Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

Numerical validation of some wave height distributions

M.A. Sharaki
Faculty of Engineering, Alexandria-EGYPT

Abstract

This paper studies, numerically, the validity of different wave height distributions and the minimum number of components needed to validate the distribution under study. Also, the effect of bandwidth of the wave spectrum is considered. Rayleigh distribution, the modified Rayleigh distribution and the Beta- Rayleigh distribution are considered in this study

Introduction

It was proved analytically that, if the sea surface elevation can be represented by a sum of an infinite number of harmonic components uniformly distributed about a central frequency and having the same amplitude., then the wave height can be represented by Rayleigh distribution, assuming that the bandwidth of the wave energy spectrum is narrow. It is not clear what is meant by narrow band spectrum. More clearly, what is the limit of the bandwidth that allows Rayleigh distribution to represent the wave height distribution accurately.

In actual situations, we restore to numerical methods, where the energy spectrum is divided into finite number of components. In this paper we try to find the minimum number of components, if any, that makes Rayleigh distribution a good representative of the wave heights.

We numerically synthesize a collection of wave trains from a uniform spectrum using different numbers of components and bandwidths and compare the different statistical quantities of the synthesized data with Rayleigh

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)

distribution, which is a one-parameter model. We also introduce two proposed distributions, namely the Beta-Rayleigh distribution, and the modified Rayleigh distribution. The former was introduced for the first time by Hughes and Borgman (1983) [1], for the distribution of the shallow water wave heights. It is a three-parameter distribution. The latter is a two-parameter distribution .The statistical quantities used to validate any distribution are H_1, H_3 , H_{10} and H_{100} , which are the average of the whole train, the highest one third, one tenth and one hundredth respectively.

The importance of this study comes from the fact that for engineering design purposes it is useful to have a statistical description of the heights of the sea waves so that a probability can be assigned to a particular water level. This will help in the design of shore protection projects and find the likelihood of a project survival.

Statistical model

We will use three different Probability Density Functions, PDFs, as candidates to represent wave height distributions in the numerical analysis. Those PDFs are
1-Rayleigh distribution, which depends on the root mean square H_{rms}, defined by

$$H_{rms} = \left[\frac{1}{N} \sum_{n=1}^{N} H_n^2 \right]^{0.5} \tag{1}$$

where N is the number of waves in the record. In this study we will adopt the zero up crossing definition for the wave height. The Rayleigh PDF is given by

$$P_R(H) = \frac{2H}{H_{rms}^2} \exp -\left(\frac{H}{H_{rms}}\right)^2 \tag{2}$$

To normalize H with respect to H_{rms}., we introduce the dimensionless quantity H^*, given by

$$H = H^* H_{rms} \tag{3}$$

Equation (2) has the dimension (1/L), where L is the length, then we can write

$$P_R(H^*) = 2 H^* \exp - H^* \tag{4}$$

which is a dimensionless quantity.

As can be seen from equations (2) and (4), Rayleigh PDF has the disadvantage that, there is no upper bound for the wave height.
2-The Beta-Rayleigh, which takes the breaking effect into account. It has the advantage that there is an upper limit for the wave height. It takes the form

$$P_{BR}(H) = \frac{2\,\Gamma(\alpha + \beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \frac{H^{2\alpha-1}}{H_m^{2\alpha}} \left[1 - \frac{H^2}{H_m^2} \right]^{\beta-1} \tag{5}$$

where

$$\alpha = K_1 (K_2 - K_1) / (K_1^2 - K_2) \tag{6}$$

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

$$\beta = (1-K_1)(K_2-K_1)/(K_1^2-K_2) \tag{7}$$

$$K_1 = H_{rms}^2 / H_m^2 \tag{8}$$

$$K_2 = H_{rmq}^2 / H_m^4 \tag{9}$$

$$H_{rmq} = \left[\frac{1}{N} \sum_{i=1}^{N} H_i^4 \right]^{0.5} \tag{10}$$

In the original work H_m is defined as the maximum height for the breaking wave. In this study we define it as the maximum wave height. Two alternatives are introduced to represent H_m. Firstly, we will take it as the maximum wave height in the record. Secondly, we will take it as the maximum wave height as predicted by Rayleigh distribution that is given by H_{1000}, according to Chakrabarti [2].
A dimensionless form of eqn (5) takes the form

