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$0(h^{2n+2-\ell})$ Bounds on Some Spline Interpolation Errors

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by

Blair Swartz

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ABSTRACT

Certain spline interpolations (of odd degree) to smooth functions at uniformly spaced joints are considered. Certain cubic spline interpolations at arbitrarily spaced joints are also discussed. Good error bounds are obtained in all cases, for the errors are essentially those of a local two-point Polya interpolation.

Introduction.

It had been felt for perhaps three or four years that C^{2n}, $2n + 1$ degree polynomial spline interpolation of a sufficiently smooth function would yield $O(h^{2n+2-\ell})$ accuracy in approximating its ℓ^{th} derivative. Ahlberg, et al.[1] (p. 151) showed this to be so for $n \geq 1$, for equally spaced joints, and under periodic boundary conditions (see also Theorem 9). We show a bit more under two other boundary conditions: the error in spline interpolation is bounded by the error in a local two-point interpolation (of a sort we call Polya interpolation) plus a higher order term (Theorems 1.1 and 1.2). These higher order terms are zero when interpolating a polynomial of degree $2n + 2$.

In spite of the lack of generality of the results, we feel that they may be of some use in practical problems and hope that the methods may aid in arriving at more general conclusions.

The foundations of the proofs arose from a study of cubic spline interpolation for variably spaced joints; the numerical bounds which resulted from that study are presented in Section 10. The role of the mesh ratio is also discussed (and down-graded somewhat). Some of the bounds are sharp, in the sense mentioned in Theorems 1.1 and 1.2. Bounds of this sort for the cubics were first disclosed by Birkhoff and de Boor[2] and improved by Sharma and Meir.[3] Our assumption of four or five derivatives has yielded bounds which are one to two orders of magnitude smaller than those of Sharma and Meir,[3] one better than those of Nord,[4] and two or more better than those in Ref. 1, p. 32. We have not compared our numbers with the $O(h^{2n+2-\ell})$ bounds associated with spline approximation by moments.[5,6]

While this was being written, it was pointed out to us that $O(h^{2n+2-\ell})$ error bounds for periodic spline interpolation of odd degree follow immediately from Theorem 4 of Subbotin.[7] We feel that the constants derivable from that paper lie between those in Ref. 1 and the bounds that follow here. It should be pointed out, however, that Subbotin's Theorems 1 and 2 consider the C^{2n-1}, $2n$ degree spline interpolations of Schoenberg[8,9] (which interpolate <u>between</u> the joints), indicating error bounds from which $O(h^{2n+1-\ell})$ bounds follow for sufficiently smooth periodic functions. We have no comparable results for this case. The argument sharpening Theorem 2 of Subbotin[7] is presented in the last paragraph of this paper. We must also add that the results of our Sections 7 and 8 may be found in Subbotin[10] as well, but we feel that our proofs may be sufficiently succinct to warrent inclusion here.

We have assumed many derivatives of the function interpolated; it seems likely, however, that Sharma and Meir[3] can now be extended using lower order Green's functions and Polya interpolations (for example, Corollaries 9.1 and 9.2 consider reasonably rough functions).

The local basis of the vector space of splines (with given joints) is discussed in Section 5. We there take the opportunity to comment on its great usefulness in practical problems, a usefulness of which some numerical analysts may be unaware.

Because this is, in a sense, a practical paper, we apologize for not supplying more numbers in the place of letters. The numbers, particularly those associated with the $\mathcal{L}_2$ norm, can be of much use in practice; see, for example, Birkhoff et al.[11] which this paper now extends (in theory) to obtaining $O(h^{4n+2})$, two-sided bounds on Sturm-Liouville eigenvalues with the numerical solution of one $(4n + 3)$-diagonal $1/h \times 1/h$ matrix eigenvalue problem. Although we have not presented the numbers concerning the strict $\mathcal{L}_2$ bounds here, we hope we have clearly indicated how to compute them. We do indicate their asymptotic form for the boundary conditions of Theorem 1.1 (see Example 3.4).

We do not know how to obtain bounds for boundary conditions other than those mentioned in theorems 1.1, 1.2, and 9.

In an Appendix we discuss a stable numerical method for calculating the interpolating splines.

Finally, we note that some of the results and proofs of this paper are summarized in Swartz,[12] where certain elementary corollaries concerning the asymptotic behavior of the errors are also proven.

1. Notation and Main Results.

Unless otherwise stated, the following notation is observed. "Joints" means the numbers $x_i = i/N$, $0 \leq i \leq N$; $h = 1/N$. $C^k[a,b]$ is the set of functions, f, such that $f^{(\ell)}$, $\ell \leq k$, is continuous on $[a,b]$. $\|f\|_{p,[a,b]}$ means $\left[\int_a^b |f|^p/(b-a)\right]^{1/p}$, $1 \leq p \leq \infty$; $\|f\|_{\infty,[a,b]} = \max_{x\in[a,b]} |f|$; $[a,b]$ missing implies $[a,b] = [0,1]$. A spline, s, is a function which is a polynomial of degree $m = 2n + 1$ between the joints, $n \geq 1$, ($m = 2n$ as well in Section 5) such that $s \in C^{m-1}[0,1]$. f_i is $f(x_i)$. H is a function which is a polynomial of degree m between the joints such that H_i and $H_i^{(\ell)}$ match f_i and $f_i^{(\ell)}$ for some ℓ's and all i (see Section 2); H is referred to as a Polya interpolation of f and some of its derivatives. "s interpolates f" means $s_i = f_i$, $i = 0,\ldots,N$. $d \equiv s - H$, $e_H \equiv H - f$; thus $e \equiv s - f = e_H + d$.

Theorem 1.1. Let s be the spline of degree $2n + 1$, $n \geq 1$, interpolating $f \in C^{2n+2}[0,1]$ and matching its first n odd derivatives at 0 and 1 as well. Let $N \geq 2n + 1$. Let H, a polynomial of degree $2n + 1$ between the joints, interpolate f_i, $f_i^{(2j-1)}$, $1 \leq j \leq n$, $0 \leq i \leq N$. Then there are constants $K_{\ell,n}^{(1)}$ and $G_{\ell,n}^{(1)}$ such that, for $0 < \ell < 2n + 1$,

$$\|e_H^{(\ell)}\|_\infty \equiv \|H^{(\ell)} - f^{(\ell)}\|_\infty \leq h^{2n+2-\ell}\, G_{\ell,n}^{(1)}\, \|f^{(2n+2)}\|_\infty, \tag{1.1}$$

and

$$\|d^{(\ell)}\|_\infty \equiv \|s^{(\ell)} - H^{(\ell)}\|_\infty \leq h^{2n+2-\ell}\, K_{\ell,n}^{(1)}\, \|f^{(2n+2)}\|_\infty. \tag{1.2}$$

If $f \in C^{2n+3}[0,1]$, there are $K_{\ell,n}^{(2)}$ such that

$$\|d^{(\ell)}\|_\infty \leq h^{2n+3-\ell}\, K_{\ell,n}^{(2)}\, \|f^{(2n+3)}\|_\infty. \tag{1.3}$$

If f is a polynomial of degree $2n + 2$, $H \equiv s$.

Analogous results hold, with better constants, using $\|\cdot\|_p$.

Proof. See Section 9 and Example 3.1. The numbers $G_{\ell,1}^{(1)}$, $K_{\ell,1}^{(1)}$, and $K_{\ell,1}^{(2)}$ occur in Table I, Section 10. By example 3.3, $G_{0,n}^{(1)}$ is sharp in the sense that there is no $K < G_{0,n}^{(1)}$ such that $\|f - s\|_\infty \leq K\, h^{2n+2} \|f^{(2n+2)}\|_\infty$ for all $f \in C^{2n+3}[0,1]$. The asymptotic form of the maximum norm of the error is discussed in Swartz.[12] The asymptotic form of the $\mathcal{L}_2$ norm is described in Example 3.4.

Theorem 1.2. Let s be the spline of degree $2n + 1$, $n \geq 1$, interpolating $f \in C^{2n+2}[0,1]$ and matching its first n even derivatives at 0 and 1 as well. Let

$N \geq 2n + 1$. Let H, a polynomial of degree $2n + 1$ between the joints, interpolate f_i, $f_i^{(2j)}$, $1 \leq j \leq n$, $0 \leq i \leq N$. Then there are constants $K_{\ell,n}^{(3)}$ and $G_{\ell,n}^{(2)}$ such that, for $\ell = 0,1,\ldots,2n+1$,

$$\|e_H^{(\ell)}\|_\infty \leq h^{2n+2-\ell}\ G_{\ell,n}^{(2)}\ \|f^{(2n+2)}\|_\infty,$$

and

$$\|d^{(\ell)}\|_\infty \leq h^{2n+2-\ell}\ K_{\ell,n}^{(3)}\ \|f^{(2n+2)}\|_\infty.$$

If ϕ is defined by $\phi(x) = (-1)^i$, $x \in (x_{i-1}, x_i)$, $i = 1,\ldots,N$; and if $f \in C^{2n+1}[0,1]$ satisfies $f^{(2n+2)} - a\phi \in C^1[0,1]$ for some number, a, while $-f^{(2n+2)}(0) = (-1)^N f^{(2n+2)}(1) = a$; then there are constants $K_{\ell,n}^{(4)}$ such that

$$\|d^{(\ell)}\|_\infty \leq h^{2n+3-\ell}\ K_{\ell,n}^{(4)}\ \|(f^{(2n+2)} - a\phi)'\|_\infty.$$

If $f^{(2n+2)} \equiv a\phi$, $H \equiv s$. Analogous results hold, with better constants, for $\|\cdot\|_p$.

Proof. See Section 9 and Example 3.2. The numbers $G_{\ell,1}^{(2)}$ and $K_{\ell,1}^{(3)}$ occur in Table I, Section 10. By Example 3.3, the numbers $G_{2k,n}^{(2)}$ are sharp in the following sense: considering only even N, let $\mathcal{F}_e$ be the set of functions f such that $f \in C^{2n+1}[0,1]$ and there exists a number, a, such that (1) $f^{(2n+2)} - a\phi \in C^1[0,1]$, and (2) $-f^{(2n+2)}(0) = (-1)^N f^{(2n+2)}(1) = a$. Then, given k, $0 \leq k \leq n$, there is no $K < G_{2k,n}^{(2)}$ such that $\|(f - s)^{(2k)}\|_\infty \leq K\, h^{2n+2-2k}\, \|f^{(2n+2)}\|_\infty$ for all $f \in \mathcal{F}_e$. The same result holds for odd N with respect to the class $\mathcal{F}_o$ defined in the same way.

