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NUMERICALLY FLAT PRINCIPAL BUNDLES

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Abstract. Generalizing the notion of a numerically flat vector bundle over a Kähler manifold M, we define a numerically flat principal G-bundle over M, where G is a semisimple complex algebraic group. It is proved that a principal G-bundle E_G is numerically flat if and only if $\mathrm{ad}(E_G)$ is numerically flat. Numerically flat bundles are also characterized using the notion of semistability.

1. Introduction. Let M be a compact Kähler manifold. In [DPS], the notion of a numerically effective holomorphic vector bundle over M was introduced (see Section 2).

Let G be a semisimple complex algebraic group. Let P be a parabolic subgroup of G and χ a character of P anti-dominant with respect to some Borel subgroup of G contained in P. So the line bundle over the projective variety G/P defined by χ is numerically effective.

For a holomorphic G-bundle E_G over M, the quotient map $E_G \to E_G/P$ defines a holomorphic principal P-bundle over E_G/P. The G-bundle E_G will be called numerically flat if for all pairs (P, χ) the line bundle over E_G/P defined by the anti-dominant character χ is numerically effective.

A principal $\mathrm{SL}(n, \boldsymbol{C})$-bundle is numerically flat if and only if the vector bundle associated to it by the standard representation is numerically flat (Proposition 2.3). For a numerically flat G-bundle, any associated vector bundle is also numerically flat (Theorem 2.4). A G-bundle E_G is numerically flat if and only if its adjoint vector bundle $\mathrm{ad}(E_G)$ is numerically flat (Theorem 2.5).

A numerically flat G-bundle E_G is semistable and all the (rational) characteristic classes of E_G of positive degree vanish. In the converse direction, if M is a projective manifold and E_G a semistable G-bundle over M such that all the characteristic classes of E_G of positive degree vanish, then E_G is numerically flat (Theorem 3.1).

For a parabolic subgroup P of G, its Levi quotient will be denoted by $L(P)$. For a principal P-bundle E_P, the principal $L(P)$-bundle obtained by extending the structure group using the projection of P to $L(P)$ will be denoted by $E_{L(P)}$. A G-bundle E_G over a Kähler manifold M is numerically flat if and only if there is a parabolic subgroup $P \subset G$ and a reduction $E_P \subset E_G$ of structure group such that the principal P-bundle E_P admits a flat holomorphic connection ∇ with the property that the monodromy of the flat connection on $E_{L(P)}$ induced by ∇ is contained in a maximal compact subgroup of $L(P)$ (Proposition 3.2).

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2. Numerically flatness. Let M be a compact connected Kähler manifold equipped with a Kähler metric ω. Let L be a holomorphic line bundle over M. We recall from [DPS] the definition of a numerically effective line bundle.

DEFINITION 2.1 (DPS, Definition 1.2). The line bundle L is called *numerically effective* if for any $\varepsilon > 0$, there is a Hermitian metric h_ε on it such that the curvature Θ_{h_ε} of the Chern connection on L satisfies the inequality

$$\Theta_{h_\varepsilon} \geq -\varepsilon\omega\,.$$

Since M is compact, the above definition clearly does not depend on the choice of ω.

A vector bundle E over M is called *numerically effective* if the tautological line bundle $\mathcal{O}_{\boldsymbol{P}(E)}(1)$ over the projective bundle $\boldsymbol{P}(E)$ is numerically effective (cf. [DPS, p. 305, Definition 1.9]). A vector bundle E over M is called *numerically flat* if both E and its dual E^* are numerically effective (cf. [DPS, p. 311, Definition 1.17]).

Let G be a semisimple linear algebraic group over the field $\boldsymbol{C}$ of complex numbers. A Zariski closed proper subgroup P of G is called a *parabolic subgroup* if G/P is compact.

Let E_G be a holomorphic principal G-bundle over M. For a parabolic subgroup $P \subset G$, the projection $E_G \to E_G/P$ defines a holomorphic principal P-bundle. Given any character χ of P, let

$$E_G \times^P \boldsymbol{C} := (E_G \times \boldsymbol{C})/P$$

be the associated line bundle over E_G/P; the quotient is for the action of P defined as follows: the action of any $g \in P$ sends any point $(z, c) \in E_G \times \boldsymbol{C}$ to $(zg, \chi(g^{-1})c) \in E_G \times \boldsymbol{C}$. This associated line bundle $E_G \times^P \boldsymbol{C}$ will be denoted by L_χ.

