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ω-CHAOS AND TOPOLOGICAL ENTROPY

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ABSTRACT. We present a new concept of chaos, ω-chaos, and prove some properties of ω-chaos. Then we prove that ω-chaos is equivalent to positive entropy on the interval. We also prove that ω-chaos is equivalent to the definition of chaos given by Devaney on the interval.

1. INTRODUCTION

Chaotic behavior has recently been the focus of considerable study by mathematicians and other scientists. Definitions of chaos have been given by Li and Yorke [LY], Devaney [D], and others.

It is known that if a continuous map of the interval has positive topological entropy, then it is chaotic according to the definition of Li and Yorke [M, O, LY, N]. The converse of this result is false; Xiong [X3] and Smitál [Sm] have given counterexamples. Here, we provide a definition of chaos which is similar to the one given by Li and Yorke. However a continuous map f of the interval is chaotic in our sense if and only if f has positive topological entropy. Furthermore, we prove that such a map has positive topological entropy if and only if it is chaotic in the sense of Devaney.

Let X be a compact metric space with metric d, and let $f: X \to X$ be continuous. The following definition is based on the work of Li and Yorke [LY].

Definition 1.1. A subset S of X containing no periodic points is called a *scrambled set* if for any $x, y \in S$ with $x \neq y$, and any periodic point $p \in X$ of f,

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$;
(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$; and
(3) $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$.

We say that f is *chaotic in the sense of Li and Yorke*, if there exists an uncountable scrambled set.

Let $\omega(x, f)$ denote the set of ω-limit points of f. Then $\omega(x, f)$ is a closed invariant subset of X.

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Definition 1.2. Let $S \subset X$. We say that S is an *ω-scrambled set* if, for any $x, y \in S$ with $x \neq y$,

(1) $\omega(x, f) \setminus \omega(y, f)$ is uncountable;

(2) $\omega(x, f) \cap \omega(y, f)$ is nonempty; and

(3) $\omega(x, f)$ is not contained in the set of periodic points.

We say that f is *ω-chaotic*, if there exists an uncountable ω-scrambled set.

Remark. J. Smitál (personal communication) has proved that in the case of a compact interval $\omega(x, f) \subset P(f)$ implies that $\omega(x, f)$ is finite (see also [S, 1965], [S,1966]). Thus, in this case, condition (3) is not needed in Definition 1.2.

A continuous map $f : X \to X$ is called *topologically transitive* if f is onto and has a dense orbit on X. $f : X \to X$ has *sensitive dependence on initial conditions* if there exists a $\delta > 0$ such that, for any $x \in X$, there exists a sequence $\{y_k\}$ of points in X and a sequence $\{n_k\}$ of positive integers such that $\lim_{k\to\infty} y_k = x$ and $d(f^{n_k}(y_k), f^{n_k}(x)) > \delta$.

Definition 1.3 [D]. f is said to be *chaotic in the sense of Devaney* if there is a closed invariant set $D \subset X$ such that the following conditions hold.

1. $f|_D$ is topologically transitive;
2. $f|_D$ has sensitive dependence on initial conditions; and
3. The periodic points of f in D are dense in D.

We say the set D is *chaotic.*

Remark. It has been proved recently that if conditions (1) and (3) hold then (2) holds (see [BBCDS, Li]). Thus condition (2) is not needed.

We will prove the following theorem.

Theorem. *Let f be a continuous map of a compact interval I to itself. The following statements are equivalent.*

(I) *f has positive topological entropy.*

(II) *There is an uncountable ω-scrambled set S such that*

$$\bigcap_{x\in S} \omega(x, f) \neq \phi.$$

(III) *f is ω-chaotic.*

(IV) *There is an ω-scrambled set containing exactly two points.*

(V) *f is chaotic in the sense of Devaney.*

(VI) *There is a chaotic set D and an uncountable ω-scrambled set $S \subset D$.*

When we say that f satisfies *statement* (II) we mean that the second statement in the above theorem is satisfied.

