^{-2} + 26E^{-1} + 66 + 26E^1 + E^2]U'''(x_i) = 60[-E^{-2} + 2E^{-1} - 2E^1 + E^2]u(x_i). \tag{47}$$

Using $E = e^{hD}$, $D = \frac{d}{dx}$, we have the following expressions, respectively (Lodhi and Mishra, 2016; Xu and Lang, 2014):

$$U'(x_i) = u'(x_i) + \frac{h^6}{5040}u^{(7)}(x_i) - \frac{h^8}{21600}u^{(9)}(x_i) + \frac{h^{10}}{1036800}u^{(11)}(x_i) + \cdots, \tag{48}$$

$$U''(x_i) = u''(x_i) + \frac{h^4}{720}u^{(6)}(x_i) - \frac{h^6}{3360}u^{(8)}(x_i) + \frac{h^8}{86400}u^{(10)}(x_i) + \cdots, \tag{49}$$

$U'''(x_i) = u'''(x_i) - \frac{h^2}{240}u^{(7)}(x_i) - \frac{11h^6}{30240}u^{(9)}(x_i) - \frac{h^8}{28800}u^{(11)}(x_i) + \cdots.$ (50)

Similarly, writing Eq. 44 in operator notation, we get:

$h^4U^{(4)}(x_i) = \frac{h^4}{40}[-E^{-2} + 11E^{-1} - 142 + 30E^1 - 30E^2]u''(x_i) + \frac{7h^3}{10}[E^{-1} + 2]u'''(x_i).$ (51)

Again, using $E = e^{hD}$ in Eq. 51, we obtain:

$h^4U^{(4)}(x_i) = \frac{h^2}{40}[-e^{-2hD} + 114e^{-hD} - 142 + 30e^{hD} - e^{2hD}]U''(x_i) + \frac{7h^3}{10}[e^{-hD} + 2]U'''(x_i).$ (52)

Expanding in powers of hD, we get:

$h^4U^{(4)}(x_i) = \frac{h^2}{40}\left[-84hD - 68h^2D^2 - 14h^3D^3 \mp \frac{14}{3}h^4D^4 + \cdots\right]U''(x_i) + \frac{7h^3}{10}\left[-3 - hD + \frac{1}{2}h^2D^2 - \frac{1}{6}h^3D^3 + \frac{1}{24}h^4D^4\right]U'''(x_i).$ (53)

Simplifying the above relation, we have:

$U^{(4)}(x_i) = u^{(4)}(x_i) + \frac{7h^3}{600}u^{(7)}(x_i) - \frac{19h^4}{3600}u^{(8)}(x_i) + \cdots.$ (54)

We define the error term at i^{th} knot as $e(x_i) = U(x_i) - u(x_i)$. Using the Eqs. 48-50 and 54 in Taylor series expansion of error term, we get:

$e(x_i + \theta h) = \frac{\theta^2}{1440}h^6u^{(6)}(x_i) + \frac{\theta(20+7\theta^2(-10+7\theta))}{100800}h^7u^{(7)}(x_i) + \cdots,$ (55)

where, $0 \le \theta \le 1$. From Eq. 55, it is clear that the truncation error in new QnBS approximation is $O(h^6)$.

5. Numerical results and discussion

In this segment, the experimental outcomes of the new quintic B-spline approximation method are presented. The accuracy of the presented numerical scheme is verified by L_∞, as (Abbas et al., 2014):

$L_\infty = \|U_i - u_i\|_\infty = \max_i |U_i - u_i|,$

where, U_i and u_i represent the numerical and true solutions at the i^{th} nodal point, respectively.

Problem 1: Consider the fourth-order Emden-Flower type equation (Wazwaz, 2015):

$u^{(4)}(x) + \frac{2}{x}u^{(3)}(x) = -\frac{3(3-2x^2)}{u^7(x)},\ 0 \le x \le 1,$
$u(0) = 1,\ u'(0) = 0,\quad u''(0) = 1,\quad u'''(0) = 0.$

The analytical solution is $\sqrt{1 + x^2}$. Table 1 displays a comparison of computational outcomes with VIM (Wazwaz, 2015). It is observed that our approximate results are better than VIM as $x \to 1$. The numerical error norm corresponding to four different choices of step size is presented in Fig. 1.