DEFINITION 2.2. The G-bundle E_G is called *numerically flat* if for every parabolic subgroup $P \subset G$ and every character $\chi : P \to \boldsymbol{C}^*$ dominant with respect to some Borel subgroup of G contained in P, the dual line bundle L_χ^* over E_G/P is numerically effective in the sense of the above definition (Definition 2.1).

See [Ra] for the details on dominant characters of P. A character χ of P is dominant if and only if the dual of the line bundle over G/P associated to χ is numerically effective.

Since the pullback of a numerically effective line bundle is also numerically effective (see [DPS, p. 302, Proposition 1.8(i)]), and a line bundle L is numerically effective if $L^{\otimes n}$ is numerically effective for some $n \geq 1$, it suffices to check the condition in Definition 2.2 only for maximal parabolic subgroups P. To explain this in more detail, for an arbitrary parabolic subgroup Q of G there are only finitely maximal parabolic subgroups P_i of G that contain Q. The ensuing map

$$G/Q \to \prod_{Q \subset P_i} G/P_i$$

is an embedding. Given any dominant character χ of Q, there are dominant characters χ_i of P_i such that $\prod_i \chi_i$ on Q coincides with χ. Therefore, it is enough to check the condition in Definition 2.2 only for maximal parabolic subgroups.

Let E be a holomorphic vector bundle of rank n over M with $\bigwedge^n E \cong \mathcal{O}_M$. So E defines a holomorphic principal $\mathrm{SL}(n, \boldsymbol{C})$-bundle over M. The principal $\mathrm{SL}(n, \boldsymbol{C})$-bundle defined by E will be denoted by E_{SL}.

PROPOSITION 2.3. *The vector bundle E is numerically flat if and only if the principal $\mathrm{SL}(n, \boldsymbol{C})$-bundle E_{SL} is numerically flat.*

PROOF. Assume that E_{SL} is numerically flat. Let $P \subset \mathrm{SL}(n, \boldsymbol{C})$ be the parabolic subgroup that fixes a given line $V_0 \subset \boldsymbol{C}^n$. Let χ be the character of P defined by its action on V_0. So the quotient E_{SL}/P is $\boldsymbol{P}(E)$ and $\mathcal{O}_{\boldsymbol{P}(E)}(-1)$ is the line bundle associated to χ. Therefore, from the definition of numerically flatness of E_{SL} we conclude that the vector bundle E is numerically effective. Since E is numerically effective and $\bigwedge^n E \cong \mathcal{O}_M$, we conclude that E is numerically flat (see [DPS, p. 311, Definition 1.17], [DPS, p. 307, Proposition 1.14(iii)]).

Now assume that E is numerically flat. Maximal parabolic subgroups of $\mathrm{SL}(n, \boldsymbol{C})$ are those that preserve some subspace of $\boldsymbol{C}^n$. The quotient is a Grassmannian. Let $\mathrm{Gr}(E, k)$ be the Grassmann bundle over M consisting of all k dimensional subspaces in the fibers of E, where $k \in [1, n-1]$. The condition that E is numerically effective implies that the vector bundle $\bigwedge^k E$ is also numerically effective [DPS, p. 307, Proposition 1.14(ii)]. The line bundles over $\mathrm{Gr}(E, k)$ corresponding to the dominant characters are the nonnegative powers of the line bundle over $\mathrm{Gr}(E, k)$ defined by the determinant of the tautological vector bundle of rank k. Since the determinant of the rank k tautological vector bundle over $\mathrm{Gr}(E, k)$ is the pullback of $\mathcal{O}_{\boldsymbol{P}(\wedge^k E)}(-1)$ using the Plücker embedding, the numerically effectiveness of $\bigwedge^k E$ implies that the dual of the determinant of the tautological vector bundle over $\mathrm{Gr}(E, k)$ is numerically effective. □

Proposition 2.3 justifies Definition 2.2.

Let V be a finite-dimensional complex G-module. For any G-bundle E_G, the quotient $E_G \times^G V := (E_G \times V)/G$ for the twisted diagonal action is a vector bundle, which is called the associated vector bundle.