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2. Shift maps satisfy statement (II)

Let Σ_2 denote the set of sequences $a_0a_1a_2\ldots$, where $a_n = 0$ or 1 for each n with the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}$$

if $x = x_0x_1\dots$ and $y = y_0y_1\dots$. Then Σ_2 is a compact metric space, and the 'one-sided shift' $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(a_0a_1a_2\dots) = a_1a_2\dots$ is continuous.

Proposition 2.1 [MH]. *There is an uncountable collection* $\{M_\alpha : \alpha \in \mathscr{I}\}$ *of pairwise disjoint minimal sets in* Σ_2 *under the one-sided shift map* σ .

Recall that if M and N are minimal sets, then either $M \cap N = \phi$ or $M = N$. For any $s \in M_\alpha$ with $s = s_0s_1s_2\dots$, define $t = t(s) \in \Sigma_2$ by

$$t = 0s_000s_0s_1000s_0s_1s_20\dots\overbrace{0\dots0}^{n \text{ many}}s_0s_1\dots s_{n-1}0\dots .$$

Let $\underline{0} = 00\dots0\dots$. Set

$$\mathscr{L} = \mathscr{L}(s) = \{x \in \Sigma_2 ;\ x = s_i\dots s_n0\dots0\dots \text{ where } n \geq i \geq 0\},$$

$$\mathscr{O} = \mathscr{O}(s) = \{x \in \Sigma_2 ;\ x = \overbrace{0\dots0}^{k \text{ many}}s_0s_1s_2\dots \text{ where } k \geq 1\}.$$

If we take a different s in M_α, we may get different $\mathscr{L}$ and $\mathscr{O}$. But the following property does not depend on the choice of s.

Lemma 2.2. $\omega(t, \sigma) = \{\underline{0}\} \cup M_\alpha \cup \mathscr{L} \cup \mathscr{O}$, *and* $\omega(t, \sigma)$ *contains only two minimal sets.*

Lemma 2.2 can be proved by carefully listing all possible choices, and using the fact that M_α is a minimal set. One can easily observe the following property.

Lemma 2.3. *Let* $s_\alpha \in M_\alpha$ *and* $s_\beta \in M_\beta$ *where* α *and* β *are distinct elements of* $\mathscr{I}$. *Then* $\omega(t(s_\alpha), \sigma) \cap M_\beta = \phi$.

Theorem 2.4. σ *satisfies statement* (II).

Proof. Define $S = \{t_\alpha ;\ t_\alpha = t(s_\alpha)$ for some $s_\alpha \in M_\alpha, \alpha \in \mathscr{I}\}$. We leave it to the reader to use the previous two lemmas to verify that S is an uncountable ω-scrambled set and $\bigcap_{x\in S}\omega(x, \sigma) \neq \phi$. □

We remark that it similarly follows that one- or two-sided shifts on any number of symbols satisfy statement (II).

3. Some properties of ω-chaos

Let X and Y be compact metric spaces. Let $f : X \to X$ and $p : Y \to Y$ be continuous maps.

Lemma 3.1. *If* f *and* p *are semiconjugate, i.e., there is a continuous onto map* $h : X \to Y$ *such that* $hf = ph$, *then* $h(\omega(x, f)) = \omega(h(x), p)$ *for each* $x \in X$.

Theorem 3.2. *If* f *is countable to one semiconjugate to* p *with semiconjugacy* $h : X \to Y$, *then* p *satisfies statement* (II) *implies that* f *satisfies statement* (II). *Also we can take an* ω*-scrambled set in* X *from the preimage under* h *of some* ω*-scrambled set in* Y .