Problem 2: Consider the fourth-order Emden-Flower type equation (Wazwaz et al., 2015):

$u^{(4)}(x) + \frac{3}{x}u^{(3)}(x) = 96(1 - 10x^4 + 5x^8)e^{-4u(x)},\ 0 \le x \le 1,$
$u(0) = 0,\ u'(0) = 0,\quad u''(0) = 0,\quad u'''(0) = 0.$

The exact solution is $u(x) = \log(1 + x^4)$. The approximate results are listed in Table 2. It is obvious that the obtained results are well balanced as compared to the Adomian decomposition method (ADM) (Wazwaz et al., 2015) and QnBSM used in Lodhi and Mishra (2016). The absolute computational error for $n = 10, 20, 40, 80$ is displayed in Fig. 2. The computational error decreases as the mesh size is decreased, which confirms the convergence of the proposed numerical technique.

Problem 3: Consider the fourth-order singular boundary value problem:

$u^{(4)}(x) + \frac{1}{x}u^{(3)}(x) + u''(x) + u'(x) + u(x) = f(x),\ 0 \le x \le 1,$
$u(0) = 0,\ u'(0) = 0,\quad u(1) = \sin 10,\quad u'(1) = 10\cos 10 + \sin 10.$

where, $f(x) = 10(-498 + x)\cos 10x + (-300 + x + 9901x^2)\frac{\sin 10x}{x}$.

The exact solution is $u(x) = x\sin 10x$. The approximate results are listed in Table 3. It is observed that the obtained approximate results are better than the quintic B-spline collocation method (QnBSM) used in Lodhi and Mishra (2016). Fig. 3 portrays a close agreement between the approximate and exact solution when h = 1/20.

Table 1: Approximate results for problem 1 when h=1/20

x	VIM (Wazwaz, 2015)	Proposed method	Exact solution
0.0	1	1.0000000000	1
0.1	1.0049875621	1.0049875593	1.0049875621
0.2	1.0198039027	1.0198038892	1.0198039027
0.3	1.0440306502	1.0440306154	1.0440306509
0.4	1.0770329231	1.0770328880	1.0770329614
0.5	1.1180331707	1.1180338570	1.1180339887
0.6	1.1661805962	1.1661901642	1.1661903790
0.7	1.2205772258	1.2206552361	1.2206555616
0.8	1.2801571267	1.2806243813	1.2806248475
0.9	1.3431296012	1.3453617662	1.3453624047
1.0	1.4052734375	1.4142127186	1.4142135624
L_∞	8.94×10^{-3}	8.44×10^{-7}	⋯