The following theorem, which is proved using a basic result due to C. Mourougane, is similar in spirit to the characterization of semistable G-bundles in terms of the semistability of the associated vector bundles (see [RS, Theorem 3], [AB, Proposition 2.10]).

THEOREM 2.4. *Let E_G be a numerically flat G-bundle over M. For any finite dimensional complex G-module V the associated vector bundle $E_G \times^G V$ is numerically flat.*

PROOF. Since V is a direct sum of irreducible G-modules, and a direct sum of numerically flat vector bundles is again numerically flat, it suffices to prove the theorem for irreducible G-modules. So assume V to be irreducible.

From the Borel-Weil-Bott theorem we know that there is a parabolic subgroup P of G and an anti-dominant character (inverse of a dominant character) χ of P such that the associated line bundle $\mathcal{L}_\chi = G \times^P \boldsymbol{C}$ over G/P is ample, and the induced representation of G on $H^0(G/P, L_\chi)$ coincides with the G-module V (cf. [Bo]).

Let $p : E_G/P \to M$ be the natural projection. The above form of the Borel-Weil-Bott theorem immediately implies that the associated line bundle $L_\chi = E_G \times^P \mathbf{C}$ over E_G/P has the property that

$$p_* L_\chi \cong E_G \times^G V. \tag{2.1}$$

Let $K_{\mathrm{rel}}^{-1} := K_{E_G/P}^{-1} \otimes p^* K_M$ be the relative anti-canonical line bundle over E_G/P. Note that the anti-canonical bundle $K_{G/P}^{-1}$ over G/P corresponds to an anti-dominant character (as $K_{G/P}^{-1}$ is ample) $\hat{\chi}$ of P, that is, $K_{G/P}^{-1}$ is the line bundle associated to $\hat{\chi}$. So the character $\chi\hat{\chi}$ of P is anti-dominant.

The line bundle $L_{\chi\hat{\chi}} = E_G \times^P \mathbf{C}$ over E_G/P associated to the character $\chi\hat{\chi}$ is clearly $L_\chi \otimes K_{\mathrm{rel}}^{-1}$. If E_G is numerically flat, we know from the definition that $L_\chi \otimes K_{\mathrm{rel}}^{-1}$ is numerically effective. Since the restriction of $L_\chi \otimes K_{\mathrm{rel}}^{-1}$ to any fiber of the projection p is an ample line bundle, the direct image

$$p_*(K_{\mathrm{rel}} \otimes L_\chi \otimes K_{\mathrm{rel}}^{-1}) \cong p_* L_\chi$$

is numerically effective (cf. [Mo, p. 895, Théorème 2]). This theorem of [Mo] says that if L is a numerically effective line bundle over E_G/P whose restriction to any fiber of p is ample, then the direct image $p_*(K_{\mathrm{rel}} \otimes L)$ is a numerically effective vector bundle over M. The above assertion is obtained by setting $L = L_\chi \otimes K_{\mathrm{rel}}^{-1}$.

Finally using the isomorphism in (2.1) we conclude that the associated vector bundle $E_G \times^G V$ is numerically effective if E_G is numerically flat. Since G does not have any nontrivial character (as it is semisimple) we have $\bigwedge^{\mathrm{top}} E_G \times^G V \cong \mathcal{O}_M$. Therefore, it follows that $\mathrm{ad}(E_G)$ is numerically flat if E_G is so. □

Let $\mathfrak{g}$ denote the Lie algebra of G, on which G acts by conjugation. Since G is semisimple, the kernel of the homomorphism

$$\rho : G \to \mathrm{SL}(\mathfrak{g}) \tag{2.2}$$

is a finite group. For a principal G-bundle E_G, the associated adjoint bundle $E_G \times^G \mathfrak{g}$ will be denoted by $\mathrm{ad}(E_G)$.

For a parabolic subgroup P of G, let $R_u(P)$ denote the *unipotent radical* of P. So $R_u(P)$ is the (unique) maximal connected unipotent normal subgroup of P. The quotient $L(P) := P/R_u(P)$ is called the *Levi factor* of P. The group $L(P)$ is reductive. (See [Bor].)