Example 1. Cubic spline interpolation on [0,1] with N interior joints x_i such that $x_{i+1} - x_i = \delta$, $i = 1,2,\ldots,N-1$ spaced so that $x_1 = \delta^2 = 1 - x_N$, to a function $f \in C^5[0,1]$, matching f and f" at 0 and 1, is almost the same as cubic Hermite interpolation to f_i, f_i', $1 \leq i \leq N$, $f(0)$, $f'(0)$, $f(1)$, $f'(1)$; for the difference between the ℓ^{th} derivatives of the two interpolations may be bounded by $\delta^{5-\ell} K_\ell \|f^{(5)}\|_\infty$ except in the two end intervals where $|d^{(2)}| = O(\delta^2)$, $|d^{(3)}| = O(1)$. The proof follows from Section 10 and is omitted. A similar example for a higher order spline is unknown.

In Section 3 we begin a sequence of lemmas leading up to the proofs of the theorems. Results will be stated only for the maximum norm for clarity; extensions to other norms and further remarks are made in comments which follow each lemma. Section 3 bounds the norm of the derivatives of f - H. Sections 5 through 8 bound s - f and its derivatives at the joints. Section 9 then bounds the norm of the derivatives of the piecewise polynomial H - s, and Theorems 1.1 and 1.2 are proved. Periodic spline interpolation is then considered as well, and Theorem 9 develops some error estimates for all three spline types simultaneously.

2. Existence and Uniqueness.

Regarding Polya interpolation, we shall have reason to refer to the following definition: the set of integers $[\ell(i,j)]$ satisfying

$$\left.\begin{array}{l} 1 \leq \ell(0,1) < \ldots < \ell(0,j) < \ldots \leq m,\ 1 \leq j \leq j(0); \\ 1 \leq \ell(1,1) < \ldots < \ell(1,j) < \ldots \leq m,\ 1 \leq j \leq j(1); \\ 0 \leq j(i),\ i = 0,\ 1;\ \ j(0) + j(1) = m - 1, \end{array}\right\} \tag{2.1}$$

are said to be a Polya set if, and only if, given any m + 1 numbers a_j, $j = 0,\ldots,j(0)$; b_j, $j = 0,\ldots,j(1)$, there exists a unique polynomial, P, of degree m satisfying $P(0) = a_o$, $P^{(\ell(0,j))}(0) = a_j$, $1 \leq j \leq j(0)$; $P(1) = b_o$, $P^{(\ell(1,j))}(1) = b_j$, $1 \leq j \leq j(1)$.

If $[\ell(i,j)]$ is a Polya set, the type of polynomial interpolation it describes can be performed on [a,b] as well as on [0,1], and there exists a unique Green's function for the boundary value problem (See Ince,[13] pp. 254-255)

$$\left.\begin{array}{l} y^{(m+1)} \equiv 0 \text{ in } (0,h), \\ y(0) = 0,\ y^{(\ell(0,j))}(0) = 0,\ 1 \leq j \leq j(0), \\ y(h) = 0,\ y^{(\ell(1,j))}(h) = 0,\ 1 \leq j \leq j(1). \end{array}\right\} \tag{2.2}$$

For example, the set $[(i,j)]$; $j = 1,\ldots,n = j(i)$; $i = 0, 1$ is a Polya set since it describes ordinary two-point Hermite interpolation. (See Wendroff,[14] pp. 1-7). The following sets, occurring in Theorems 1.1 and 1.2, are Polya sets.

$$(i,\ 2j - 1);\ j = 1,\ldots,n = j(i);\ i = 0,\ 1, \tag{2.3}$$

$$(i,\ 2j);\ j = 1,\ldots,n = j(i);\ i = 0,\ 1. \tag{2.4}$$

For, in the language of Schoenberg[15], $[\ell,(i,j)]$ is a Polya set if, and only if, it can be converted, in an obvious way, into the "incidence matrix" (with 1's in its first column) for a "poised two-point Hermite-Birkhoff" interpolation problem. A necessary and sufficient condition that $[\ell,(i,j)]$ is a Polya set was determined by Polya and is given in Ref. 15, p. 540: for each ℓ, $1 \leq \ell \leq m-1$, ℓ should be less than two plus the number of $\ell(i,j) \in [\ell,(i,j)]$ which do not exceed ℓ. Equations 2.3 and 2.4 satisfy this criterion.

For the purposes of this paper we have referred to this type of Hermite-Birkhoff interpolation as "Polya" interpolation; this serves to emphasize that we only consider two points per polynomial.

The existence and uniqueness of the spline interpolations in Theorems 1.1 and 1.2 follow from the existence and uniqueness of periodic spline interpolation[16] (however, it also follows from Lemma 8). The spline interpolation in Theorem 1.1 can be constructed as follows: because Eq. 2.3 is a Polya set, we may assume that f and its first n odd derivatives vanish at 0 and 1. Reflect f as an even function in [-1,1] and extend it by periodicity of period 2. Interpolate f at the joints (and their reflections) in [-1,1] with a (unique) periodic spline of period 2. This spline and its first n odd derivatives vanish at x = 0 and x = 1. The existence of the spline interpolation of Theorem 1.2 follows similarly from Eq. 2.4 using the odd reflection of f.

3. Bounding the Polya Part of the Error. It is well known that ordinary two-point 2n + 1 degree Hermite interpolation to $f \in C^{2n+2}[0,1]$ and its first n derivatives at all the joints gives an error, e_H, whose ℓ^{th} derivative is $O(h^{2n+2-\ell})$.

Lemma 3. Let $[\ell(i,j)]$ be a Polya set. Let $f \in C^{m+1}[0,h]$ be given. Let P be the polynomial accomplishing the Polya interpolation of f at 0 and h described by $[\ell(i,j)]$. Then there are constants G_ℓ, independent of h but dependent on $[\ell(i,j)]$, such that, for $0 \leq \ell \leq m$,

$$\|P^{(\ell)} - f^{(\ell)}\|_{\infty,[0,h]} \leq G_\ell \, h^{m+1-\ell} \, \|f^{(m+1)}\|_{\infty,[0,h]}. \tag{3.1}$$

The G_ℓ are defined in Eq. 3.2.

Proof. f - P solves the boundary value problem $y^{(m+1)} = f^{(m+1)}$ with the boundary conditions of Eq. 2.2. Let $G_h(x,t)$ be the Green's function for Eq. 2.2. Then $(f - P)(x) = \int_0^h G_h(x,t) f^{(m+1)}(t)dt$. With G the Green's function for h = 1, $G_h(x,t) = h^m G(x/h, t/h)$. Equation 3.1 follows with

$$G_\ell = \max_{x\in[0,1]} \int_0^1 \left| \frac{\partial^{(\ell)} G}{\partial x^\ell}(x,t) \right| dt. \tag{3.2}$$

Corollary 3. Suppose only one type of interpolation between the joints is used in forming H(x), and suppose $f \in C^{m+1}[0,1]$. Then

$$\|e_H^{(\ell)}\|_\infty \leq h^{m+1-\ell} \, G_\ell \, \|f^{(m+1)}\|_\infty. \tag{3.3}$$

Proof. Immediate from Lemma 3.

Comment. The result for $\|e_H\|_p$ in terms of $\|f^{(m+1)}\|_p$, $1 \leq p \leq \infty$ is similar:

$$\|e_H\|_p \leq h^{m+1-\ell} \, G_{\ell,p} \, \|f^{(m+1)}\|_p, \quad 1 \leq p \leq \infty,$$

where

$$G_{\ell,p} = \left\{ \int_0^1 \left[\int_0^1 \left| \frac{\partial^{(\ell)} G}{\partial x^\ell}(x,t) \right|^{p/(p-1)} dt \right]^{p-1} dx \right\}^{1/p}, \quad p > 1;$$

$$G_{\ell,1} = \int_0^1 \max_{t\in[0,1]} \left| \frac{\partial^{(\ell)} G}{\partial x^\ell}(x,t) \right| dx, \quad p = 1.$$

Example 3.1. We now show that spline interpolation of degree 2n + 1 to a polynomial, P, of degree 2n + 2, matching the first n odd derivatives of P at 0 and 1, is the same as 2n + 1 degree Polya interpolation (between the equally spaced joints) to P_i and $P_i^{(\ell)}$, $\ell = 1, 3, \ldots, 2n-1$; $0 \leq i \leq N$. It suffices to show that $e_H \in C^{2n}[0,1]$. Because of the equal joint spacing this follows if

$$\int_0^1 \frac{\partial G^{(2k)}}{\partial x^{2k}}(0,t)dt = \int_0^1 \frac{\partial G^{(2k)}}{\partial x^{2k}}(1,t)dt, \quad 1 \leq k \leq n,$$

where G is the Green's function for the boundary value problem on [0,1]. The first integral is $y^{(2k)}(0)$; the second, $y^{(2k)}(1)$; where $y^{(2n+2)} = 1$ in (0,1) with y and its first n odd derivatives vanishing at 0 and 1. But $y(1-x)$ also solves this boundary value problem. By uniqueness, $y(x) \equiv y(1-x)$.

Example 3.2. Similarly, if $f^{(2n+2)}(x) = \phi(x)$ (defined in Theorem 1.2), then spline interpolation of degree 2n + 1 to f, matching its first n even derivatives at 0 and 1, is the same as Polya interpolation of degree 2n + 1, between the joints, to f_i and $f_i^{(2k)}$, k = 1,...,n. Indeed, $e_H \in C^{2n+1}[0,1]$ which implies that H, and the spline, is a polynomial.

Example 3.3. The Green's functions for Eq. 2.2 with m = 2n + 1, under the boundary conditions specified by either Eq. 2.3 or 2.4, do not change sign in the square which is their domain of definition; even more is true under Eq. 2.4 conditions. The proof uses Rolle's theorem and is given for Eq. 2.3 conditions. For fixed interior $t = t_o$, $\partial^{(2n)}G/\partial x^{2n}$ is piecewise linear and continuous with a jump of 1 in its derivative at $x = t_o$. Hence it vanishes at no more than two interior points. Thus $\partial^{(2n-1)}G/\partial x^{2n-1}$, which vanishes at the ends, has at most one interior zero. Consequently $\partial^{(2n-2)}G/\partial x^{2n-2}$ vanishes at no more than two interior points, etc. Thus $\partial G/\partial x$ vanishes at no more than one interior point, and $G(x,t_o)$ has no interior zeroes (since it vanishes at the ends). By continuity, G does not change sign in the square.

Under Eq. 2.4 conditions we may show similarly that $\partial^{(2k)}G/\partial x^{2k}$ does not change sign in the square, $0 \leq k \leq n$.