$$P_{BR}(H^*) = \frac{2\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{H^{*2\alpha-1}}{H_m^{*2\alpha}} \left[1 - \left(\frac{H^*}{H_m^*} \right)^2 \right]^{\beta-1} \tag{11}$$

where

$$H_m = H_m^* H_{rms} \tag{12}$$

The modified Beta-Rayleigh distribution will be tested once with H_m given by H_{1000} and once by the maximum wave height in the record. This gives 2 alternatives.
3-The modified Rayleigh distribution given by, in dimensionless form

$$P_{mr}(H^*) = \frac{2H^{*2\alpha-1}}{b_o^{\alpha ao}\Gamma(\alpha_0)} \exp\left(\frac{-H^{*2}}{b_o} \right)^2 \tag{13}$$

where

$$\alpha_o = \frac{1}{H_{rms}^{*2} - 1} \tag{14}$$

$$b_o = H_{rmq}^{*2} - 1 \tag{15}$$

Equation (13) is nothing but eqn (11) when H_m tends to infinity.

Data generation

A computer program is written in MATLAB to generate the required wave train and carry out the statistical analysis and comparison processes. To check the validity of the program two methods are utilized. Firstly, a wave train with just two components is generated and its distribution is compared with the theoretical distribution, given by Longuet–Higgins (1952) [3]

$$P(H^*) = \begin{cases} \dfrac{2}{\pi\,(2-H^*)^{0.5}} & H^* \le \sqrt{2} \\ 0 & H^* > \sqrt{2} \end{cases} \tag{16}$$

Secondly, H_{rms} as given by eqn (1), is checked versus its value as calculated from the wave spectrum by

$$H_{rms} = 2\sqrt{m_o} \tag{17}$$

where m_i is the i th spectrum moment, given by

$$m_i = \int_0^\infty f^i\, S(f)\, df \tag{18}$$

where f is the cyclic frequency, cycles per second,

To measure the bandwidth of the wave spectrum used to generate the wave train, two methods are in use, that due to Vanmarcke [2]

$$\varsigma_V = \left(1 - \frac{m_1^2}{m_o\ m_2}\right)^{0.5} \tag{19}$$

and that due to Cart wright and Longuet–Higgins [2]

$$\varsigma_L = \left(1 - \frac{m_2^2}{m_o\ m_4}\right)^{0.5} \tag{20}$$

The wave train is generated using the formula

$$\eta_k = \sum_{i=1}^{n} C_i \cos\left(2\pi f_i\, k\,\Delta t + \phi_i\right) \tag{21}$$

where

$C_i = (2\,\Delta f\, S_i)^{0.5}$	is the amplitude of the ith component with cyclic frequency f_I
$\phi_i = 2\pi$ U[0,1]	is a phase angle uniformly distributed in the interval [0, 2 ™]
η_k	is the surface elevation at time $t = k\,\Delta t$
S_i	is the amplitude of the wave spectrum at frequency f_i
$\Delta t = T/20$	is the time step, and
T	is the period of the shortest wave in the record

Δt is selected equal T/20 so that the maximum error in calculating the crest or trough height is not more than 1.23% = [cos (0) – cos (9)] 100 %.

The simulation time is selected so that the longest wave component in the train appears 200 times. This allows a long number of wave heights to be obtained for statistical analysis.

We used a white spectrum centered at f_c = 0.125 Hz (T = 8 sec.) and different values for $\Delta f = f_k - f_1$, as shown in Table (1). Where f_1 and f_k are the smallest and the largest frequencies of the train,

To compensate the effect of the random phase angle ϕ, each rum , given f_c, Δf, and the number of components is repeated 16 times and the analysis is carried out for the total sum of them. As shown in table(1), the period of the longest wave is 10 sec, so the maximum simulation time is about 535 min.

