Numerical valuation of options under Kou's model

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Numerical methods are developed for pricing European and American options under Kou's jump-diffusion model which assumes the price of the underlying asset to behave like a geometrical Brownian motion with a drift and jumps whose size is log-double-exponentially distributed. The price of a European option is given by a partial integro-differential equation (PIDE) while American options lead to a linear complementarity problem (LCP) with the same operator. Spatial differential operators are discretized using finite differences on nonuniform grids and time stepping is performed using the implicit Rannacher scheme. For the evaluation of the integral term easy to implement recursion formulas are derived which have optimal computational cost. When pricing European options the resulting dense linear systems are solved using a stationary iteration. Also for pricing American options similar iterations can be employed. A numerical experiment demonstrates that the described method is very efficient as accurate option prices can be computed in a few milliseconds on a PC.

1 Kou's model and discretization

Under Kou's model [7] the value v of a European option satisfies a final value problem defined by a partial integro-differential equation (PIDE)

$$v_t + \frac{1}{2}\sigma^2 x^2 v_{xx} + (r - \lambda\zeta) x v_x - (r+\lambda) v + \lambda \int_{\mathbb{R}_+} v(t, xy) f(y)\, dy = 0 \quad \forall\, (t,x) \in [0,T) \times \mathbb{R}_+, \tag{1}$$

where t and x denote the time and the value of the underlying asset, respectively. The risk free interest rate r and the jump intensity λ are constants while we let the volatility σ to be a function of t and x as in [2]. The log-double-exponential density f is defined by

$$f(y) = \begin{cases} q\alpha_2 y^{\alpha_2 - 1}, & y < 1 \\ p\alpha_1 y^{-\alpha_1 - 1}, & y \geq 1, \end{cases}$$

where p, q, $\alpha_1 > 1$, and α_2 are positive constants such that $p + q = 1$. The coefficient ζ in (1) is given by $\zeta = \frac{p\alpha_1}{\alpha_1 - 1} + \frac{q\alpha_2}{\alpha_2 + 1} - 1$. The final value $v(T, x)$ is given by the payoff function. The value of an American option satisfies a variational inequality based on (1); see [3], for example. We discretize the spatial derivatives using finite differences on nonuniform grids. Our temporal discretizations are based on the implicit Rannacher time stepping.

2 Recursion formulas for the integral term

A straightforward treatment of the integral term leads the computational cost to be $\mathcal{O}(n^2)$ flops for evaluating the integral on all grid points, where n is the number of grid points. Using FFT this cost can be reduced to be $\mathcal{O}(n \log n)$ flops; see [1, 2, 4, 5]. With FFT nonuniform grids lead to more involved implementation. Instead we propose recursion formulas for the evaluation [9] which are easy-to-implement also on nonuniform grids and they lead to the optimal computation cost $\mathcal{O}(n)$ flops. In order to describe the recursion formulas, we divide the integral into two parts as

$$\int_{\mathbb{R}_+} v(t, xy) f(y)\, dy = \int_0^x v(t, xy) f(y)\, dy + \int_x^\infty v(t, xy) f(y)\, dy.$$

We consider the first integral while the second one can be treated in the same way. By making the change of variable $y = z/x$, we get

$$I^- = \int_0^x v(t, xy) f(y)\, dy = q\alpha_2 x^{-\alpha_2} \int_0^x v(t, z) z^{\alpha_2 - 1} dz.$$

We denote the value of the integral I^- at a grid point $x = x_i$ by I_i^-. The value of I_{i+1}^- can obtain using I_i^- as

$$I_{i+1}^- = \frac{x_{i+1}^{-\alpha_2}}{x_i^{-\alpha_2}} I_i^- + q\alpha_2 x_{i+1}^{-\alpha_2} \int_{x_i}^{x_{i+1}} v(t, z) z^{\alpha_2 - 1} dz.$$

Hence, it is necessary to integrate only once over each grid interval in order evaluate all integrals I_i^-, $i = 1, \ldots, n$.

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3 Iterative solution

For European options we use a stationary iterative method proposed in [8] and analyzed in [5]. Following the notations in [1], we form a regular splitting for the coefficient matrix $\mathbf{A}$ as $\mathbf{A} = \mathbf{T} - \mathbf{R}$, where $-\mathbf{R}$ corresponds the integral term in (1). With this choice $\mathbf{T}$ is a tridiagonal matrix. Then a rapidly converging iterative method for the linear system $\mathbf{Av} = \mathbf{b}$ reads

$$\mathbf{v}^{i+1} = \mathbf{T}^{-1}\left(\mathbf{R}\mathbf{v}^i + \mathbf{b}\right), \qquad i = 0, 1, \ldots, \tag{2}$$

where the initial guess $\mathbf{v}^0$ is taken to be the solution from the previous time step. Each iteration requires a solution with the tridiagonal matrix $\mathbf{T}$ and the multiplication of a vector by the full matrix $\mathbf{R}$ which corresponds to the evaluation of the integral term.

When pricing American options we solve the linear complementarity problems (LCPs) arising at each time step using either an operator splitting method [6] or a penalty method [4]; see also [9]. With the splitting method the above iteration can be used while the penalty method uses its generalization.

4 Numerical example

In our example, we price European call options by computing the prices of corresponding put options and employing the put-call parity. We use the model parameters [5]:

$$\sigma = 0.15, \quad r = 0.05, \quad T = 0.25, \quad K = 100,$$
$$\lambda = 0.1, \quad \alpha_1 = 3.0465, \quad \alpha_2 = 3.0775, \quad \text{and} \quad p = 0.3445.$$

The truncation boundary is at $X = 400$ and we use highly refined grid in the x-direction. On a very fine time-space grid we obtain the reference prices: 0.672677 at $x = 90$, 3.973479 at $x = 100$, and 11.794583 at $x = 110$ which are the same as in [5].

The numerical results in Table 1 show that the discretization is approximately second-order accurate when the number of time and space steps are increased at the same rate. We obtained accurate prices in a few milliseconds on a PC. More experiments with a local volatility function and American options are presented in [9]. The behavior also in these experiments is similar to the one in here.

m	n	error at 90	error at 100	error at 110	rate	iter.	time
6	40	4.4×10^{-3}	-1.6×10^{-2}	-6.1×10^{-3}		19	0.1
10	80	6.0×10^{-4}	-3.2×10^{-3}	-1.8×10^{-3}	4.8	26	0.4
18	160	6.2×10^{-4}	-6.9×10^{-4}	-1.7×10^{-4}	4.0	43	1.0
34	320	-1.4×10^{-4}	-1.5×10^{-4}	-2.1×10^{-4}	3.2	68	3.0
66	640	-7.2×10^{-6}	-3.6×10^{-5}	-3.7×10^{-5}	5.6	132	10.3

Table 1 The columns are: 'm' the number of time steps, 'n' the number of space steps, errors computed using the reference prices, 'rate' the ratio of consecutive l_2 errors, 'iter.' the number of iterations (2), and 'time' the CPU time in milliseconds on a 3.8GHz PC.

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