Thus, if $f^{(2n+2)}$ is constant in (0,h), $e_H(x)$ attains its bound, Eq. 3.1, under Eq. 2.3 boundary conditions. Similarly, under Eq. 2.4 conditions, $e_H^{(2k)}(x)$ attains its bound (Eq. 3.1), $0 \leq k \leq n$.

Example 3.4 (April, 1968). The asymptotic form of the maximum norm of the error, for periodic boundary conditions, is indicated in Swartz,[12] Corollary 3, as

$$\|e\|_{\infty,i} = h^{2n+2-\ell}\left\{\|f^{(2n+2)}\|_{\infty,i}\, C_{n,\infty} + o\left[\omega(f^{(2n+2)},h)\right]\right\},$$

$$\|e^{(\ell)}\|_{\infty,i} = h^{2n+2-\ell}\left\{\|f^{(2n+2)}\|_{\infty,i}\, D_{2n+2-\ell,\infty} + o\left[\omega(f^{(2n+2)},h)\right]\right\}. \tag{3.4}$$

Here

$$\omega(f^{(2n+2)},h) \equiv \sup_{|x-y|\leq h} |f^{(2n+2)}(x) - f^{(2n+2)}(y)|,$$

while

$$C_{k,\infty} = 2|B_{2k+2}|(1 - 1/2^{2k+2})/(2k+2)! < 2D_{2k+2,\infty},$$

$$D_{k,\infty} = \begin{Bmatrix} |B_k|/k! & , k \text{ even} \\ \|B_k(x)\|_\infty/k!, & k \text{ odd}\end{Bmatrix} < 2/[(2\pi)^k(1 - 2^{1-k})].$$

$B_k[B_k(x)]$ is the k^{th} Bernoulli number [polynomial].

To find corresponding numbers relating the $\mathcal{L}_2$ norms, we first recall from Swartz,[12] Eq. 4, that for any x and y in [0,h]

$$e_H^{(\ell)}(x) = h^{2n+2-\ell}\left\{f^{(2n+2)}(y)Q_{2n+2}^{(\ell)}(x/h) + o\left[\omega(f^{(2n+2)},h)\right]\right\}, \tag{3.5}$$

where $Q_{2n+2}(t) \equiv [B_{2n+2}(t) - B_{2n+2}]/(2n+2)!$. (Thus $Q_{2n+2}^{(\ell)}(t) \equiv B_{2n+2-\ell}(t)/(2n+2-\ell)!$.) Squaring (3.5) and picking y such that

$$[f^{(2n+2)}(y)]^2 = \int_0^h \left[f^{(2n+2)}(t)\right]^2 dt/h \equiv \|f^{(2n+2)}\|_{2,[0,h]}^2,$$

we see, upon integration between 0 and h, that

$$\|e_H^{(\ell)}\|_{2,[0,h]}^2 = h^{4n+4-2\ell}\left\{\|f^{(2n+2)}\|_{2,[0,h]}^2\, \|Q_{2n+2}^{(\ell)}\|_2^2 + o\left[\omega(f^{(2n+2)},h)\right]\right\}.$$

Since similar results hold between each pair of adjacent joints, we multiply by h and add them all up, obtaining

$$\|e_H^{(\ell)}\|_2^2 = h^{4n+4-2\ell}\left\{\|f^{(2n+2)}\|_2^2\, \|Q_{2n+2}^{(\ell)}\|_2^2 + o\left[\omega(f^{(2n+2)},h)\right]\right\}.$$

Now $(s - H)^{(\ell)} = O[h^{2n+2-\ell}\omega(f^{(2n+2)},h)]$ (see Swartz[12]). Thus we have, in analogy to Eq. 3.4, for odd derivative or periodic boundary conditions:

$$\|e\|_2^2 = h^{4n+4}\left\{\|f^{(2n+2)}\|_2^2(C_{n,2})^2 + o\left[\omega(f^{(2n+2)},h)\right]\right\},$$

$$\|e^{(\ell)}\|_2^2 = h^{4n+4-2\ell}\left\{\|f^{(2n+2)}\|_2^2\,(D_{2n+2-\ell,2})^2 + o\left[\omega(f^{(2n+2)},h)\right]\right\};$$

where

$$(C_{k,2})^2 = |B_{4k+4}|/(4k+4)! + B_{2k+2}^2/[(2k+2)!]^2,$$

$$(D_{k,2})^2 = |B_{2k}|/(2k)! < 2/(2\pi)^{2k}.$$

We note that $D_{k,\infty} \approx \sqrt{2}\, D_{k,2}$, while $C_{k,\infty} \approx \sqrt{8/3}\, C_{k,2}$.

4. Bounding Polynomials in Terms of Bounds on Some Derivatives at the Ends of an Interval.

If the spline interpolating f, and the function H formed from local Polya interpolation, are both of degree 2n + 1, then $d = s - H$ is piecewise polynomial of degree 2n + 1 between the joints. Sections 4 through 9 are concerned with bounding d. Section 9 exhibits bounds on certain derivatives of d at the joints, and the following result will be needed.

Lemma 4. Let $\mathcal{P}_h$ be the (m-1)-dimensional real vector space of polynomials of degree $\le m$ which vanish at 0 and h, with the topology induced from C[0,h]. Let $[\ell,(i,j)]$ be a Polya set (Section 2), and let m-1 positive numbers A(i,j), $1 \le j \le j(i)$, i = 0, 1 be given as well. Define a parallelepiped

$$\mathcal{B}_h = \left\{ \begin{array}{l} P \in \mathcal{P}_h \text{ such that} \\ \quad |P^{(\ell(0,j))}(0)| \le A(0,j)/h^{\ell(0,j)} \\ \quad \text{for } 1 \le j \le j(0), \text{ while} \\ \quad |P^{(\ell(1,j))}(h)| \le A(1,j)/h^{\ell(1,j)} \\ \quad \text{for } 1 \le j \le j(1). \end{array} \right\} \tag{4.1}$$

Then there are constants, B_ℓ, $0 \le \ell \le m$, (depending on ℓ, $[\ell(i,j)]$, and $[A(i,j)]$, but not on h) such that

$$\max_{P \in \mathcal{B}_h} \|P^{(\ell)}\|_{\infty,[0,h]} = B_\ell/h^\ell.$$

For computation, we note that B_ℓ is attained at a vertex of $\mathcal{B}_1$.

Proof: Let q be the continuous seminorm, $q(P) = \|P^{(\ell)}\|_{\infty,[0,h]}$, $\ell \ge 0$. The map $\mathcal{J}$: $\mathcal{P}_h \to \mathcal{P}_1$ defined by $\mathcal{J}(P)(x) = P(hx)$ is an isomorphism, and $\left[d^{(\ell)}(\mathcal{J}(P))/dx^\ell\right](x) = h^\ell \left[d^{(\ell)}P/dx^\ell\right](hx)$. Thus $\mathcal{J}(\mathcal{B}_h) = \mathcal{B}_1$, and $P_M \in \mathcal{B}_1$ maximizes q over $\mathcal{B}_1$ if, and only if, $\mathcal{J}^{-1}(P_M)$ maximizes q over $\mathcal{B}_h$. Setting $B_\ell = M = q(P_M)$, we have $q\left[\mathcal{J}^{-1}(P_M)\right] = B_\ell/h^\ell$. It remains to show that q attains B_ℓ at a vertex of $\mathcal{B}_1$.

Let $\psi_{ij} \in \mathcal{P}_1$ be the basis for $\mathcal{P}_1$ defined by

$$\left.\begin{array}{l} \psi_{oj}^{(\ell(0,k))}(0) = \delta_{jk}, \quad 1 \le j,\ k \le j(0), \\ \psi_{oj}^{(\ell(1,k))}(1) = 0, \ 1 \le j \le j(0),\ 1 \le k \le j(1), \end{array}\right\} i = 0,$$

and similarly for i = 1. (See, e.g., Appendix B).

The (compact) convex symmetric parallelepiped $\mathcal{B}_1$ is seen to be

$$\mathcal{B}_1 = \left\{ \begin{array}{l} \sum_{j=1}^{j(0)} t_{oj}\psi_{oj} + \sum_{j=1}^{j(1)} t_{1j}\psi_{1j}; \\ |t_{ij}| \le A_{(i,j)},\ j = 1,\dots,j(i);\ i = 0,1 \end{array} \right\}.$$

Now, $P_M \in \partial\mathcal{B}_1$; for if $P_M \in \operatorname{Int}(\mathcal{B}_1)$ then there is an α, $|\alpha| > 1$, such that $\alpha P_M \in \partial\mathcal{B}_1$; but $q(\alpha P_M) = |\alpha|\, q(P_M) > M$. Hence the boundary, ∂S_M, of the (closed, convex) M-ball for q intersects $\partial\mathcal{B}_1$. In these circumstances there is a vertex of $\mathcal{B}_1$ in ∂S_M; for if a compact, convex polyhedron, $\mathcal{B}$, is contained in a closed, convex body, C (all in E^{m-1}), and there is a vector $v \in \partial\mathcal{B} \cap \partial C$, then there is a vertex of $\mathcal{B}$ in ∂C. (Let $\mathcal{H}$ be a support hyperplane for C through v. Then $\mathcal{H}$ is a support hyperplane for $\mathcal{B}$ through v, thus there is a vertex of $\mathcal{B}$ in $\mathcal{H}$ and consequently in ∂C.)

Comment. The lemma remains true, with different constants $B_{\ell,p}$, for the seminorms $q_{\ell,p,h}(P) = \|P^{(\ell)}\|_{p,[0,h]}$, $1 \le p < \infty$.

5. A Local Relation between a Spline and its Deritives at the Joints.

$d_i^{(\ell)} = s_i^{(\ell)} - H_i^{(\ell)}$ is also $s_i^{(\ell)} - f_i^{(\ell)}$ if H interpolates $f_i^{(\ell)}$. To estimate this we will need a localized relation between neighboring s_i and $s_i^{(\ell)}$. For the cubic splines, for example, Eq. 4 in de Boor[17] relates three adjacent s_i's with the three corresponding $s_i^{(1)}$'s, while Eq. 4 in Walsh et al.[18] relates three s_i's with three corresponding $s_i^{(2)}$'s. In similar fashion, for splines of degree m, Eq. 7 in Loscalzo and Talbot[19] relates m $s_i^{(1)}$'s, while Eqs. 7 and 15 in Ahlberg et al.[16] relate m s_i's, with m $s_i^{(m-1)}$'s. Like the last two, our result is for splines of odd or even degree.