THEOREM 2.5. *Let V be as in Theorem 2.4 such that the kernel of the homomorphism $G \to \mathrm{SL}(V)$ is a finite group. A G-bundle E_G is numerically flat if and only if the associated vector bundle $E_G \times^G V$ is numerically flat. In particular, E_G is numerically flat if and only if $\mathrm{ad}(E_G)$ is numerically flat.*

PROOF. If E_G is numerically flat, then Theorem 2.4 implies that the vector bundle $E_G \times^G V$ is numerically flat.

Let $E_{\mathrm{SL}(V)}$ be the principal $\mathrm{SL}(V)$-bundle over M obtained by extending the structure group of E_G using the homomorphism

$$\tau : G \to \mathrm{SL}(V) \tag{2.3}$$

defined by the action of G on V.

For any maximal parabolic subgroup P of G and any dominant character χ of P, there is a parabolic subgroup Q (not necessarily maximal) of $\mathrm{SL}(V)$ and a dominant character χ' of Q such that $\tau(P) = \tau(G) \cap Q$ and $\tau^*\chi' = \chi^n$ for some $n \geq 1$. To prove this first note that since P is maximal parabolic the group of characters of P is isomorphic to $\boldsymbol{Z}$. If $\tau(P) = \tau(G) \cap Q$, then G/P embeds into $\mathrm{SL}(V)/Q$. Since the pullback of a numerically effective line bundle is numerically effective, we have $\tau^*\chi' = \chi^n$ for some $n \geq 1$. So all we need to show is that there is a parabolic subgroup Q with $\tau(G) \cap Q = \tau(P)$.

Let $N_1 \subset \mathrm{SL}(V)$ be the normalizer of the subgroup $\tau(R_u(P))$, where τ is defined in (2.3). Let $R_1 \subset N_1$ be its unipotent radical. Inductively define N_{k+1} to be the normalizer of R_k, and R_{k+1} to be the unipotent radical of N_{k+1}. Both $\{N_i\}_{i\geq 1}$ and $\{R_i\}_{i\geq 1}$ are increasing subgroups of $\mathrm{SL}(V)$. Note that each N_i is a proper subgroup of $\mathrm{SL}(V)$, as it is a normalizer of a nontrivial unipotent subgroup (the unipotent subgroup is nontrivial as the kernel of the homomorphism $G \to \mathrm{SL}(V)$ is finite). The limiting group, call it Q, of $\{N_i\}_{i\geq 1}$ has the property that the normalizer of the unipotent radical of Q is Q itself. This implies that Q is a parabolic subgroup of $\mathrm{SL}(V)$. (The assumption that the kernel of the homomorphism $G \to \mathrm{SL}(V)$ is finite ensures that Q is a proper subgroup of $\mathrm{SL}(V)$.) The parabolic group Q clearly has the property that $Q \cap \tau(G) = \tau(P)$.

Consequently, we have an embedding of E_G/P in $E_{\mathrm{SL}(V)}/Q$, and the line bundle over E_G/P defined by χ^n coincides with the restriction of the line bundle over $E_{\mathrm{SL}(V)}/Q$ defined by χ'. Therefore, E_G is numerically flat if $E_{\mathrm{SL}(V)}$ is so. □

REMARK 2.6. Let

$$\sigma : G \to H$$

be a homomorphism to a complex semisimple group H. Using σ the Lie algebra $\mathfrak{h}$ of H is a left G-module. Consider the principal H-bundle $E_H := E_G \times^G H$ obtained by extending the structure group of E_G using σ. Since that adjoint vector bundle $\mathrm{ad}(E_H)$ is the one associated to E_G for the G-module $\mathfrak{h}$, if E_G is numerically flat then Theorem 2.4 and Theorem 2.5 combine together to imply that E_H is numerically flat.

3. Semistability and numerical flatness. Let F be a holomorphic vector bundle defined on a dense open subset $U \subset M$ such that the complement $M \setminus U$ is a complex analytic subset of (complex) codimension at least two. Let $\iota : U \hookrightarrow M$ be the inclusion map. The condition on the codimension of $M \setminus U$ implies that the direct image $\iota_* F$ is a coherent sheaf on M. The degree of F is defined as

$$\deg(F) := \int_M c_1(\iota_* F)\omega^{d-1},$$

where $d = \dim M$ and ω is the fixed Kähler form on M.