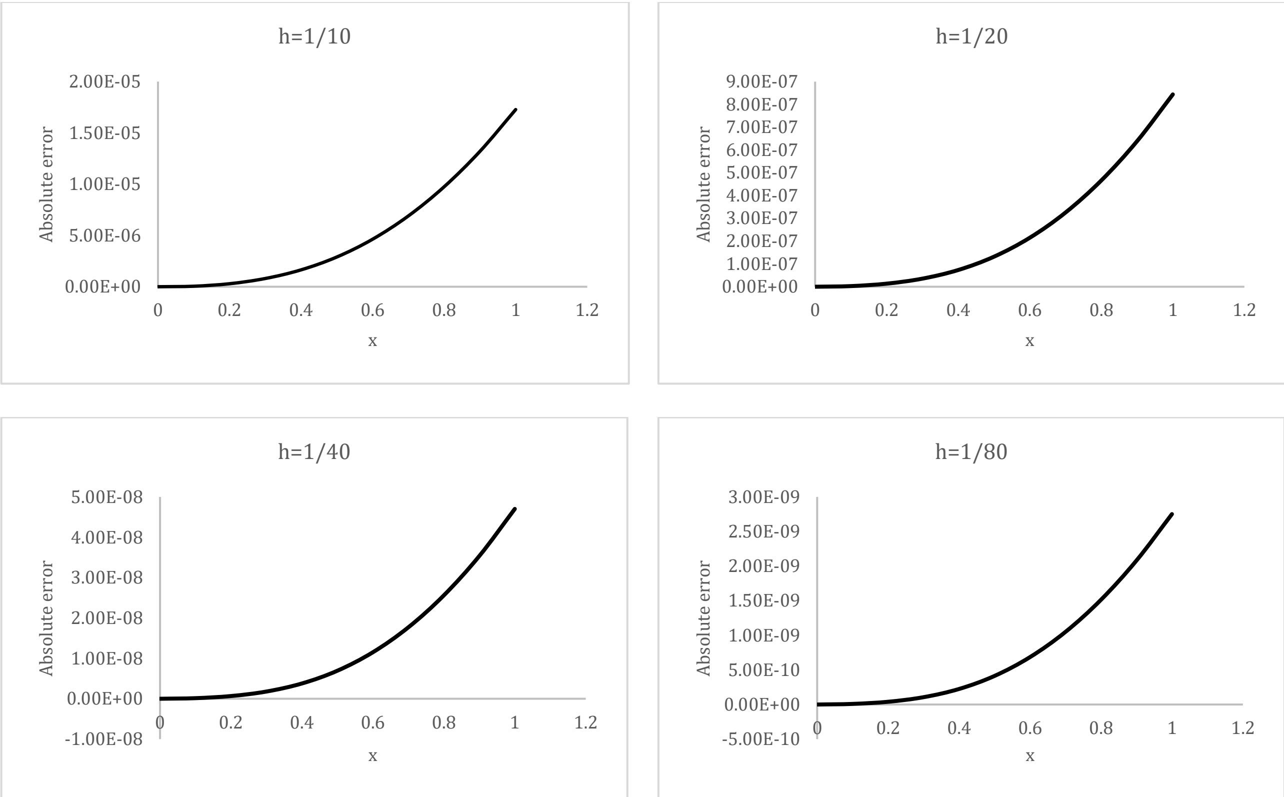

Fig. 1: Absolute error for problem 1

Table 2: Numerical results for problem 2 when h=1/20

x	ADM (Wazwaz et al., 2015)	Proposed method	Exact solution
0.0	0	0.0000000000	0
0.1	0.0000999950	0.0000999841	0.0000999950
0.2	0.0015987214	0.0015986513	0.0015987214
0.3	0.0080673721	0.0080671401	0.0080673711
0.4	0.0252779124	0.0252772347	0.0252778072
0.5	0.0606282552	0.0606234649	0.0606246218
0.6	0.1219275141	0.1218616201	0.1218635878
0.7	0.2158897574	0.2151891479	0.2151920215
0.8	0.3486204122	0.3433022850	0.3433059762
0.9	0.5350095738	0.5044611160	0.5044654406
1.0	0.8333333333	0.6931423256	0.6931471806
L_∞	1.40×10^{-1}	4.85×10^{-6}	...

Problem 4: Consider the following fourth-order singularly perturbed SBVP (Lodhi and Mishra, 2016).

$$\epsilon u^{(4)}(x) + \frac{1}{x}u(x) = e^x[1 - x - \epsilon(8 + 7x + x^2) - \frac{2}{3}e(1 - x^2), \; 0 \le x \le 1,$$
$$u(0) = 0, \; u''(0) = 0, \quad u(1) = 0, \quad u''(1) = 0.$$

The exact solution is $u(x) = xe^x(1 - x) - \frac{2}{3}e(1 - x^2)$. The computational outcomes are listed in Table 4, when $\text{h} = \frac{1}{20}$ and $\epsilon = \frac{1}{16}$. t is obvious that the approximate results are in good agreement with the exact solution.

Table 5 portrays a comparison of absolute numerical error with QnBSM (Lodhi and Mishra, 2016) using different values of $\in$ and h. It is revealed that our approximate results are superior to those obtained by QnBSM. In Fig. 4, the analytical and numerical solutions are exhibited when $h = \epsilon = \frac{1}{6}$. Fig. 5 displays the absolute computational error corresponding to four different step sizes with $\epsilon = \frac{1}{6}$.