Lemma 5. For any spline $s(x)$, of degree $m \geq 2$ and in $C^{m-1}[0,1]$; for each ν, $0 \leq \nu \leq N + 1 - m$; and for each ℓ, $1 \leq \ell \leq m - 1$, there is a linear relation between the m quantities, $s_{j+\nu}$, and the m quantities, $s^{(\ell)}_{j+\nu}$, $0 \leq j \leq m - 1$. This relation is given by

$$\sum_{j=0}^{m-1} a_j^{(m,\ell)} s_{j+\nu} = h^\ell \sum_{j=0}^{m-1} b_j^{(m)} s^{(\ell)}_{j+\nu}. \tag{5.1}$$

The coefficients may be written as

$$a_j^{(m,\ell)} = (-1)^\ell \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} Q_{m-\ell+1} (j + 1 - i), \tag{5.2}$$

$$b_j^{(m)} = Q_{m+1} (j + 1), \tag{5.3}$$

where

$$Q_m(x) = \frac{1}{(m-1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x - i)_+^{m-1}. \tag{5.4}$$

Proof. The proof is a straightforward generalization of the proof found in Ref. 20, pp. 435-436, there attributed to Schoenberg. In the notation of Ref. 20 the generalization is made by replacing the last two equations on page 436 with

$$Q^{(\ell)}_{m+1} (m - x) = (-1)^\ell Q^{(\ell)}_{m+1} (x + 1), \tag{5.5}$$

and

$$Q^{(\ell)}_{m+1} (x + 1) = \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} Q_{m+1-\ell} (x + 1 - i).$$

Lemma 5 and its proof originally generalized Loscalzo and Talbot.[21]

Comments: The following properties of the coefficients $b_j^{(m)}$ and $a_j^{(m,\ell)}$ should be noted: (1.) $m!\, b_j^{(m)}$ satisfy a simple recursion, see Quade and Collatz,[22] p. 414 and Ahlberg et al.,[16] Eq. 18. (2.) $m!\, b_j^{(m)}$ and $(m - \ell)!\, a_j^{(m,\ell)}$ are integers; for each ℓ and n the smallest nonzero of these is 1 in absolute value ($j = 0$, $j = m - 1$). (3.) For fixed m and even ℓ, $b_j^{(m)}$ and $a_j^{(m,\ell)}$ are symmetric in j about $(m - 1)/2$; for odd ℓ and $a_j^{(m,\ell)}$ are antisymmetric in j about $(m - 1)/2$. (For the a's this is just restating Eq. 5.5.) (4.) The integers $m!\, b_j^{(m)}$ are positive and add to $m!$.

The $Q_m(x)$, called B-splines by Schoenberg, have generalizations to variable mesh spacings; see, for example, Schoenberg,[9] and de Boor,[23] § 5. For a uniform mesh on [0,1], $Q_{m+1}[(x - x_k)/h]$, $-m \leq k \leq N - 1$, form a basis for the $(N + m)$-dimensional vector space of all C^{m-1} splines of degree m with the given $N + 1$ joints (see Ref. 8, § 3.1; Ref. 23, p. 28; and Ref. 24, Eq. 2.16). The basis is local in the sense that each element vanishes identically outside an interval of width $(m + 1)h$. Thus, for example, the interpolation problem may be solved by setting up an about $N \times N$, m-diagonal, linear system with $\vec{f} = [f_i]^T$ its right-hand side.[8,24] The use of the basis in problems involving variational principles often sets up band matrix problems whose solutions determine approximations of high order accuracy. Local two-point Hermite interpolation of the same degree, $m = 2n + 1$, has a similar basis, with about $(n + 1)N$ elements. The use of this basis yields matrices of the same band width $(4n + 3)$, but $(n + 1)$ times bigger.[11,23,25] The result of this paper, then, is that the smaller matrix problem will give the same order of accuracy.

6. A Truncation Error for Odd Degree Splines.

Lemma 6. Let $s \in C^{2n}[0,1]$ be a spline of odd degree $m = 2n + 1$ interpolating f. Define truncation errors (for as many ℓ as f permits)

$$T_{n,\ell}(f,x_k) \equiv \sum_{j=-n}^{n} b_{n+j}^{(2n+1)} (f^{(\ell)}_{k+j} - s^{(\ell)}_{k+j}), \tag{6.1}$$

$k = n,\ldots,N - n$; $\ell = 1, 2,\ldots,2n$.

Then

$$\sum_{j=-n}^{n} b_{n+j}^{(2n+1)} f^{(\ell)}_{k+j} - \Big(\sum_{j=-n}^{n} a_{n+j}^{(2n+1,\ell)} f_{k+j}\Big)/h^\ell$$

$$= T_{n,\ell} (f,x_k). \tag{6.2}$$

Various assumptions about f yield various estimates of $|T_{n,\ell}|$; those of interest here are

$$|T_{n,\ell}(f,x_k)| \leq h^{2n+2-\ell} A^{(0)}_{n,\ell} M_{2n+2}$$

$$\text{if } \|f^{(2n+2)}\|_\infty = M_{2n+2}; \tag{6.3}$$

$$|T_{n,\ell}(f,x_k)| \leq h^{2n+2-\ell} (A^{(1)}_{n,\ell} M_{2n+2} + h A^{(2)}_{n,\ell} M_{2n+3})$$

$$\text{if } \|f^{(2n+3)}\|_\infty = M_{2n+3}, \tag{6.4a}$$

$$A^{(1)}_{n,\ell} = 0 \text{ for odd } \ell; \tag{6.4b}$$

and

$$|T_{n,\ell}(f,x_k)| \leq h^{2n+2-\ell} (A^{(3)}_{n,\ell} |M_{2n+2}|$$

$$+ h A^{(4)}_{n,\ell} M_{2n+3}), \tag{6.5a}$$

where

$$A^{(3)}_{n,\ell} = 0 \quad \text{for even } \ell, \text{ if } f \in C^{2n+1}[0,1] \tag{6.5b}$$

and there exists M_{2n+2} with

$$\|(f^{(2n+2)} - M_{2n+2}\phi)'\|_\infty = M_{2n+3}. \tag{6.5c}$$

$A^{(0)}_{n,\ell}$, $A^{(1)}_{n,\ell}$, and $A^{(2)}_{n,\ell}$ are defined in Eqs. 6.8, 6.10, and 6.11; ϕ in Theorem 1.2.

Proof. Equation 6.2 is verified by replacing j with j + n and ν with k - n in Eq. 5.1, dividing by h^ℓ, subtracting the derivative term of Eq. 6.2 from both sides, and noting that $s_{k+j} = f_{k+j}$.

Since Eq. 5.1 is an identity for polynomials of degree $\leq 2n + 1$, we now assume that

$$f(x) = P_{2n+1}(x) + R(x), \tag{6.6}$$

where $P_{2n+1}(x)$ is the Taylor polynomial of degree 2n + 1 for f about x_k. Then, with $m = 2n + 1$,

$$R(x) = \int_{x_k}^{x} (x - s)^m f^{(m+1)}(s)\, ds/m!.$$

It follows that

$$R_{k+j} = h^{m+1} \int_0^j (j - t)^m f^{(m+1)} (x_k + ht)dt/m!,$$

$$R^{(\ell)}_{k+j} = h^{m+1-\ell} \int_0^j (j - t)^{m-\ell} f^{(m+1)} (x_k + ht)dt/(m - \ell)!.$$

Thus, substituting Eq. 6.6 into Eq. 6.2, we have

$$m!\,(m - \ell)!\, T_{n,\ell}(f,x_k)/h^{m+1-\ell}$$

$$= \sum_{j=1}^{n} \int_0^j \Big\{\Big[m!\, b^{(m)}_{n+j} (j - t)^{m-\ell} - (m - \ell)!\, a^{(m,\ell)}_{n+j} (j - t)^m\Big] f^{(m+1)} (x_k + ht)$$

$$- \Big[m!\, b^{(m)}_{n-j} (t - j)^{m-\ell} - (m - \ell)!\, a^{(m,\ell)}_{n-j} (t - j)^m\Big] f^{(m+1)} (x_k - ht)\Big\}dt.$$

Since $m = 2n + 1$ is odd, the symmetry properties of the $b^{(m)}_j$ and the $a^{(m,\ell)}_j$ (see the comments in Section 5) imply that

$$m!\,(m - \ell)\, T_{n,\ell}(f,x_k)/h^{m+1-\ell}$$

$$= \sum_{j=1}^{n} \int_0^j \Big[m!\, b^{(m)}_{n+j} (j - t)^{m-\ell} - (m - \ell)!\, a^{(m,\ell)}_{n+j} (j - t)^m\Big]$$

$$\Big[f^{(m+1)} (x_k + ht) + (-1)^\ell f^{(m+1)} (x_k - ht)\Big]dt. \tag{6.7}$$

If all we know is that $\|f^{(m+1)}\|_\infty = M_{m+1}$, Eq. 6.7 gives the estimate Eq. 6.3 where

$$m!\,(m - \ell)!\, A^{(0)}_{n,\ell}$$

$$= 2 \sum_{\nu=1}^{n} \int_{\nu-1}^{\nu} \Big| \sum_{j=\nu}^{n} \Big[m!\, b^{(m)}_{n+j} (j - t)^{m-\ell} - (m - \ell)!\, a^{(m,\ell)}_{n+j} (j - t)^m\Big]\Big| dt. \tag{6.8}$$

If we know, however, that $\|f^{(m+2)}\|_\infty = M_{m+2}$, then

$$f^{(m+1)} (x_k + ht) = f^{(m+1)} (x_k) + h \int_0^t f^{(m+2)} (x_k + hs)ds, \tag{6.9}$$

and Eqs. 6.4a and b follow from Eq. 6.7 with

$$m!\,(m - \ell)!\, A^{(1)}_{n,\ell}$$

$$= 2 \Big| \sum_{j=1}^{n} \int_0^j \Big[m!\, b^{(m)}_{n+j} (j - t)^{m-\ell} - (m - \ell)!\, a^{(m,\ell)}_{n+j} (j - t)^m\Big]dt\Big|, \text{ even } \ell; \tag{6.10}$$

$$m!\,(m - \ell)!\, A^{(2)}_{n,\ell}$$

$$= 2 \sum_{\nu=1}^{n} \int_{\nu-1}^{\nu} \Big| \sum_{j=\nu}^{n} \Big[m!\, b_{n+j} (j - t)^{m-\ell} - (m - \ell)!\, a^{(m,\ell)}_{n+j} (j - t)^m\Big]\Big| t\,dt. \tag{6.11}$$

Equations 6.5a, b, and c follow in similar fashion from Eq. 6.7, formulae are not given.