A principal G-bundle E_G over M is called *semistable* (respectively, *stable*) if for any reduction of structure group $E_P \subset E_G|_U$ to any parabolic subgroup P over an open subset U, with $\mathrm{codim}(M \setminus U) \geq 2$, and any nontrivial character χ of P dominant with respect to some Borel subgroup contained in P, the associated line bundle $L_\chi = E_P \times^P \boldsymbol{C}$ over U satisfies the condition

$$\deg(L_\chi) \leq 0$$

(respectively, $\deg(L_\chi) < 0$) (see [Ra], [RS], [AB]).

Take P to be a maximal parabolic subgroup in the above definition. Let ι be the inclusion map of U in M and $\sigma : U \to E_G/P$ the section of the projection $E_G/P \to M$ defining the reduction of structure group to P. The above the inequality can be replaced by the inequality

$$\deg(\iota_* \sigma^* T_{\mathrm{rel}}) \geq 0$$

(respectively, $\deg(\iota_* \sigma^* T_{\mathrm{rel}}) > 0$), where T_{rel} is the relative tangent bundle for the projection $E_G/P \to M$; see [Ra, Lemma 2.1] for a proof that the two formulations of the definition of (semi)stability are equivalent.

THEOREM 3.1. *Let E_G be a principal G-bundle over a Kähler manifold M. If E_G is numerically flat, then E_G is semistable and all the* (*rational*) *characteristic classes of E_G of degree at least one vanish.*

If E_G is semistable and all the (*rational*) *characteristic classes of E_G of degree at least one vanish, then E_G is numerically flat provided M is a projective manifold.*

PROOF. Let E_G be a numerically flat G-bundle over M. Theorem 2.5 says that the adjoint vector bundle $\mathrm{ad}(E_G)$ is numerically flat. From [DPS, p. 311, Theorem 1.18] it follows that $\mathrm{ad}(E_G)$ is semistable. The semistability of $\mathrm{ad}(E_G)$ implies that the G-bundle E_G is semistable (cf. [AB, Proposition 2.10]).

Writing G as a product of simple groups we see that it is enough to prove that all the higher characteristic classes (higher than degree zero) of E_G vanish assuming that G is simple. But for G simple, all the characteristic classes of E_G are contained in the characteristic classes of the adjoint vector bundle $\mathrm{ad}(E_G)$. As $\mathrm{ad}(E_G)$ is numerically flat, all the higher Chern classes (higher than degree zero) of $\mathrm{ad}(E_G)$ vanish (cf. [DPS, p. 311, Corollary 1.19]). Consequently, all the characteristic classes of E_G of positive degree vanish.

Now assume that M is a projective manifold and E_G a semistable principal G-bundle over M such that all the characteristic classes of E_G of positive degree vanish.

The semistability of E_G implies that the vector bundle $\mathrm{ad}(E_G)$ is semistable (cf. [RS, Theorem 3], [AB, Proposition 2.10]). Since all the characteristic classes of E_G of positive degree vanish, it follows immediately that $c_i(\mathrm{ad}(E_G)) = 0$ for all $i \geq 1$.

Since $\mathrm{ad}(E_G)$ is semistable with vanishing Chern classes, Theorem 2 of [Si, p. 39] says that there is a filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{k-1} \subset F_k = \mathrm{ad}(E_G) \tag{3.1}$$

of $\mathrm{ad}(E_G)$ by holomorphic subbundles such that each quotient vector bundle F_i/F_{i-1}, $i \in [1, k]$, is a stable vector bundles with $c_j(F_i/F_{i-1}) = 0$ for all $j \geq 1$. To deduce this from [Si, p. 39, Theorem 2] simply set the Higgs field to be zero in [Si, Theorem 2]; we need the assumption that M is projective to be able to use this result of Simpson.