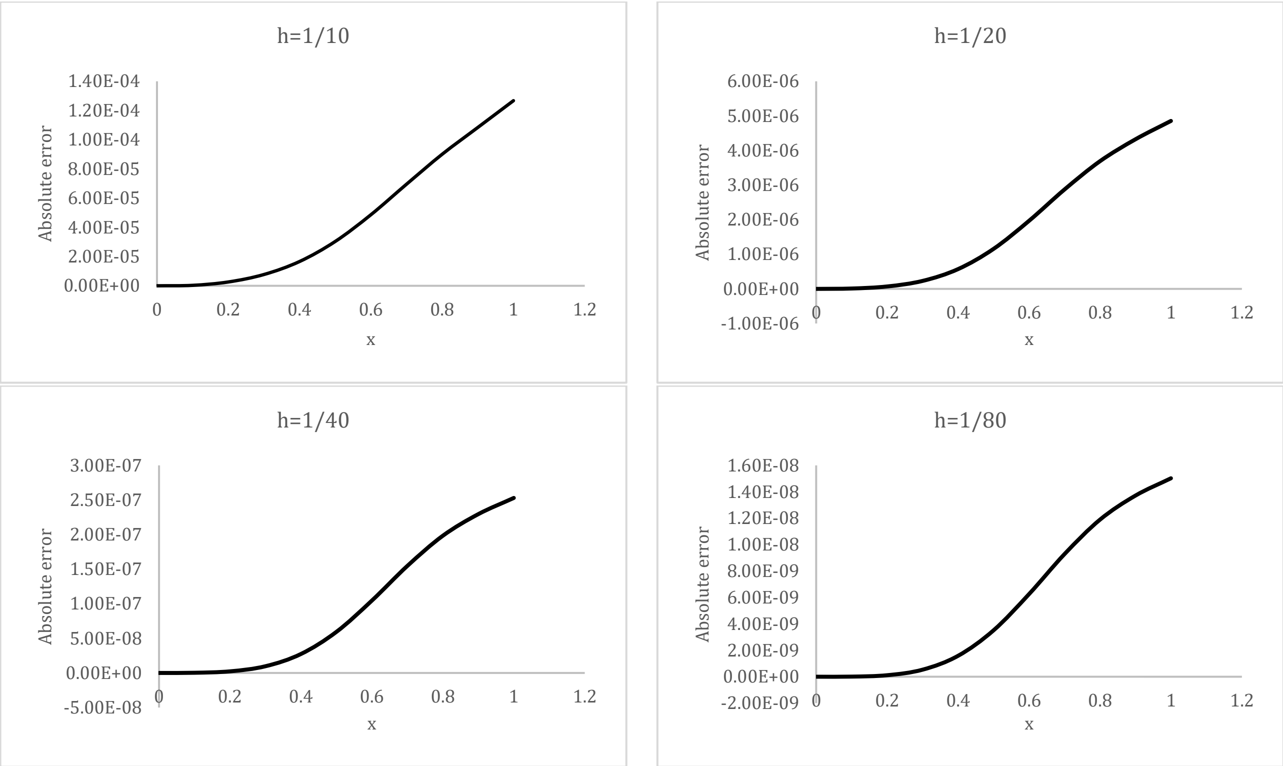

Fig. 2: Absolute error for problem 2

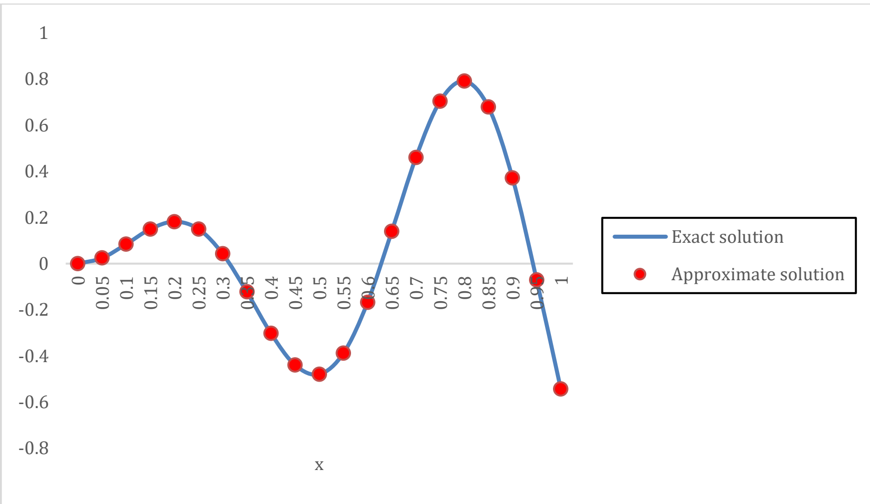

Fig. 3: Exact and approximate solutions for problem 3 using h=1/20

Table 3: Absolute numerical error for problem 3

n	8	16	32	64	128
Proposed method	1.38E-02	1.14E-03	1.01E-04	8.92E-06	8.54E-07
QnBSM used in Lodhi and Mishra (2016)	3.06E-01	6.79E-02	1.63E-02	4.02E-03	1.00E-03