Comments. Bounds of the same order result when $T_{n,\ell}(f,x_k)$ is estimated in terms of $\|f^{(2n+2)}\|_{p,[x_{k-n},x_{k+n}]}$ or $\|f^{(2n+3)}\|_{p,[x_{k-n},x_{k+n}]}$. Similar results hold if f has only lower order derivatives; the powers of h drop accordingly. Perhaps the magnitude of the $A^{(i)}_{n,\ell}$ can be improved by using some sort of 2n + 1 degree Polya interpolation on $[x_{k-n}, x_{k+n}]$ instead of the Taylor interpolation, Eq. 6.6, at x_k.

7. A Polynomial-Like Ring.

Suppose the same type of Polya interpolation to f_i and $f_i^{(\ell(j))}$, j = 1,...,n is used in all the intervals $[x_i, x_{i+1}]$. Then d = s - H is not only piecewise polynomial of degree 2n + 1, but its $\ell(j)$-th derivatives at the joints are also $s_i^{(\ell(j))} - f_i^{(\ell(j))}$, thus satisfying Relation 6.1. To bound $\max_i |s_i^{(\ell(j))} - f_i^{(\ell(j))}|$ in terms of $\max_i |T_{n,\ell(j)}(f,x_i)|$ now becomes desirable, for then Lemma 4 will bound d and its derivatives. Relation 6.1 is a band matrix taking a vector in E^{N+1} into a vector in E^{N+1-2n}. More precisely, let $\mathfrak{m}$ be the (N + 1 - 2n) x (N + 1) matrix $[\mathfrak{m}_{ij}]$, where the integers $\mathfrak{m}_{ij}$ are given by

$$\mathfrak{m}_{ij} = \begin{cases} (2n+1)!\, b^{(2n+1)}_{n-|i-j|}, & |i - j| \le n \\ 0, & \text{otherwise} \end{cases},$$

$$n \le i \le N - n,\quad 0 \le j \le N. \tag{7.1}$$

With e ≡ f - s,

let $\vec{e}^{\,(\ell)} \equiv \left[e_0^{(\ell)},\dots,e_N^{(\ell)}\right]^T$ and $\vec{T}^{(\ell)} \equiv \left[T_{n,\ell}(f,x_n),\dots,T_{n,\ell}(f,x_{N-n})\right]^T$. Then (Eq. 6.1)

$$\mathfrak{m}\,\vec{e}^{\,(\ell)} \equiv (2n + 1)!\,\vec{T}^{(\ell)}. \tag{7.2}$$

This band matrix $\mathfrak{m}$ has a simple structure, independent of ℓ; it is (2n + 1)- diagonal, if that term may be applied here, and each row is the translation of one generic row which, in turn, is symmetric about is central element. If $\mathfrak{m}$ were diagonally dominant, standard arguments could come into play. The matrices for odd-degree splines of degrees 7 to 15 are not diagonally dominant; probably none beyond 7 are. We now show that such matrices can be factored into a product of tri-diagonal matrices, and will make use of this fact in Section 8.

Consider the class C of doubly infinite (2n + 1)-diagonal matrices, n = 0, 1,... of complex numbers $(c_0,\dots,c_n)$ given by $C = [c_{ij}]$, $-\infty < i, j < +\infty$, where $c_{ij} = c_{n-|i-j|}$, $|i - j| \le n$, and $c_{ij} = 0$ otherwise.

c ∈ C implies that the rows of c are identical (except for translation), and the generic row is symmetric about its diagonal element, c_n. Evidently each c ∈ C may be represented by the notation $c = \{c_0,\dots,c_n\}$, $c_0 \ne 0$, for some n. Let C_k be the set of $c = \{c_0,\dots,c_k\}$, $c_0 \ne 0$. $C_0 \cup \{0\}$ is isomorphic to the complex numbers. C itself is a commutative ring with identity {1} under matrix multiplication. The class $D_n = \bigcup_{k=0}^{n} C_k$ is analogous to the polynomials $\sum_{i=0}^{n} c_i z^{n-i}$ of degree ≤ n, for if $c_m \in C_m$ and $c_n \in D_n$, $c_m \circ c_n \in D_{m+n}$. The rule of combination of coefficients, however, appears to be different from that for polynomial multiplication.

We now show that the ring of polynomials with complex coefficients is isomorphic to C. The correspondence, Θ, is set up as follows: Θ[a] = {a}, Θ[z] = {1,0}, $\Theta[z^n] = (\Theta[z])^n$, $\Theta[\sum_{i=0}^{n} \sigma_i z^{n-i}] = \sum_{i=0}^{n} \sigma_i \Theta(z^{n-i})$. In particular we note that

$$\begin{aligned}
\Theta[a] &= \{a\}\\
\Theta[z] &= \{1,0\}\\
\Theta[z^2] &= \{1,0,2\}\\
\Theta[z^3] &= \{1,0,3,0\}\\
\Theta[z^4] &= \{1,0,4,0,6\}\\
&\;\;\vdots
\end{aligned}$$

$$\Theta[z^n] = \{c_0,c_1,\dots,c_n\},\ \text{where } c_i = \begin{cases} \binom{n}{\ell}, & i = 2\ell \\ 0, & \text{otherwise.} \end{cases} \tag{7.3}$$

Indeed, the full generic row of $\Theta[z^n]$ is the nth full row of Pascal's triangle with 0's interspersed; in fact, it consists of the coefficients of $(z^2 + 1)^n$. We will use this in a moment. The map Θ clearly preserves addition and multiplication. Furthermore, for any $\{c_0, c_1,\ldots,c_n\} = c \in C$ there exists a polynomial $P_n(z) \equiv \sum_{i=0}^{n} \sigma_i z^{n-i}$ such that $\Theta[P_n] = c$. (One simply starts with $\sigma_0 = c_0$, $\sigma_1 = c_1$, looks at $c - \Theta[\sigma_0 z^n] - \Theta[\sigma_1 z^{n-1}]$ to find σ_2, σ_3, etc). This is equivalent to solving the pair of uncoupled lower triangular linear systems (with 1 on the diagonals)

$$\left.\begin{aligned} c_{2k} &= \sum_{i=0}^{k} \binom{n-2i}{k-i} \sigma_{2i}, \qquad 0 \le k \le [n/2], \\ c_{2k+1} &= \sum_{i=0}^{k} \binom{n-1-2i}{k-i} \sigma_{2i+1}, \quad 0 \le k \le [(n-1)/2], \end{aligned}\right\} \tag{7.4}$$

where $[x]$ is the largest integer not bigger than x.

We note that if the coefficients, instead of being complex numbers, were simply the elements of a commutative ring, R, with unity, then Θ would be an isomorphism of the ring of polynomials with coefficients in R onto the ring C with coefficients in R. In our case, however, R is the complex numbers, and we have shown

<u>Lemma 7.1</u>. $\{1, c_1,\ldots,c_n\} = \prod_{i=1}^{n} \{1, r_i\}$ if, and only if, $\sum_{i=0}^{n} \sigma_i z^{n-i} \equiv \prod_{i=1}^{n} (z + r_i)$, where the σ_i and c_i are related by Eq. 7.4.

For a diagonally dominant tridiagonal matrix $\{1,r\}$, the important quantity is $r - 2$ if $r > 2$. For $\{1, c_1,\ldots,c_n\} = \prod_{i=1}^{n} \{1, r_i\}$, the important quantity will be its "excess," $E_n \equiv \prod_{i=1}^{n} (r_i - 2)$, if $r_i > 2$, $1 \le i \le n$ (Lemma 8). We now show that the matrices $(2n+1)!\,\{b_0^{(2n+1)},\ldots,b_n^{(2n+1)}\}$ have such a factorization, and that E_n is computable directly from $b_0^{(2n+1)},\ldots,b_n^{(2n+1)}$ via Eq. 7.6 or from the Bernoulli numbers via Eq. 7.7.

To show this we have another isomorphism in mind: define

$$\psi\left[\sum_{i=0}^{n} \sigma_i z^{n-i}\right] \equiv \sum_{i=0}^{n} \left[\sigma_i\,(z^2+1)/z\right]^{n-i}$$

$$\equiv \sum_{i=0}^{2n} c_i^* z^{n-i}. \tag{7.5}$$

We note that if $R_{2n}(z) = \psi[P_n]$, then $R_{2n}(1/z) \equiv R_{2n}(z)$; thus c_i^* and c_{2n-i}^*, $0 \le i \le n$. Furthermore, $\psi[z^n](z) \equiv (z^2+1)^n/z^n$; and ψ is an isomorphism onto its range. It follows that $\Theta[P_n(z)] = \{c_0,\ldots,c_n\}$ if, and only if, $\psi[P_n(z)](z) \equiv \sum_{i=0}^{2n} c_i^* z^{n-i}$, where $c_i^* = c_{2n-i}^* = c_i$, $0 \le i \le n$. We also have $\psi[\prod_{i=1}^{n} (z + r_i)](z) \equiv \prod_{i=1}^{n} (z + r_i + 1/z)$. We now deduce

<u>Lemma 7.2</u>. Let $(2n+1)!\,\{b_0^{(2n+1)},\ldots,b_n^{(2n+1)}\} = \prod_{i=1}^{n} \{1,r_i^{(n)}\}$, where $b_j^{(m)}$ are defined by Eq. 5.3. Then $r_i^{(n)} > 2$, $1 \le i \le n$, $n = 1, 2, \ldots$.

<u>Proof</u>. Let $\Theta^{-1}\left[(2n+1)!\,\{b_0^{(2n+1)},\ldots,b_n^{(2n+1)}\}\right] = \sum_{i=0}^{n} \sigma_i^{(n)} z^{n-i} \equiv P_n(z)$; i.e. $\sigma_i^{(n)}$ are defined from from the $c_i = (2n+1)!\,b_i^{(2n+1)}$ by Eq. 7.4. Then $\psi[P_n](z) \equiv \prod_{i=1}^{n} (z + r_i^{(n)} + 1/z)$. But it has been shown that $z^n\psi[P_n](z)$ has roots, w_j, which are all real, distinct, and negative (Ref. 22, § 17); and they occur in reciprocal pairs: $w_{2i+1} = 1/w_{2i}$, $1 \le i \le n$ (see Ref. 10, p. 33; Ref. 16, near Eq. 23; Ref. 24, p. 101). It follows that $r_i^{(n)}$ may be taken to be $-(w_{2i} + 1/w_{2i}) > 2$, $1 \le i \le n$.

The values of $r_i^{(n)}$, $1 \le i \le n$, $1 \le n \le 7$ have been computed as the roots of the polynomials of Lemma 7.1 (see the Appendix, Table AI).