Proof. Since p satisfies statement (II), there is an uncountable ω-scrambled set $S(p)$ in Y with $\bigcap_{y\in S(p)}\omega(y, p) \neq \phi$. Let $y_0 \in \bigcap_{y\in S(p)}\omega(y, p)$. For each $y \in S(p)$, choose one point $x = x(y) \in h^{-1}(y)$ and let $T = \{x(y) : y \in S(p)\}$. By Lemma 3.1, $\omega(x, f) \cap h^{-1}(y_0) \neq \phi$ for every $x \in T$. Since h is countable to one, there exists $x_0 \in h^{-1}(y_0)$ such that $x_0 \in \omega(x, f)$ for uncountably many

$x \in T$. Then $S(f) = \{x \in T : x_0 \in \omega(x, f)\}$ is an uncountable ω-scrambled set with $\bigcap_{y \in S(f)} \omega(y, f) \neq \phi$. □

Lemma 3.3 [C, X]. *Let n be a positive integer.*

$$\omega(x, f) = \bigcup_{i=0}^{n-1} f^i(\omega(x, f^n)) = \bigcup_{i=0}^{n-1} \omega(f^i(x), f^n),$$

and $f^i(\omega(x, f^n)) = \omega(f^i(x), f^n)$ for each $i = 0, 1, 2, \dots, n-1$.

Let $X_0, X_1, \dots, X_{n-1}$ be compact subspaces of X, which are invariant under f^n. Suppose $f : X \to X$ is a continuous map such that $f(X_i) \subset X_{i+1(\mathrm{mod}(n))}$ for each $i = 0, 1, \dots, n-1$. Then we have the following property.

Lemma 3.4. *Let $\mathscr{M}_i$ denote the collection of f^n-minimal sets of X_i. Then f^{j-i} sets up a one-to-one correspondence between $\mathscr{M}_i$ and $\mathscr{M}_j$ for given positive integers i and j with $i < j$.*

Theorem 3.5. *Suppose that f^m satisfies statement* (II), *and let $S(f^m)$ be an ω-scrambled set as in statement* (II). *Suppose also that for any $x \in S(f^m)$ the following conditions are satisfied.*

(1) *$\omega(x, f^m)$ contains a finite, nonzero number of infinite minimal sets.*

(2) *$\omega(x, f^m)$ contains only countably many points which are not in these minimal sets.*

Then f satisfies statement (II).

Proof. For any $x \in S(f^m)$ and any f^m-minimal set M either $M \subset \omega(x, f)$ or $M \cap \omega(x, f) = \phi$. By the above two lemmas and hypothesis 1), $\omega(x, f)$ contains only finitely many f^m-minimal sets. Since $S(f^m)$ is uncountable, there exists an uncountable subset $S_1(f^m)$ such that $\omega(x, f)$ contains the same number of f^m-minimal sets for each $x \in S_1(f^m)$.

For $x, y \in S_1(f^m)$, say $x \sim y$ if $\omega(x, f)$ and $\omega(y, f)$ contain the same f^m-minimal sets. It is easy to see that this is an equivalence relation. Note that for distinct x and y $\omega(x, f^m) \setminus \omega(y, f^m)$ contains an infinite f^m-minimal set by Definition 1.2 and hypothesis (2). Thus each equivalence class is finite. Let $S(f)$ be a subset of $S_1(f^m)$ which contains exactly one representative of each equivalence class. Then $S(f)$ is uncountable. Also for any pair of distinct points $x, y \in S(f)$, $\omega(x, f) \setminus \omega(y, f)$ contains an infinite minimal set and hence is uncountable. We leave the rest of the proof to the reader. □

4. Proof of Theorem

In this section, we let I denote a compact interval, and we suppose that $f : I \to I$ is a continuous map. Let $C(2^\infty)$ denote the set of maps f with no periodic points of periods not a power of two. Let P, AP, R, and Λ denote the sets of periodic points, almost periodic points, recurrent points, and ω-limit points, respectively. Let $\Lambda^2 = \bigcup_{x \in \Lambda} \omega(x, f)$.