Now a theorem due to Donaldson [Do] and Uhlenbeck-Yau [UY] says that F_i/F_{i-1} admits a unitary flat connection. Consequently, the vector bundle F_i/F_{i-1} is numerically flat by [DPS, p. 311, Theorem 1.18]. Since an extension of a numerically flat vector bundle by a numerically flat vector bundle is again numerically flat by [DPS, p. 308, Proposition 1.15(ii)], using (3.1) it follows immediately that $\mathrm{ad}(E_G)$ is numerically flat. Now Theorem 2.5 says that the G-bundle E_G is numerically flat. □

Let E_P be a holomorphic principal P-bundle over the compact Kähler manifold M equipped with a holomorphic flat connection ∇, where P is a parabolic subgroup of G. Let $E_{L(P)} := E_P \times^P L(P)$ be the corresponding principal $L(P)$-bundle, where $L(P)$ is the Levi factor (defined prior to Theorem 2.5); here P acts on $L(P)$ on the left using the projection of P to $L(P)$. The connection ∇ on E_P induces a connection $\nabla^{L(P)}$ on $E_{L(P)}$, which is flat holomorphic as ∇ is so.

PROPOSITION 3.2. *A holomorphic G-bundle E_G over a compact Kähler manifold M is numerically flat if and only if there is a parabolic subgroup $P \subset G$, a reduction of structure group $E_P \subset E_G$, and a flat holomorphic connection ∇ on the P-bundle E_P such that the monodromy of the connection $\nabla^{L(P)}$ on $E_{L(P)}$ is contained in some maximal compact subgroup of $L(P)$.*

PROOF. Let $E_P \subset E_G$ be a reduction of structure group to a parabolic subgroup P and ∇ a flat holomorphic connection on E_P with the above property. There is a parabolic subgroup $Q \subset \mathrm{SL}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, such that for the adjoint representation ρ (defined in (2.2)) we have $\rho(P) = \rho(G) \cap Q$. The construction of Q is given in the proof of Theorem 2.5.

Let $E_Q = E_P \times^P Q$ be the principal Q-bundle obtained by extending the structure group of E_P using ρ. Note that the adjoint vector bundle $\mathrm{ad}(E_G)$ is the vector bundle associated (by the standard action) to the $\mathrm{SL}(\mathfrak{g})$-bundle obtained by extending the structure group of E_G using ρ. The Q-bundle E_Q is a reduction of structure group to Q of $\mathrm{ad}(E_G)$, as E_Q is the extension of structure group of E_P using ρ. So the reduction E_Q defines a filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathrm{ad}(E_G) \tag{3.2}$$

of the vector bundle $\mathrm{ad}(E_G)$ by holomorphic subbundles.

Let ∇^Q denote the flat holomorphic connection on E_Q defined by the connection ∇ on E_P. So ∇^Q preserves the filtration in (3.2). Consequently, ∇^Q induces a flat holomorphic connection on each subsequent quotient V_i/V_{i-1} in (3.2). The corresponding connection on the graded vector bundle $\bigoplus_{i=1}^k V_i/V_{i-1}$ coincides with the one induced by the connection $\nabla^{L(P)}$ on $E_{L(P)}$. Indeed, the homomorphism $\rho : P \to Q$ (in (2.2)) induces a homomorphism $L(P) \to L(Q)$ of the Levi quotients. Using this induced homomorphism the $L(P)$-bundle $E_{L(P)}$ gives a $L(Q)$-bundle $E_{L(Q)}$ (by extension of structure group). The vector bundle associated to $E_{L(Q)}$ for the action of $L(Q)$ on the graded vector space (for the filtration of $\mathfrak{g}$ that Q preserves) is identified with $\bigoplus_{i=1}^k V_i/V_{i-1}$. So a connection on $E_{L(P)}$ gives a connection on $\bigoplus_{i=1}^k V_i/V_{i-1}$ by inducing a connection on $E_{L(Q)}$. Clearly, the connection on $\bigoplus_{i=1}^k V_i/V_{i-1}$ obtained from ∇^Q (constructed as above) coincides with the one given by $\nabla^{L(P)}$.

Since the monodromy of the flat connection $\nabla^{L(P)}$ is contained in some maximal compact subgroup of $L(P)$ it follows immediately that the connection on V_i/V_{i-1} preserves some Hermitian structure on V_i/V_{i-1}. In other words, each V_i/V_{i-1} admits a unitary flat connection. This implies that $\mathrm{ad}(E_G)$ is numerically flat by [DPS, p. 311, Theorem 1.18]. Consequently, by Theorem 2.5 the G-bundle E_G is numerically flat.

To prove the converse, let E_G be a numerically flat G-bundle over M. So $\mathrm{ad}(E_G)$ is numerically flat (Theorem 2.5). Now [DPS, p. 311, Theorem 1.18] says that $\mathrm{ad}(E_G)$ admits a filtration by holomorphic subbundles such that each successive quotient is a stable vector bundle with vanishing Chern classes of every positive degree.