Table 4: Numerical results for problem 4, when $h = \frac{1}{20}$ and $\epsilon = \frac{1}{16}$

x	Exact solution	Proposed method	Absolute error
0.0	0	0.0000000000	2.95×10^{-16}
0.1	-0.0799412181	-0.0799412319	1.38×10^{-8}
0.2	-0.1525156327	-0.1525156584	2.57×10^{-8}
0.3	-0.2112569432	-0.2112569787	3.55×10^{-8}
0.4	-0.2508572021	-0.2508572451	4.29×10^{-8}
0.5	-0.2673901394	-0.2673901869	4.75×10^{-8}
0.6	-0.2585716360	-0.2585716846	4.86×10^{-8}
0..7	-0.2240630066	-0.2240630520	4.53×10^{-8}
0.8	-0.1658235625	-0.1658235994	3.96×10^{-8}
0.9	-0.0885198484	-0.0885198706	2.21×10^{-8}
1.0	0	0.0000000000	3.25×10^{-8}

Table 5: Computational error norm for problem 4 with $\epsilon = 10^{-k}$

n	k	0	2	4	6	8
8	Proposed Method	2.60E-06	8.85E-07	1.01E-07	3.05E-09	3.19E-11
	QnBSM (Lodhi and Mishra, 2016)	8.51E-04	2.62E-04	7.52E-06	1.30E-07	1.35E-09
16	Proposed Method	1.60E-07	5.51E-08	6.09E-09	5.20E-10	8.14E-12
	QnBSM (Lodhi and Mishra, 2016)	2.12E-04	6.62E-05	1.86E-06	2.87E-08	3.82E-10
32	Proposed Method	9.96E-09	4.43E-09	3.83E-10	4.07E-11	1.86E-12
	QnBSM (Lodhi and Mishra, 2016)	5.30E-05	1.65E-05	4.65E-07	6.92E-09	9.40E-11
64	Proposed Method	6.21E-10	2.13E-10	2.41E-11	2.53E-12	242E-13
	QnBSM (Lodhi and Mishra, 2016)	1.33E-05	4.13E-06	1.16E-07	1.72E-09	2.04E-11
128	Proposed Method	2.50E-11	1.68E-11	1.87E-12	1.59E-13	1.52E-14
	QnBSM (Lodhi and Mishra, 2016)	3.31E-06	1.03E-06	2.90E.08	4.30E-10	4.98E-12