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

Table 1 Properties of the wave trains used in the analysis

Cf	$\Delta f/f_c$	fi	f_k	T_{max}	T_{min}	4_L	#0fComp
0.003125	0.025	0.1234	0.1265	8.1	7.901	0.0014	8,16,32,
0.00625	0.05	0.1218	0.1281	8.205	7.805	0.0054	64,128,
0.0125	0.1	0.1187	0.1313	8.421	7.619	0.0212	256,512,
0.025	0.2	0.1125	0.1375	8.889	7.273	0.0775	1024
0.05	0.4	0.1	0.15	10.00	6.667	0.2312	

Numerical test result

Equation (16) is used to validate the program. It is shown that there is good agreement between the theoretical distribution and the one obtained from the synthesized wave train with two components. This is true when the difference between the frequency of the two components is small. Also, comparing the value of H_{rms} as calculated by equns (1)and (17) shows a very close agreement, especially when both the simulated time is long and the number of components is large.

Figures (1) and (2) show bar graphs for the wave heights distribution versus Rayleigh distribution. The number of components used in the analysis is 8 and the bandwidth equals 0.0013626 as calculated using equn (20). The figures show that there is a pronounced deviation from Rayleigh distribution, in spite of the limited bandwidth. This may be attributed in part to the limited number of components used in the analysis and in part to effect of the random phase angle.

Figure (3) shows the statistics of a wave train with 1024 components and bandwidth 0.23123 versus Rayleigh distribution Figure (3) shows less deviation in spite of the larger bandwidth used. This reflects the importance of the number of components.

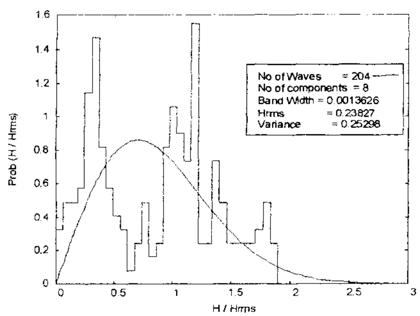

Figure 1 :Wave height distribution versus Raleigh distribution

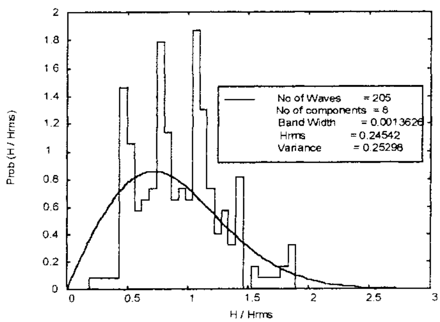

Figure 2 :Wave height distribution versus Rayleigh distribution

Figure(4) shows a typical distribution of the wave heights versus the different theoretical distributions used in this study. Here, R stands for Rayleigh distribution, BRI for Beta-Rayleigh distribution using the H_{max} from the record, BRII for Beta-Rayleigh distribution using H_{max} as calculated by Rayleigh distribution, and finally, MR stands for modified Rayleigh distribution.

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

A total of 40 cases are studied. The primary variables are the bandwidth and the number of components. All wave heights are normalized with respect to H_{rms}. The comparison of the different distributions are based on the following equation

$$\% \, difference = \frac{H_i^D - H_i^M}{H_i^D} \tag{22}$$

where, H^D and H^M are the generated and distribution wave heights respectively

The results of the comparisons are shown in figures (5) through (21). The ordinate shows the outcome of equn (22), while the abscissa shows the log, for base2, of the number of components.

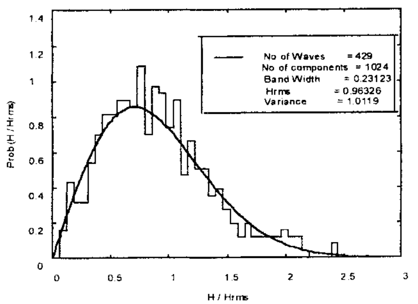

Figure 3 :Wave height distribution and Rayleigh distribution

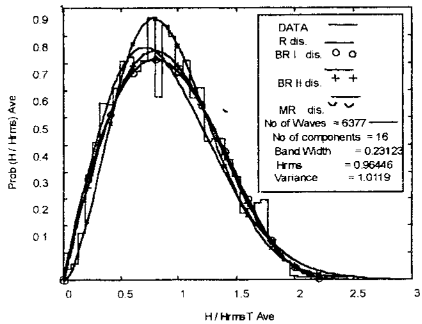

Figure 4 :Wave height distribution and different theoretical distributions

Figure (5) through figure (8) show the difference between H_{mean} of the wave train and the distributions under study. The deviation is less than 5% for all bandwidths and number of components. It is less than 1.5% for the modified Rayleigh distribution. For wide band spectrum, the modified Rayleigh distribution represents the wave data better than the other three distributions and the Rayleigh distribution gives the second best results.