Let

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{k-1} \subset W_k = \mathrm{ad}(E_G) \tag{3.3}$$

be the filtration of the vector bundle $\mathrm{ad}(E_G)$ defined by socle. In other words, W_i/W_{i-1} is the (unique) maximal polystable subsheaf of degree zero (the socle) of $\mathrm{ad}(E_G)/W_{i-1}$ (see [AB, p. 211, Lemma 2.5] for properties of the socle).

Since $\mathrm{ad}(E_G)$ admits a filtration by holomorphic subbundles such that each successive quotient is a stable vector bundle with vanishing higher Chern classes, it follows immediately that each subsheaf W_i in (3.3) is a subbundle of $\mathrm{ad}(E_G)$. Furthermore, for the same reason $c_j(W_i/W_{i-1}) = 0$ for all $i, j \geq 1$. (If F is a subbundle of degree zero of a polystable vector bundle E with $c_j(E) = 0$ for all $j \geq 1$, then $c_j(F) = 0 = c_j(E/F)$ for all $j \geq 1$.)

From the above properties of $\mathrm{ad}(E_G)$ it follows that it has a natural flat holomorphic connection ∇ that preserves the filtration in (3.3), and the connection on each successive quotient W_i/W_{i-1} induced by ∇ is unitary flat (cf. [Si, p. 40, Corollary 3.10]). The construction of the connection ∇ in [Si] needs the base manifold to be projective. But this assumption is only needed to conclude that the vector bundle admits a filtration by subbundles such that each successive quotient is stable of degree zero, that is, to have [Si, p. 39, Theorem 2] valid for the vector bundle. But using the assumption that $\mathrm{ad}(E_G)$ is numerically flat we already have such a filtration in (3.3).

Let $\mathcal{V} \subset \mathrm{End}(\mathrm{ad}(E_G))$ be the subbundle that preserves the filtration in (3.3). So for any point $x \in M$ and any endomorphism $T \in \mathrm{End}(\mathrm{ad}(E_G)_x)$ we have $T \in \mathcal{V}_x$ if and only if $T((W_i)_x) \subset (W_i)_x$ for all $i \geq 1$. The Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$ define a homomorphism of vector bundles

$$\tau : \mathrm{ad}(E_G) \to \mathrm{End}(\mathrm{ad}(E_G)).$$

Since G is semisimple, its Lie algebra $\mathfrak{g}$ has trivial center. Hence the above homomorphism τ is pointwise injective.

Consider the intersection $\mathcal{V} \cap \tau(\mathrm{ad}(E_G))$ inside $\mathrm{End}(\mathrm{ad}(E_G))$. For each $x \in M$, this intersection defines a parabolic subalgebra of $\mathrm{ad}(E_G)_x$. Since the normalizer (inside G) of a parabolic subgroup $P \subset G$ is P itself, the subalgebra bundle $\mathcal{V} \cap \tau(\mathrm{ad}(E_G))$ defines a reduction of structure group $E_P \subset E_G$ to a parabolic subgroup P such that $\mathrm{ad}(E_P) = \mathcal{V} \cap \tau(\mathrm{ad}(E_G))$ (see the construction of this reduction in the last paragraph of p. 341 in [ABi]).

Using the Killing form on $\mathfrak{g}$, the vector bundle $\mathrm{ad}(E_G)$ gets identified with $\mathrm{ad}(E_G)^*$. Indeed, since the Killing form is G-invariant, it defines a nondegenerate symmetric bilinear form on $\mathrm{ad}(E_G)$. Consider the vector bundle $\mathcal{W} := \bigwedge^2 \mathrm{ad}(E_G)$. The Lie algebra structure on the fibers of $\mathrm{ad}(E_G)$ gives a nowhere vanishing section

$$s \in H^0(M, \mathcal{W})$$

using the identification of $\mathrm{ad}(E_G)$ with its dual. Let $L_s \subset \mathcal{W}$ be the trivial line subbundle generated by s.