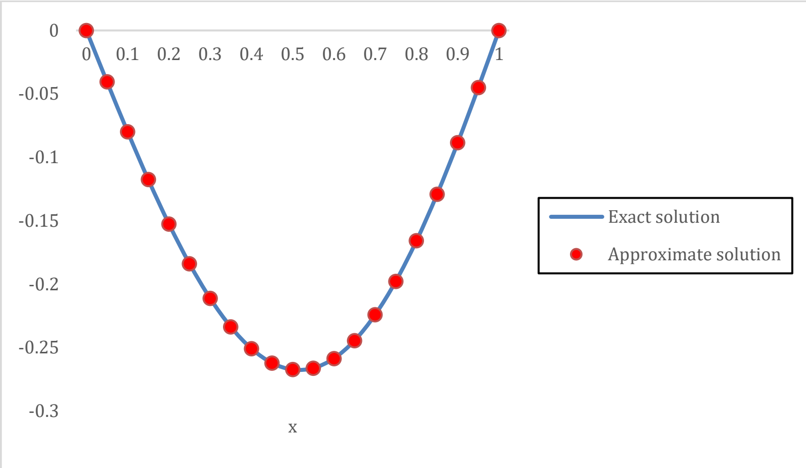

Fig. 4: Exact and approximate solution of problem 4 using h=ϵ=1/16

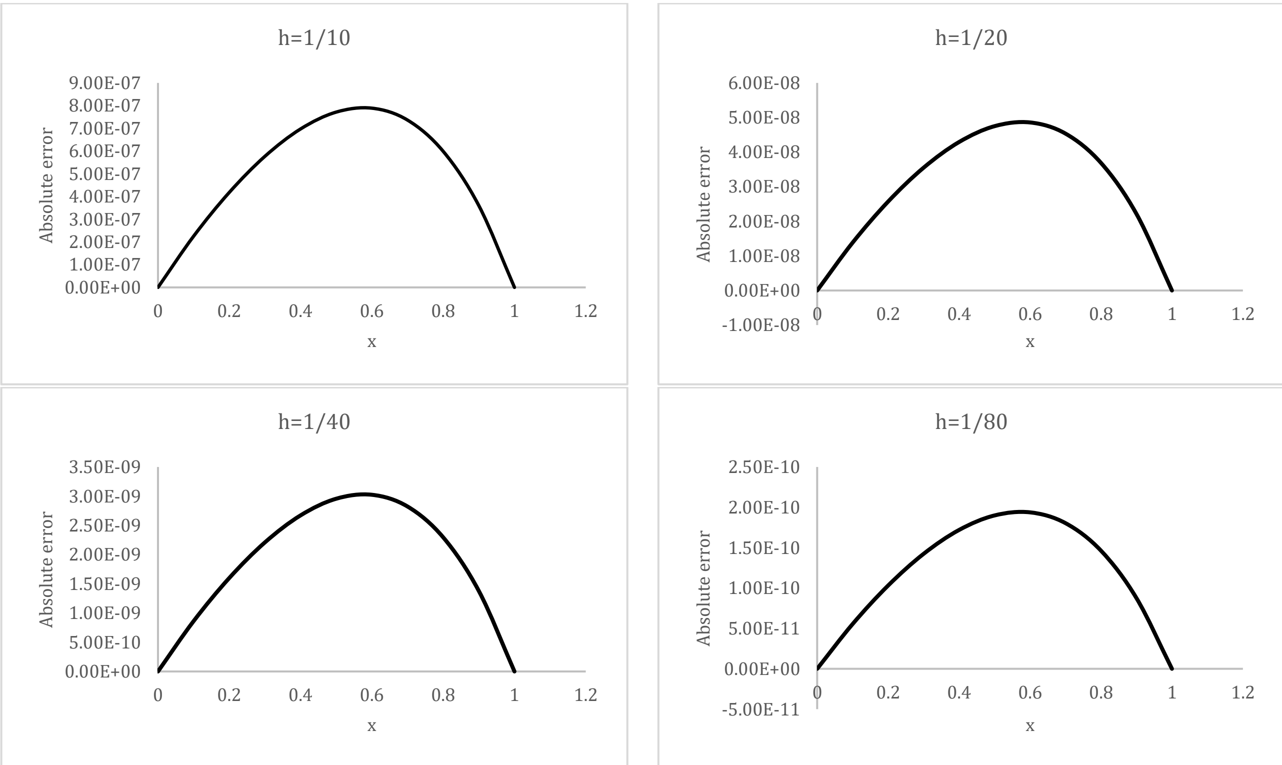

Fig. 5: Absolute error from problem 4 when ϵ=1/16

6. Conclusion

In this paper, a new quintic B-spline approximation technique is developed for solving fourth-order singular boundary value problems. We conclude this work as:

1. The proposed numerical approach is based on a new quintic B-spline approximation for the fourth-order derivative.
2. The presented technique is novel for fourth-order singular boundary value problems.
3. The scheme is uniformly convergent in the entire domain.

4. As the step size is decreased, the approximate solution approaches the exact analytical solution, which ensures the convergence of the proposed algorithm.

By virtue of simple implementation, it produces more accurate outcomes as compared to VIM (Wazwaz, 2015), ADM (Wazwaz et al., 2015), and QnBSM (Lodhi and Mishra, 2016).