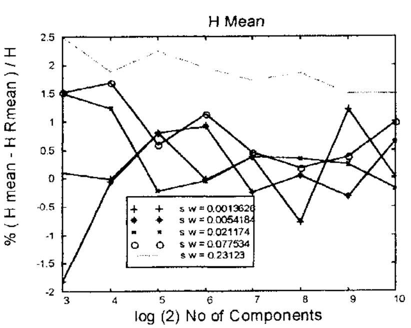

Figure 5:Wave data and Rayleigh distribution(R)

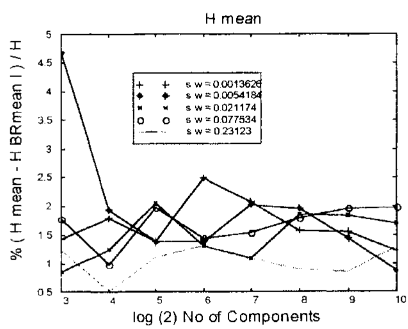

Figure 6: Wave data and Beta - Rayleigh distribution (BRI)

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

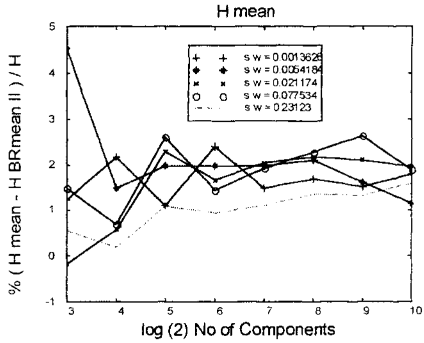

Figure 7: Wave data and Beta -Rayleigh Distribution (BRII)

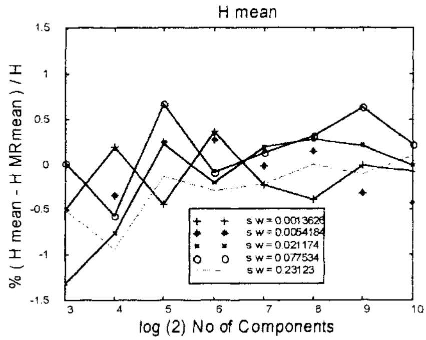

Figure 8: Wave data and modified Rayleigh distribution (MR)

Figure (9) through figure (12) show the difference between $H_{significant}$ of the wave train and the distributions under consideration. Again, it is shown that the maximum difference is less than 5% when using 8 components and it decrease with increasing the number of wave components. The modified Rayleigh distribution has the advantage that, the error never exceeds 1.5% for all the cases and again the Rayleigh distribution follows it in the accuracy of representation

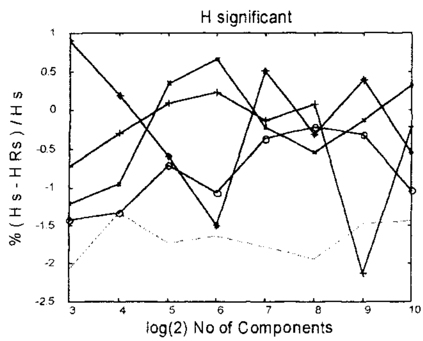

Figure 9:Wave date and Rayleigh Distribution (R)

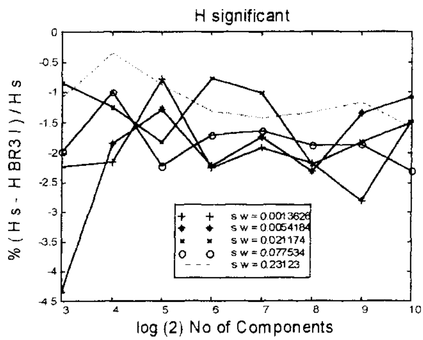

Figure 10: Wave data and Beta-Rayleigh Distribution (BRI)

Figure (13) through figure (16) show the difference between H_{10} of the generated wave train and the distributions under study. Excluding the results given by Rayleigh distribution, which gives a deviation up to 6%, all other distributions give deviation not more than 1.5. For BRI, and BRII distributions, this may be attributed to the use of H_{max} as a parameter in the distribution.