The canonical flat connection ∇ on $\mathrm{ad}(E_G)$ constructed in [Si] induces a flat connection $\widetilde{\nabla}$ on $\mathcal{W}$. Since L_s is a trivial line subbundle of $\mathcal{W}$, the connection $\widetilde{\nabla}$ preserves L_s, and the induced connection on L_s is the trivial connection, that is, it has trivial monodromy; this property of $\widetilde{\nabla}$ follows immediately from the general properties of the connection constructed in [Si].

Consequently, the connection ∇ on $\mathrm{ad}(E_G)$ is compatible with the Lie algebra structure of the fibers. The following lemma shows that the connection ∇ induces a connection on the G-bundle E_G.

LEMMA 3.3. *Let E'_G be a smooth principal G-bundle over a smooth manifold M', where G is a semisimple linear algebraic group defined over $\mathbf{C}$. Let ∇' be a connection on the adjoint vector bundle* $\mathrm{ad}(E'_G)$ *such that*

$$[\nabla'_v(s), t] + [s, \nabla'_v(t)] = \nabla'_v([s, t])$$

for all locally defined sections s, t of $\mathrm{ad}(E'_G)$ *and all locally defined vector field on M'.* (*The Lie algebra structure of the fibers of* $\mathrm{ad}(E'_G)$ *gives a smooth section of the vector bundle* $\mathrm{ad}(E'_G) \otimes \mathrm{ad}(E'_G)^* \otimes \mathrm{ad}(E'_G)^*$; *the above condition on ∇' is equivalent to the condition that this section is flat with respect to the connection on* $\mathrm{ad}(E'_G) \otimes \mathrm{ad}(E'_G)^* \otimes \mathrm{ad}(E'_G)^*$ *induced by ∇'.*) *Then there is a unique connection ∇'' on the principal G-bundle E'_G such that ∇' is obtained from ∇'' by extension of structure group.*

PROOF. Let $E'_{\mathrm{GL}(\mathfrak{g})}$ be the smooth principal $\mathrm{GL}(\mathfrak{g})$-bundle over M' defined by $\mathrm{ad}(E'_G)$. Since $E'_{\mathrm{GL}(\mathfrak{g})}$ is obtained by extending the structure group of E'_G using the adjoint representation of G, a connection on E'_G induces a connection on $E'_{\mathrm{GL}(\mathfrak{g})}$. Since the kernel of the adjoint representation of G is finite, there can be at most one connection on E'_G inducing a given connection on $E'_{\mathrm{GL}(\mathfrak{g})}$.

Let ∇'_0 be the smooth one-form on $E'_{\mathrm{GL}(\mathfrak{g})}$ defining the connection ∇'. Let

$$\rho : E'_G \to E'_{\mathrm{GL}(\mathfrak{g})}$$

be the natural map. We will show that $\rho^*\nabla'_0$ is a connection on E'_G. Let

$$\rho_G : G \to \mathrm{GL}(\mathfrak{g})$$

be the adjoint representation of G. The kernel of ρ_G coincides with the center $Z(G) \subset G$. Hence the image $\rho(E'_G)$ is a principal $G/Z(G)$-bundle over M'. Furthermore, $\rho_G(G)$ is a connected component of the subgroup $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$. Therefore, the condition

$$[\nabla'_v(s), t] + [s, \nabla'_v(t)] = \nabla'_v([s, t])$$

in the lemma means that the pullback $\rho^*\nabla'_0$ on E'_G is a $\mathfrak{g}$-valued one-form. Consequently, $\rho^*\nabla'_0$ defines a connection on E'_G. It is easy to see that this connection on E'_G defined by $\rho^*\nabla'_0$ induces the connection ∇' on $E'_{\mathrm{GL}(\mathfrak{g})}$. $\square$

Continuing with the proof of the proposition, let ∇^G be the connection on E_G obtained from the connection ∇ on $\mathrm{ad}(E_G)$ using Lemma 3.3. Since ∇ is flat holomorphic, it follows immediately that the connection ∇^G on E_G is also flat holomorphic.

Finally, since ∇ preserves the filtration in (3.3), and $\mathrm{ad}(E_P) = \mathcal{V} \cap \tau(\mathrm{ad}(E_G))$, the connection ∇^G on E_G induces a connection on E_P. In other words, the connection ∇^G is the extension of a connection on the P-bundle E_P. This connection on E_P clearly has the property stated in the proposition. $\square$

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