<u>Lemma 7.3</u>. Let $\{1, c_1,\ldots,c_n\} = \prod_{i=1}^{n} \{1,r_i\}$, and set $c_0 = 1$. Then

$$\prod_{i=1}^{n} (r_i - 2) = c_n + 2 \sum_{i=1}^{n} (-1)^i c_{n-i}. \tag{7.6}$$

<u>Proof</u>. Let $\Theta^{-1}\{c_0,\ldots,c_n\} = \sum_{i=0}^{n} \sigma_i z^{n-i} \equiv P_n(z)$; then $\psi[P_n](z) \equiv \sum_{i=0}^{2n} c_i^* z^{n-i}$, $c_i^* = c_{2n-i}^* = c_i$, $0 \le i \le n$. We have

$$\prod_{i=1}^{n} (-2 + r_i) = \sum_{i=0}^{n} \sigma_i (-2)^{n-i}$$

$$= \sum_{i=0}^{n} \sigma_i \left[(z_1^2 + 1)/z_1\right]^{n-i},$$

where $(z_1^2 + 1)/z_1 = -2$. Thus

$$\prod_{i=1}^{n} (r_i - 2) = \sum_{i=0}^{2n} c_i^* (-1)^{n-i}$$

$$= c_n + 2 \sum_{i=1}^{n} (-1)^i c_{n-i}.$$

Comment. Subsequently, it has been pointed out that the isomorphism $\psi^{-1}\theta$ is the map taking $R(x) = R_{2n}(e^{ix})$, $-\pi \leq x \leq \pi$, onto the matrix c whose generic row consists of its Fourier coefficients. The tridiagonal factorization then follows from the proof of the Fejer-Riez representation theorem (Ref. 26, p. 21). In connection with Lemma 8; the Wiener theorem (Ref. 27, p. 246) states that if R(x) does not vanish, c: $\ell_\infty \to \ell_\infty$ is invertible, an estimate of $\|c^{-1}\|$ being provided in Ref. 27, p. 247. The inverse of a general c may be found in Ref. 10, p. 26; the matrix of Lemma 7.2 is inverted in Ref. 24.

(May, 1968). The function $R_{\mathfrak{m}}(x)$ associated with $\mathfrak{m}$ was also shown to be positive, hence $\mathfrak{m}$ invertible, in Schoenberg.[8] (There $R_{\mathfrak{m}}(x)/(2n + 1)!$ is called $\phi_{2n+2}(x)$; see Eqs. III (17), III (18), IV (6), and IV (7) of Part A, and Section I of Part B.[8]) That $\min \phi_{2n+2}(x) = \phi_{2n+2}(\pi)$ was recently shown in Schoenberg,[28] Lemma 6. As an alternative to calculating the integers $(2n + 1)!\, b_k^{(2n+1)}$ in order to evaluate $R_{\mathfrak{m}}(\pi)$, i.e., Eq. 7.6, we have from Eq. 2.19 in Ref. 28 that

$$R_{\mathfrak{m}}(\pi) = (2n + 1)!\; 2\, (2/\pi)^{2n+2} \sum_{i=1}^{\infty} 1/(2i - 1)^{2n+2}.$$

We now observe, using Eqs. 23.2.20 and 23.2.16 of Ref. 29, that

$$R_{\mathfrak{m}}(\pi) = (1 - 1/2^{2n+2})\, 4^{2n+2}\, |B_{2n+2}|/(2n + 2)$$

$$> (2n + 1)!\; 2\, (2/\pi)^{2n+2}, \qquad (7.7)$$

thus giving the "excess", Eq. 7.6, of $\mathfrak{m}$ quite explicitly together with a good lower bound. Here B_k is the k^{th} Bernoulli number.

8. A Use for the Tridiagonal Factorization.

If $\vec{w} = \mathfrak{m}\vec{v}$ where $\mathfrak{m}$ has a diagonally dominant factorization, and if $\|\vec{w}\|_\infty$ is at hand, one can obtain a bound on $\|\vec{v}\|_\infty$ in some circumstances. In our case we will need the following.

Lemma 8. Suppose $\vec{v} = [v_I, v_{I+1}, \ldots, v_J]^T$, $J \geq I$, and $\vec{w} = [w_{I-n}, \ldots, w_{J+n}]^T$ satisfy $\vec{v} = \mathcal{T}_1 \mathcal{T}_2 \cdots \mathcal{T}_n \vec{w}$, where each $\mathcal{T}_k$ is the appropriate $(J - I - 1 + 2k) \times (J - I + 1 + 2k)$ segment of $\{1, r_k\}$,

$$\mathcal{T}_k = \begin{pmatrix} 1 & r_k & 1 & & & \\ & 1 & r_k & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & r_k & 1 \end{pmatrix}, \quad 1 \leq k \leq n.$$

Define $\vec{v}^{(n)} = \vec{w}$; $\vec{v}^{(k-1)} = \mathcal{T}_k \vec{v}^{(k)}$, $k = n, \ldots, 1$. Then $\vec{v}^{(0)} = \vec{v}$. Suppose further that for each k, $1 \leq k \leq n$, there exists m(k) such that

$$\max_{I-k \leq i \leq J+k} |v_i^{(k)}| = |v_{m(k)}^{(k)}|,$$

$$I - k < m(k) < J + k. \qquad (8.1)$$

Then, if $|r_k| > 2$, $1 \leq k \leq n$,

$$\max_{I \leq i \leq J} |v_i| \geq \prod_{i=1}^{n} (|r_i| - 2) \max_{I-n \leq i \leq J+n} |w_i|. \qquad (8.2)$$

Proof. For $1 \leq k \leq n$

$$|v_{m(k)}^{(k-1)}| = |v_{m(k)-1}^{(k)} + r_k v_{m(k)}^{(k)} + v_{m(k)+1}^{(k)}|$$

$$\geq (|r_k| - 2)\, |v_{m(k)}^{(k)}| \geq (|r_k| - 2)\, |v_{m(k+1)}^{(k)}|,$$

the last inequality being omitted if k = n.

Comment. The crucial assumption here (besides diagonal dominance of each factor) is that as one moves down from $\vec{v}$ to $\vec{w}$ one needs an interior maximum, Eq. 8.1, at each stage; this is where the boundary conditions enter so strongly in the next section.

In the case of interest, Lemma 7.2, all the r_k are greater than 2. Hence the bound $1/\prod_{k=1}^{n} (r_k^{(n)} - 2)$ on $\|c^{-1}\|$ (c: $\ell_\infty \to \ell_\infty$), which is estimated by Eqs. 8.2 and 7.6, is actually attained at the vector $\left[(-1)^i\right]^T$. This result may also be found in Theorem 1 and Lemma 3 of Ref. 10, p. 27 and pp. 33-36.

By Eqs. 5.3 and 5.4 and the remarks following Lemma 5, we have also proved the existence of the periodic spline interpolating periodic data given at equally spaced joints. Appendix A applies the tridiagonal factorization to the numerically stable calculation of this spline.

9. Proofs of Theorems 1.1 and 1.2; Theorem 9.

We concentrate first on Theorem 1.1. Pick $n \geq 1$. Set $\ell(j) = 2j - 1$, $j = 1,\ldots,n$. Let s be the spline of degree $2n + 1$ interpolating f and satisfying $s^{(\ell(j))}(0) = f^{(\ell(j))}(0)$, $s^{(\ell(j))}(1) = f^{(\ell(j))}(1)$, $1 \leq j \leq n$ (Section 2). Let H be given in each $[x_{i-1}, x_i]$ by polynomial interpolation of degree $2n + 1$ of f_{i-1}, $f_{i-1}^{(\ell(j))}$, f_i, $f_i^{(\ell(j))}$, $1 \leq j \leq n$, $1 \leq i \leq N$ (Section 2). Equations 7.1 and 7.2 suggest applying Lemma 8 to bound $\|\vec{e}^{\ell(j)}\|_\infty$ in terms of $\|\vec{T}^{\ell(j)}\|_\infty$, but Eq. 8.1 cannot be verified. We now extend the domain of definition of $e = s - f$ and $T_{\ell(j),n}(f,x_k)$ so that Eq. 8.1 can be verified.

Let P be the Taylor polynomial of degree $2n + 1$ for f about $x = 0$. Define $f^* = f - P$, $s^* = s - P$, $H^* = H - P$, and extend their domain of definition by even reflection in 0: thus

$$f^*(x) = \begin{cases} f(x) - P(x), & x \in [0,1], \\ f(-x) - P(-x), & x \in [-1,0]. \end{cases}$$

We note that $f^* \in C^{2n+2}[-1,1]$, s^* is a spline of degree $2n + 1$ in $C^{2n}[-1,1]$ interpolating f^*, and H^* interpolates f^* as H interpolated f. Furthermore, $e_H = H - f$ is identical with $H^* - f^*$ in [0,1] and may be extended to [-1,0] by $e_H \equiv H^* - f^*$; the same sort of extension defines $d = s - H$ and e on [-1,1]. Set $\ell = \ell(j)$, $1 \leq j \leq n$. The truncation errors of Section 6, $T_{n,\ell}(f,x_k)$, $n \leq k \leq N - n$ are identical with $T_{n,\ell}(f^*,x_k)$, $n \leq k \leq N - n$, and are now extended by that identity for $-(N - n) \leq k \leq N - n$. The bounds, Eq. 6.3 or 6.4, on $|T_{n,\ell}(f,x_k)|$ in terms of $\|f^{(2n+2)}\|_\infty$ or $\|f^{(2n+3)}\|_\infty$ are unaffected (in terms of $\|\cdot\|_p$ they will be affected by a root of 2). In other words, we have effected an extension of Eqs. 7.1 and 7.2 to $\vec{e}^{(\ell)} = \left[e_{-N}^{(\ell)},\ldots,e_N^{(\ell)}\right]^T$, $\vec{T}^{(\ell)} = \left[T_{n,\ell}(f,x_{-(N-n)}),\ldots, T_{n,\ell}(f,x_{N-n})\right]^T$ satisfying $|e_{-k}^{(\ell)}| = |e_k^{(\ell)}|$, $k = 0,\ldots,N$ and $|T_{n,\ell}(f,x_{-k})| = |T_{n,\ell}(f,x_k)|$, $k = 0,\ldots,N - n$.