Proposition 4.1 [X1, S]. *$\omega(x, f)$ contains only one minimal set for $f \in C(2^\infty)$.*

Proposition 4.2 [X2]. *$\Lambda \setminus \Lambda^2$ is countable for any continuous map $f : I \to I$.*

Proposition 4.3 [X2]. *$\Lambda^2 = R = AP$ for $f \in C(2^\infty)$.*

Lemma 4.4. *For $f \in C(2^\infty)$ and $x \in I$, if $\omega(x) \subset \Lambda^2$, then $\omega(x)$ is a minimal set.*

Proof. By Proposition 4.1, $\omega(x)$ contains a unique minimal set M. For any $y \in \omega(x)$, we have $\omega(y) \subset \omega(x)$, since $\omega(x)$ is a closed invariant set. Then $y \in \omega(x) \subset \Lambda^2 = AP$ by Proposition 4.3. Thus $\omega(y)$ is a minimal set, and $y \in \omega(y)$. Hence $M = \omega(y)$ and $y \in M$. Since y was arbitrary, $\omega(x) = M$. □

Proposition 4.5 [Bl]. *If f has zero entropy, then $f \in C(2^\infty)$.*

Proposition 4.6. *Let y and z be distinct points of I. If $\{y, z\}$ is an ω-scrambled set, then f has positive entropy.*

Proof. Suppose that f does not have positive entropy. Then, by Proposition 4.5, $f \in C(2^\infty)$.

Suppose one of $\omega(y)$ and $\omega(z)$ is contained in Λ^2. By Lemma 4.4, if $\omega(y)$ and $\omega(z)$ have nonempty intersection, then one is contained in the other. This contradicts the definition of an ω-scrambled set. So, both $\omega(y) \cap (\Lambda \setminus \Lambda^2)$ and $\omega(z) \cap (\Lambda \setminus \Lambda^2)$ are nonempty.

By Proposition 4.1, $\omega(y)$ contains a unique minimal set $M(y)$, and $\omega(z)$ contains a unique minimal set $M(z)$. It follows from Proposition 4.3 that $\omega(y) \cap \Lambda^2 = M(y)$ and $\omega(z) \cap \Lambda^2 = M(z)$.

From the second condition of Definition 1.2, we know that $\omega(y) \cap \omega(z) \neq \phi$. Let $u \in \omega(y) \cap \omega(z)$. Then $M(y) = \omega(u) = M(z)$ by Proposition 4.1 and Lemma 4.4. Thus, $\omega(y) \cap \Lambda^2 = M(y) = M(z) = \omega(z) \cap \Lambda^2$, and $\omega(y) \setminus \omega(z) \subset \Lambda \setminus \Lambda^2$.

By Proposition 4.2, $\Lambda \setminus \Lambda^2$ is countable, and hence $\omega(y) \setminus \omega(z)$ is countable. This contradicts the definition of an ω-scrambled set. Therefore f has positive entropy. □

Proposition 4.7 [C]. *If f has positive entropy then there exists a closed set $X \subset I$ and $m > 0$ such that $f^m(X) = X$ and $f^m|_X$ is at most two-to-one semiconjugate to the one-sided shift map σ. Furthermore, there are only countably many points in Σ which have 2 preimages, and if one of the preimages is periodic, then so is the other.*

The last statement of this proposition is not included in [C], but is easily observed.

Proposition 4.8 [D]. *The one-sided shift map is chaotic in the sense of Devaney with a chaotic set Σ_2.*

Using Propositions 4.7 and 4.8, we can easily verify the following result.

Proposition 4.9. *If $f : I \to I$ has positive entropy, then there is a positive integer m such that f^m is chaotic on I in the sense of Devaney.*

Proof. Let $m > 0$ and $X \subset I$ be as in Proposition 4.7. By Proposition 4.8, $D(\sigma) = \Sigma_2$ is a chaotic set. Let $s \in D(\sigma)$ satisfy $\overline{\mathrm{Orb}(s, \sigma)} = D(\sigma)$. Let $x \in X$ be a preimage of s under the semiconjugacy in Proposition 4.7, and let $D(f^m) = \overline{\mathrm{Orb}(x, f^m)}$. Then $D(f^m) \subset X$, and $D(f^m)$ contains at least one preimage of each point in $D(\sigma)$. It is not hard to show that the periodic points in $D(f^m)$ are dense in $D(f^m)$ and $f^m|_{D(f^m)}$ has sensitive dependence on initial conditions. Thus $D(f^m)$ is a chaotic set for f^m. □

The reader can verify that the following proposition holds for a continuous map on a compact metric space.