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

References

Abbas M, Majid AA, Ismail AIM, and Rashid A (2014). Numerical method using cubic B-spline for a strongly coupled reaction-diffusion system. PloS One, 9(1): e83265. https://doi.org/10.1371/journal.pone.0083265 **PMid:24427270 PMCid:PMC3888394**

Abukhaled M, Khuri SA, and Sayfy A (2011). A numerical approach for solving a class of singular boundary value problems arising in physiology. International Journal of Numerical Analysis and Modeling, 8(2): 353-363.

Akram G (2011). Quartic spline solution of a third order singularly perturbed boundary value problem. Anziam Journal, 53: 44-58. https://doi.org/10.21914/anziamj.v53i0.4526

Akram G and Amin N (2012). Solution of a fourth order singularly perturbed boundary value problem using quintic spline. International Mathematical Forum, 7(44): 2179-2190. https://doi.org/10.21914/anziamj.v53i0.4526

Aruna K and Kanth AR (2013). A novel approach for a class of higher order nonlinear singular boundary value problems. International Journal of Pure and Applied Mathematics, 84(4): 321-329. https://doi.org/10.12732/ijpam.v84i4.2

Caglar H, Caglar N, and Ozer M (2009). B-spline solution of non-linear singular boundary value problems arising in physiology. Chaos, Solitons and Fractals, 39(3): 1232-1237. https://doi.org/10.1016/j.chaos.2007.06.007

Fyfe DJ (1969). The use of cubic splines in the solution of two-point boundary value problems. The Computer Journal, 12(2): 188-192. https://doi.org/10.1093/comjnl/12.2.188

Goh J, Majid AA, and Ismail AIM (2011). Extended cubic uniform B-spline for a class of singular boundary value problems. Science Asia, 37: 79-82. https://doi.org/10.2306/scienceasia1513-1874.2011.37.079

Goh J, Majid AA, and Ismail AIM (2012). A quartic B-spline for second-order singular boundary value problems. Computers and Mathematics with Applications, 64(2): 115-120.

Iqbal MK, Abbas M, and Wasim I (2018). New cubic B-spline approximation for solving third order Emden–Flower type equations. Applied Mathematics and Computation, 331: 319-333. https://doi.org/10.1016/j.amc.2018.03.025

Khuri SA (2001). An alternative solution algorithm for the nonlinear generalized Emden-Fowler equation. International Journal of Nonlinear Sciences and Numerical Simulation, 2(3): 299-302. https://doi.org/10.1515/IJNSNS.2001.2.3.299

Khuri SA and Sayfy A (2014). Numerical solution for the nonlinear Emden–Fowler type equations by a fourth-order adaptive method. International Journal of Computational Methods, 11(01): 1350052. https://doi.org/10.1142/S0219876213500527

Kim W and Chun C (2010). A modified Adomian decomposition method for solving higher-order singular boundary value problems. Zeitschrift für Naturforschung A, 65(12): 1093-1100. https://doi.org/10.1515/zna-2010-1213

Lodhi RK and Mishra HK (2016). Solution of a class of fourth order singular singularly perturbed boundary value problems by quintic B-spline method. Journal of the Nigerian Mathematical Society, 35(1): 257-265. https://doi.org/10.1016/j.jnnms.2016.03.002

Parand K and Delkhosh M (2017). An effective numerical method for solving the nonlinear singular Lane-Emden type equations of various orders. Jurnal Teknologi, 79(1): 25-36. https://doi.org/10.11113/jt.v79.8737

Taiwo OA and Hassan MO (2015). Approximation of higher-order singular initial and boundary value problems by iterative decomposition and Bernstein polynomial methods. Journal of Advances in Mathematics and Computer Science, 9(6): 498-515. https://doi.org/10.9734/BJMCS/2015/17157

Wazwaz AM (2015). The variational iteration method for solving new fourth-order Emden–Fowler type equations. Chemical Engineering Communications, 202(11): 1425-1437. https://doi.org/10.1080/00986445.2014.952814

Wazwaz AM, Rach R, and Duan JS (2015). Solving new fourth–order emden–fowler-type equations by the adomian decomposition method. International Journal for Computational Methods in Engineering Science and Mechanics, 16(2): 121-131. https://doi.org/10.1080/15502287.2015.1009582

Xu XP and Lang FG (2014). Quintic B-spline method for function reconstruction from integral values of successive subintervals. Numerical Algorithms, 66(2): 223-240. https://doi.org/10.1007/s11075-013-9731-x