We now do the same thing at the other end, using the Taylor polynomial there. The result is the extension of Eqs. 7.1 and 7.2 to $\vec{e}^{(\ell)} = \left[e_{-N}^{(\ell)},\ldots,e_{2N}^{(\ell)}\right]^T$ and $\vec{T}^{(\ell)} = \left[T_{n,\ell}(f,x_{-(N-n)}),\ldots, T_{n,\ell}(f,x_{2N-n})\right]^T$ satisfying $|e_{-k}^{(\ell)}| = |e_k^{(\ell)}| = |e_{2N-k}^{\ell}|$, $0 \leq k \leq N$; $|T_{n,\ell}(f,x_{-k})| = |T_{n,\ell}(f,x_k)|$, $0 \leq k \leq N - n$; and $|T_{n,\ell}(f,x_{2N-k})| = |T_{n,\ell}(f,x_k)|$, $n \leq k \leq N$. The bounds, Eqs. 6.3 or 6.4, on $\|\vec{T}^{(\ell)}\|_\infty$ have been unaffected (and essentially so for $\|\cdot\|_p$ estimates). In Lemma 8 we now take $\vec{w} = \vec{e}^{(\ell)}$, $I = -(N - n)$, $J = 2N - n$, $\vec{v} = (2n + 1)!\,\vec{T}^{(\ell)}$, $r_k = r_k^{(n)}$ occurring in the factorization of $(2n + 1)!\left\{b_0^{(2n+1)},\ldots,b_n^{(2n+1)}\right\}$. Using Lemma 7.2, we conclude from Lemma 8 that, since $\vec{e}^{(\ell)} = \vec{d}^{(\ell)}$, $\ell = \ell(j)$,

$$(2n + 1)! \max_{0 \leq k \leq N} |T_{n,\ell}(f,x_k)| \geq \prod_{i=1}^{n} (r_i^{(n)} - 2) \max_{0 \leq k \leq N} |d_k^{(\ell)}|, \quad \ell = \ell(j), \quad 1 \leq j \leq n. \tag{9.1}$$

The rest is easy. We have the excess, $E_n = \prod_{i=1}^{n} (r_i^{(n)} - 2)$, from Eqs. 7.6 or 7.7. Assuming that $\|f^{(2n+2)}\|_\infty = M_{2n+2}$, we have from Eqs. 6.3 and 9.1

$$\max_{0 \leq i \leq N} |d_i^{(\ell(j))}| \leq (2n + 1)!\, h^{2n+2-\ell(j)} A_{n,\ell(j)}^{(0)} M_{2n+2}/E_n, \quad 1 \leq j \leq n.$$

From Lemma 4, using the Polya set 2.3, and with $A(i,j) = A_{n,\ell(j)}^{(0)}$, $i = 0,1$, we calculate B_ℓ, concluding Eq. 1.2 with $K_{\ell,n}^{(1)} = (2n + 1)!\, B_\ell/E_n$. From Corollary 3, using Eq. 2.3, we conclude Eq. 1.1 with $G_{\ell,n}^{(1)} = G_\ell$. If $f \in C^{2n+3}[0,1]$ we similarly use Eqs. 6.4a and b, calculate B_ℓ from $A_{n,\ell}^{(2)}$ and validate Eq. 1.3 with $K_{\ell,n}^{(2)} = (2n + 1)!\, B_\ell/E_n$. (Although $f^{*(2n+3)}$ may jump at 0 or 1, Eq. 6.9 still holds.)

Theorem 1.2 is proved in similar fashion except that f^*, s^*, and H^* are defined by odd reflection because their first n even derivatives vanish at 0 and 1. The bounds are different because the Polya set used is Eq. 2.4; Eqs. 6.5a, b, and c replace 6.4a and b in the argument.

We now make two observations about this proof which permit the construction of bounds that may be smaller than those in Theorems 1.1 and 1.2 in some cases. The first observation is that the subtractions of the Taylor polynomials at either end and the subsequent even, or odd, reflections were performed only to extend s as a spline s^* on [-1,2] so that Lemma 8 would apply, and to ensure that f^* had enough continuous derivatives at 0 and 1 so that Lemma 6 could be applied to $s^* - f^*$. Thus for periodic spline interpolation of a function with enough periodic derivatives,[16] one may apply Lemmas 6 and 8 directly to s - f. The second observation concerns the Polya interpolations used. Lemmas 6 and 8 actually bound $(s^* - f^*)^{(\ell)}$ at the joints for each $\ell \leq 2n + 1$. Thus any Polya interpolation of degree $\leq 2n + 1$ may be used in the error decomposition on any interval, with Lemmas 3 and 4 then bounding the two parts of the error and its derivatives in that interval. We conclude

<u>Theorem 9</u>. Let $n \geq 1$, $N \geq 2n + 1$. Let $f \in C^{2n+1}[0,1]$ be such that $f^{(2n+2)}$ is continuous except, perhaps, for jump discontinuities at the joints. Let s be the spline of degree 2n + 1 interpolating f, and (a.) matching its first n odd derivatives at 0 and 1, (b.) matching its first n even derivatives at 0 and 1, or (c.) satisfying periodic boundary conditions (if f is periodic with period 1, and $f \in C^{2n+1}(-\infty,\infty)$, with possible jump discontinuities in $f^{(2n+2)}$ at the joints). For $\ell = 0, \ldots, 2n + 1$, define

$$K^{(3)}_{\ell,n} = \min_{[\ell(i,j)]} \left[G_\ell + (2n + 1)!\, B_\ell/E_n\right],$$

where E_n is defined by Eq. 7.6; $\ell[i,j]$ is any Polya set for 2n + 1 degree Polya interpolation (Section 2); and for each such Polya set, G_ℓ is determined by Lemma 3 and B_ℓ by Lemma 4 using Lemma 6. Then, for $0 \leq \ell \leq 2n + 1$,

$$\|(f - s)^{(\ell)}\|_\infty \leq h^{2n+2-\ell}\, K^{(3)}_{\ell,n}\, \|f^{(2n+2)}\|_\infty.$$

As an example of what can be done with less smooth functions, we easily show (January, 1968)

<u>Corollary 9.1</u>. Suppose f is continuous, but not differentiable, on $(-\infty,\infty)$ and periodic with period one. Let

$$\max_{|x-y| \leq \delta} |f(x) - f(y)| = \omega(f,\delta).$$

Let s be the spline of degree 2n + 1 which interpolates f at the equally spaced joints and satisfies periodic conditions at 0 and 1 (or has its first n odd, or even, derivatives vanishing at 0 and 1). Let H be piecewise linear interpolation of f at the joints. Then, with h = 1/N,

$$\|f - H\|_\infty \leq \omega(f,h), \qquad (9.2)$$

and there exist K_n such that

$$\|H - s\|_\infty \leq K_n\, \omega(f,h). \qquad (9.3)$$

<u>Proof</u>. For $x \in [x_i, x_{i+1}]$,

$$(f - H)(x)$$
$$= (x_{i+1} - x)(f(x) - f_i)/h + (x - x_i)(f(x) - f_{i+1})/h,$$

from which Eq. 9.2 follows immediately. Furthermore, $|H_i'| \leq \omega(f,h)/h$. $s_j^{(\ell)}$, ℓ odd, is bounded by $O[\omega(f,h)/h^\ell]$ using Eq. 5.1, Comment 3 after Lemma 5, suitable reflection as above, and Lemmas 7.2 and 8. Thus there are constants $A_{n,\ell}$ such that

$$\max_i |s_i^{(\ell)}| \leq A_{n,\ell}\, \omega(f,h)/h^\ell, \quad \ell = 1,3,5,\ldots,2n - 1.$$

Lemma 4 now applies to H - s, proving Eq. 9.3.

For the cubic splines with equally spaced joints, this argument yields $\|f - s\|_\infty \leq 2\,\omega(f,h)$ not quite so good as Nord's 7/4.[4]

<u>Corollary 9.2</u>. Suppose instead that f is merely bounded. Then the rest of Corollary 9.1 holds unaltered. Thus the spline interpolants are uniformly bounded for all h.

10. <u>Error Bounds for Some Cubic Spline Interpolations on Arbitrary Meshes</u>.

Let the joints $0 = x_0 < x_1 < \ldots < x_N = 1$ be given, and set $h_i = x_i - x_{i-1}$, $h_m = \min_{1 \leq i \leq N} h_i$, $h_M = \max_{1 \leq i \leq N} h_i$. With f also in hand, we first consider the spline, s, which is a cubic polynomial between the joints, is in $C^2[0,1]$, and interpolates f at the joints and f' at 0 and 1. Let H be the piecewise cubic which interpolates f_i, f_i', $0 \leq i \leq N$.

All the previous work becomes easy for the cubics. The analogue of Relation 5.1 among three successive s_i's and the corresponding $s_i^{(1)}$'s is given by de Boor's Eq. 4.[17] The analogue of the truncation error, $T_{1,1}(f,x_i)$, Eq. 6.1, is easily computed using Taylor's theorem with integral remainder. (We have not explored the consequences of estimating $T_{1,1}$ by using Polya interpolation on $[x_{i-1}, x_{i+1}]$ instead of Taylor interpolation at x_i.) The analogue of the matrix, $\mathfrak{m}$, Eq. 7.1, is tridiagonal and diagonally dominant. Since $d_0' = d_N' = 0$, the extreme value of $|d_i'|$ is attained at an interior joint. One then concludes, if $\|f^{(4)}\|_{\infty,[x_{i-1},x_{i+1}]} = M_{4,i}$ or $\|f^{(5)}\|_{\infty,[x_{i-1},x_{i+1}]} = M_{5,i}$,

$$\max_{0 \le i \le N} |d_i'| \le$$

$$\max_{1 \le i \le N-1} \left[h_{i+1}h_i\left(h_{i+1}^2 + h_i^2\right)M_{4,i}/(h_{i+1} + h_i)\right]/24,$$

or

$$\max_{1 \le i \le N-1}\left\{h_{i+1}h_i\left[\left(h_{i+1}^2 + h_i^2\right)M_{5,i}/120 + |h_{i+1} - h_i|M_{4,i}/24\right]\right\}. \qquad (10.1)$$

Let p_i bound d_i' and d_{i-1}'. Define $\|d^{(\ell)}\|_i = \|d^{(\ell)}\|_{\infty,[x_{i-1},x_i]}$. Using Lemma 4, we bound $\|d^{(\ell)}\|_i$ by p_i times the appropriate entry in the first column of Table I. Using Lemma 3, with $G(x,t)$ given in Ref. 30, p. 376, we bound $\|e_H^{(\ell)}\|_i$ by $M_{4,i}$ times the second column (the numbers are $G_{\ell,1}^{(1)}$ of Theorem 1.1). From these two results and Eq. 10.1 one may observe much; for example, (a.) the mesh ratio, h_M/h_m, has no important effect on $O(h_M^{4-\ell})$-type bounds on $\|f - s\|_\infty$ and $\|f' - s'\|_\infty$; indeed $\|f - s\|_i \le h_i\|f^{(4)}\|_\infty(4h_M^3 + h_i^3)/384$. (b.) for sufficiently smooth meshes, however, $\|d^{(\ell)}\|_\infty \le O(1/N^{5-\ell})\|f^{(5)}\|_\infty$, $0 \le \ell \le 3$. For a uniform mesh, with $h = 1/N$, $\|f^{(4)}\|_\infty = M_4$ or $\|f^{(5)}\|_\infty = M_5$, we derive the third and fourth columns of Table I from Eq. 10.1 and the first column. The numbers in columns 3 and 4 are $K_{\ell,1}^{(1)}$ and $K_{\ell,1}^{(2)}$ of Theorem 1.1.