Proposition 4.10. *If, for some* $m > 0$, f^m *is chaotic in the sense of Devaney, then* f *is also chaotic in the sense of Devaney. Furthermore, if* $D(f^m)$ *is a chaotic set for* f^m, *then* $\bigcup_{i=0}^{m-1} f^i(D(f^m))$ *is a chaotic set for* f.

Proof of the theorem. (II) $\Rightarrow$ (III) and (III) $\Rightarrow$ (IV) are obvious. (IV) $\Rightarrow$ (I) is proved in Proposition 4.6.

Now let us prove that (I) $\Rightarrow$ (II). By Proposition 4.7, there is a closed subset X of I and a positive integer m such that $f^m(X) = X$, and $f^m|_X$ is at most two-to-one semiconjugate (via a semiconjugacy h) to the one-sided shift map σ. Also, there are at most countably many points in Σ_2 which have two preimages under h. By Theorem 2.4, σ satisfies statement (II), so, by Theorem 3.2, $f^m|_X$ satisfies the statement (II). Let $S(\sigma)$ be the ω-scrambled set constructed in Theorem 2.4, and $S(f^m)$ be the ω-scrambled set constructed in Theorem 3.2. Let $x \in S(f^m)$, and let $h(x) = s \in S(\sigma)$. Then $\omega(s, \sigma)$ contains a unique infinite minimal set M, and there are only countably many points of $\omega(s, \sigma)$ not in this minimal set. Since $h^{-1}(M)$ is a closed invariant set (as $h(f^m(h^{-1}(M))) = \sigma(h(h^{-1}(M))) = \sigma(M) = M$ implies that $f^m(h^{-1}(M)) \subset h^{-1}(M)$), $h^{-1}(M)$ must contain a minimal set $\widetilde{M}$. Then h maps $\widetilde{M}$ onto M since $h(\widetilde{M})$ is a closed, invariant subset of M, and hence, $\widetilde{M}$ is infinite. Because there are only countably many points in Σ_2 which have two preimages and $\omega(s, \sigma)\backslash M$ is countable, we have that $\omega(x, f^m)\backslash\widetilde{M}$ is also countable. Since, x was arbitrary, the hypothesis of Theorem 3.5 is satisfied, and hence, statement (II) holds.

(VI) $\Rightarrow$ (V) is obvious. (V) $\Rightarrow$ (I) follows from Propositions 4.1 and 4.5.

It remains to show that (I) $\Rightarrow$ (VI). Suppose that f has positive topological entropy. By Proposition 4.9 there is an integer $m > 0$ such that f^m is chaotic in the sense of Devaney. Let $D(f^m)$ be a chaotic set for f^m as in the proof of Proposition 4.9. Set

$$D(f) = \bigcup_{i=0}^{m-1} f^i(D(f^m)).$$

By Proposition 4.10, $D(f)$ is a chaotic set for f. Clearly $D(f^m) \subset D(f)$.

Let $S(\sigma)$ be the ω-scrambled set for σ constructed in Theorem 2.4. Let $S(f^m)$ be the collection of the preimages, under the semiconjugacy as in Proposition 4.7, of the points in $S(\sigma)$ which have unique preimages. Since there are only countably many points in Σ_2 which have two preimages, $S(f^m)$ must be uncountable. Using Theorem 3.2 and its proof, it is easy to see that $S(f^m)$ is an ω-scrambled set for f^m.

By the proof of Proposition 4.9, $D(f^m)$ contains at least one preimage of each point in Σ_2. Since each point in $S(f^m)$ is the unique preimage of some point in $S(\sigma)$, $S(f^m) \subset D(f^m)$. Let $S(f)$ be the ω-scrambled set for f constructed as in Theorem 3.5. Then

$$S(f) \subset S(f^m) \subset D(f^m) \subset D(f).$$

This completes the proof. □

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