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

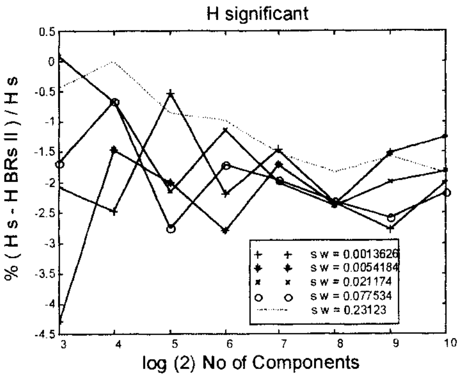

Figure 11: Wave data and Beta-Rayleigh Distribution (BRII)

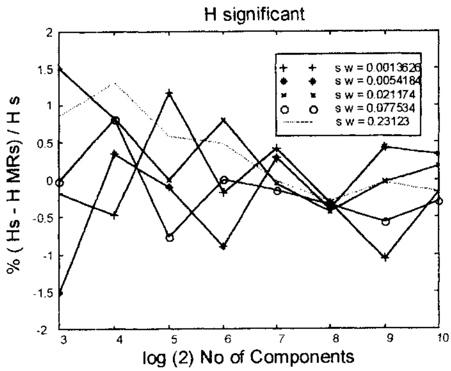

Figure 12 Wave data and modified Rayleigh Distribution (MR)

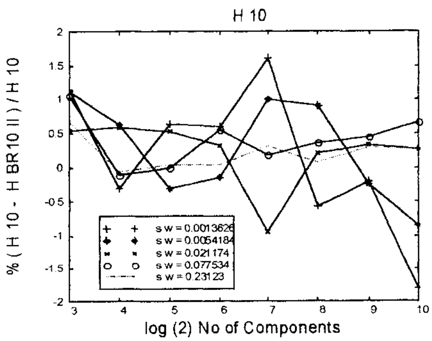

Figure 13 : Wave data and Rayleigh Distribution (R)

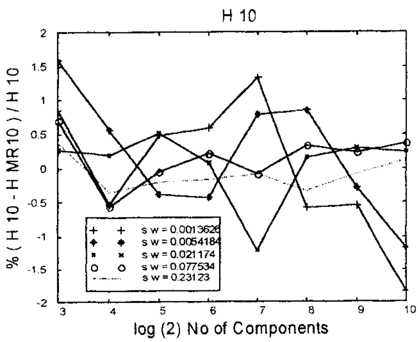

Figure 14: Wave data and Beta-Rayleigh Distribution (BRI)

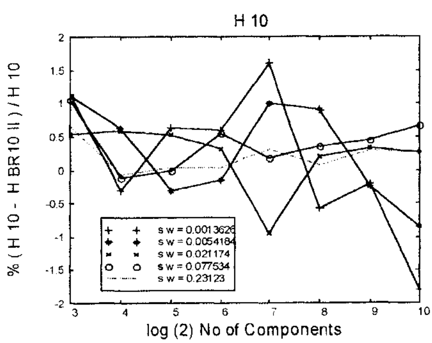

Figure 15 : Wave data and Beta-Rayleigh Distribution (BRII)

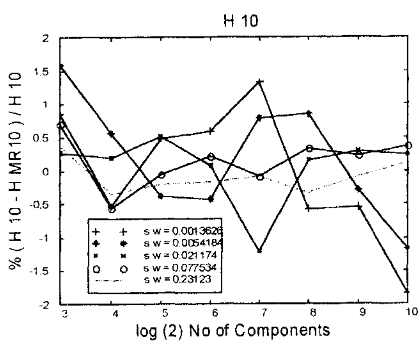

Figure 16: Wave data and modified Rayleigh Distribution (MR)

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

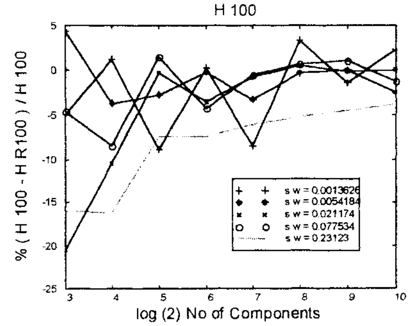

Figure 17 :Wave data and Rayleigh distribution (R)

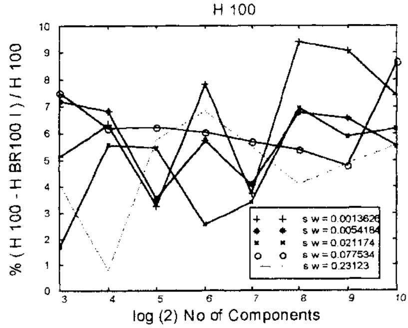

Figure 18: Wave data and Beta-Rayleigh distribution (BRI)

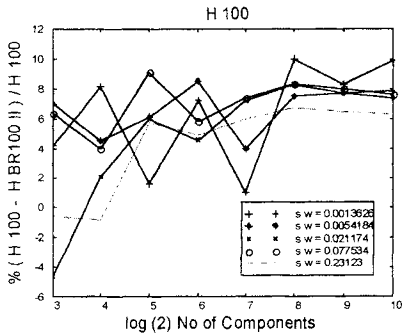

Figure 19:Wave data and Beta-Rayleigh distribution (BRII)

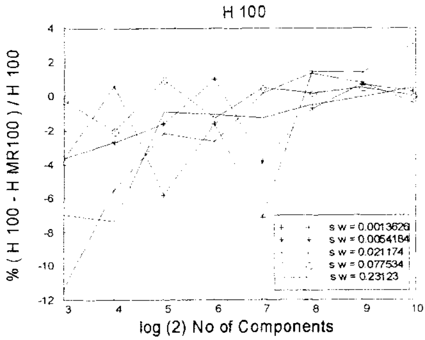

Figure 20: Wave data and modified-Rayleigh distribution (MR)

Finally, the comparison for H_{100} is shown in figure (17) through figure (20). For 8 components both Rayleigh and the modified Rayleigh distribution give error up to 20% and 12% respectively, while BRI and BRII give error not more than 8%. However, with increasing the number of components the relative error given by Rayleigh and the modified Rayleigh distribution does not exceed 5% and 2% respectively. While the error given by both BRI and BRII is in the range between 8% and 10%. From all of the above, it is clear that the modified Rayleigh distribution supercede Rayleigh distribution especially for extreme events representation.

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
© 2000 WIT Press, www.witpress.com, ISBN 1-85312-813-9

Conclusions

In this paper a study of a synthetic wave height distributions is carried out. Using a white spectrum both wide and narrow band spectra are employed. Besides Rayleigh distribution, three others proposed distributions are used. Among all of them, the modified Rayleigh distribution seems to represent the wave heights distribution better than all the others, especially for extreme events. The modified Rayleigh distribution is a two parameter model and it is the asymptotic form of the Beta-Rayleigh model.

The synthetic time series contain between 8 and 1024 wave components. For the case of 8 components with bandwidth 0.0013626 and the case of 1024 components with bandwidth 0.23123, this means 3200 of the longest wave in each run. Those numbers are enough to computer concise statistics.

Regardless the number of wave components and the bandwidth used in the analysis both Rayleigh distribution and the modified Rayleigh distribution give representations of the wave heights with error up to 5% and 1.5% respectively. This applies equally for H_1, H_3 and H_{10}.

For H_{100} and for number of components more than 64, Rayleigh distribution and the modified Rayleigh distribution gives 5 % and 1.5 % error, respectively.

Those results suggest that the modified Rayleigh distribution must replace Rayleigh distribution in representing the wave height distribution.

Reference

[1]Hughes S.A. and Borgman L. E ,Beta-Rayleigh Distribution for Shallow Water Wave Heights, Proceeding of Conference on Coastal Hydrodynamic, ASCE Newark. DE ,June 28-July 1, pp. 17-31 (1987)

[2]Chakrabarti S.K., 1994, Hydrodynamic of Offshore Structures, Computational Mechanics Publications, Southampton Boston.

[3] Longuet Higgins M. S., On the Statistical Distribution of the Heights of sea Waves , Journal of Marine Research, Vol 11,pp. 245-266, 1953