We repeat in similar fashion for the cubic spline, s, which matches f" at 0 and 1, and with H the piecewise cubic which interpolates f_i, f_i'', $0 \le i \le N$. The analogue of Relation 5.1 is now given by Walsh et al., Eq. 4.[18] The result corresponding to Eq. 10.1 is (since there is no particular advantage in assuming that $f^{(5)}$ exists)

$$\max_{0 \le i \le N} |d_i''| \le \max_{1 \le i \le N-1}$$

$$\left[\left(h_i^2 - h_ih_{i+1} + h_{i+1}^2\right)M_{4,i}\right]/4. \qquad (10.2)$$

With p_i now a bound on d_i'' and d_{i-1}'', we use Lemma 4 again to set up the first column of the second set of entries in Table I. The appropriate Green's function yields the second column. From this and Eq. 10.2 we see, among other things, that (a.) the mesh ratio only disturbs $O(h_M^{4-\ell})$ bounds on $\|f^{(\ell)} - s^{(\ell)}\|_\infty$ for $\ell = 3$, noting further that $\|f - s\|_i \le O(h_i^2h_M^2)M_4$; (b.) for sufficiently smooth meshes, $O(1/N^{4-\ell})$ bounds can be found on $\|f^{(\ell)} - s^{(\ell)}\|_\infty$, $0 \le \ell \le 3$. The third column again indicates the bounds on $\|d^{(\ell)}\|_\infty$ with a uniform mesh; the numbers there are $K_{\ell,1}^{(3)}$ of Theorem 1.2.

Finally, we observe that the ideas developed in Section 9 apply to these two types of cubic spline interpolation and to cubic periodic spline interpolation on arbitrary meshes as well. Thus, for these three types of boundary conditions, we see that the mesh ratio has no significant effect on the convergence rate (in terms of h_M) of $s^{(\ell)}$ to $f^{(\ell)}$, $0 \le \ell \le 2$. Furthermore, we see that the smaller the local mesh length, the faster the local convergence, $\ell = 0, 1$. In this last respect, cubic spline interpolation acts somewhat like cubic Polya interpolation, although with the Polya interpolation it is only the local mesh length which is significant. For example, the maximum mesh width, h_M, can remain fixed while the local length, h_i, goes to zero, and the spline error will go to zero like h_i^2 while the Polya error goes to h_i^4. This order of local convergence for the spline is not to be improved on in general, for if $f(x) = x^4$ is interpolated at $-1, -\epsilon, \epsilon, 1$ by the cubic spline, s, which matches f' at ± 1 as well, then $s(0) = \epsilon^2 + O(\epsilon^3)$.

Table I. Cubic Splines, $\| \cdot \|_\infty$ Bounds.

ℓ	$\|d^{(\ell)}\|_i \le p_i$	$\|e_H^{(\ell)}\|_i \le M_{4,i}$	$\|d^{(\ell)}\|_\infty \le M_4$	$\|d^{(\ell)}\|_\infty \le M_5$
		f' interpolation		
0	$h_i/4$	$h_i^4/384$	$4h^4/384$	$h^5/240$
1	1	$h_i^3\, G_1$*	$h^3/24$	$h^4/60$
2	$6/h_i$	$h_i^2/12$	$h^2/4$	$h^3/10$
3	$12/h_i^2$	$h_i/2$	$h/2$	$h^2/5$
		f" interpolation		
0	$h_i^2/8$	$5h_i^4/384$	$12h^4/384$	
1	$h_i/2$	$h_i^3/24$	$h^3/8$	
2	1	$h_i^2/8$	$h^2/4$	
3	$2/h_i$	$h_i/2$	$h/2$	

*$1000\, G_1 \approx 8.0187537$ at $x = 1/2 \pm 0.2886751$. G. Birkhoff and A. Priver[31] have shown subsequently that $G_1 = \sqrt{3}/216$.

The author thanks Professor Carl de Boor who inquired about the existence of tridiagonal factorization of band matrices and contributed the following argument which sharpens Theorems 2 and 4 of Subbotin.[7] Suppose $f \in C^{2n+1}(-\infty,\infty)$ with bounded $2n + 1^{st}$ derivative. According to Eq. 1 of Subbotin,[7] if s is the unique C^{2n-1} spline of degree 2n interpolating f halfway between the uniformly spaced joints, then

$$\|f^{(\ell)} - s^{(\ell)}\|_\infty \le D_\ell\, h^{2n-1-\ell}\, \omega\left(f^{(2n-1)}, h\right),$$

$$0 \le \ell \le 2n - 1.$$

Let $\bar{s}$ be a function such that $\bar{s}^{(2n-1)}$ is piecewise linear and interpolates $f^{(2n-1)}$ at the joints. Then the spline interpolating $f - \bar{s}$ is $s - \bar{s}$. Hence

$$\|f^{(\ell)} - s^{(\ell)}\|_\infty = \|(f - \bar{s})^{(\ell)} - (s - \bar{s})^{(\ell)}\|_\infty$$

$$\le D_\ell\, h^{2n-1-\ell}\, \omega\left[(f - \bar{s})^{(2n-1)}, h\right].$$

But $\|(f - \bar{s})^{(2n-1)}\|_\infty \le D\, h^2\, \|f^{(2n+1)}\|_\infty$, giving the additional two powers of h which are desired.

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Appendix A. Stable Computation of the Spline Interpolants.

(May, 1968). We now apply the tridiagonal factorization of Section 7 to the practical problem of finding numerically the periodic spline of degree $2n + 1$, of period 1, which interpolates f (also of period 1) at the equally spaced joints $x_k = k/N$. (The calculation of the interpolants involving odd or even derivative boundary conditions is then easily accompolished - see the end of the Appendix.) In particular, we wish to compute the coefficients, c_k, of the elements of the local basis for the splines,

$$\phi_k(x) \equiv Q_{2n+2}(x/h - k + n + 1),$$

such that the spline,

$$s(x) = \sum_{k=-\infty}^{\infty} c_k \, \phi_k(x),$$

satisfies $s(kh) = f(kh)$, $0 \le k \le N - 1$, and is periodic with period 1. Here Q_{2n+2} is given by Eq. 5.4 (although we do not know if that definition is the best way to compute it). Thus we wish to solve the doubly infinite matrix problem

$$\mathfrak{M}\, c = (2n + 1)!\, f \equiv g, \qquad (A.1)$$

where $\mathfrak{M}$ is given by Eq. 7.1, (but $-\infty < i,j < \infty$),

$$c = (\ldots, c_{-1}, c_0, c_1, \ldots, c_{N-1}, c_N, \ldots)^T,$$

$$f = (\ldots, f_{-1}, f_0, f_1, \ldots, f_{N-1}, f_N, \ldots)^T;$$

while f has period N, that is

$f_{k+N} = f_k$ for all k.

It is clear that finding $c_0, \ldots, c_{N-1}$ will suffice, since c also has period N.

We shall show the following: the N coefficients may be found by solving two N/2 by N/2 sets of linear equations. Each of this pair of linear equations is solved by finding, successively, the solutions of n tridiagonal N/2 by N/2 linear equations (the tridiagonal matrices are indicated below). Since each tridiagonal system is diagonally dominant, it is easily and stably solved numerically by the usual technique. The result, then, is that the N coefficients may be found by a stable numerical procedure involving about 2nN multiplications, nN divisions, and 3nN additions. The result for the other two boundary conditions is essentially the same (see the next to last paragraph of this Appendix).

To both motivate and prove the general case, we consider first the tridiagonal case and assume that $\mathfrak{M} = \mathcal{T} = \{1,r\}$, with $r > 2$.

A sequence, c, is called periodic with period N if, and only if, $c_k = c_{k+N}$, all k. A sequence c is called [anti] symmetric if, and only if, $c_{-k} = (-1)^i c_k$, $i = 0$ [$i = 1$], all k. We note that a periodic sequence is [anti] symmetric if, and only if, it is [anti] symmetric about N/2 (or, more precisely, $c_{N-k} = (-1)^i c_k$). We note further that periodicity is preserved under $\mathcal{T}$ and $\mathcal{T}^{-1}$, and that [anti] symmetry of a periodic sequence is also preserved under $\mathcal{T}$ and $\mathcal{T}^{-1}$. We assume hereafter that all sequences are periodic with period N.

To solve, now, the problem

$$\mathcal{T} c = g,$$

where the given g is either symmetric or antisymmetric, we solve the problem

$$\mathcal{T}^* \vec{c} = \vec{g},$$

where

$$\vec{c} = (c_0, c_1, \ldots, c_{[N/2]})^T,$$

$$\vec{g} = (g_0, g_1, \ldots, g_{[N/2]})^T;$$

and set

$$c_{N-k} = \begin{cases} c_k, & g \text{ symmetric} \\ -c_k, & g \text{ antisymmetric} \end{cases} \quad 1 \le k \le [N/2].$$

The diagonally dominant [N/2] + 1 by [N/2] + 1 matrix $\mathcal{T}^*$ to be used depends upon g and N.

g symmetric, N even:

$$\mathcal{T}^* = \begin{pmatrix} r & 2 & & & \\ 1 & r & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & r & 1 \\ & & & 2 & r \end{pmatrix} \qquad \text{(A.2)}$$

g antisymmetric, N even:

$$\mathcal{T}^* = \begin{pmatrix} r & & & & \\ 1 & r & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & r & 1 \\ & & & & r \end{pmatrix} \qquad \text{(A.3)}$$

g symmetric, N odd:

$$\mathcal{T}^* = \begin{pmatrix} r & 2 & & & \\ 1 & r & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & r & 1 \\ & & & 1 & r+1 \end{pmatrix}$$

g antisymmetric, N odd:

$$\mathcal{T}^* = \begin{pmatrix} r & & & & \\ 1 & r & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & 1 & r & 1 \\ & & & 1 & r-1 \end{pmatrix}.$$

To solve

$$\mathcal{T} c = g$$

given a general periodic sequence, g, we first define sequences g_{even} and g_{odd} by

$$(g_{even})_k = (g_k + g_{N-k})/2, \text{ all } k;$$

$$(g_{odd})_k = (g_k - g_{N-k})/2, \text{ all } k.$$

Then we solve the two systems

$$\mathcal{T}^* \vec{c}_{even} = \vec{g}_{even}, \quad \mathcal{T}^* \vec{c}_{odd} = \vec{g}_{odd};$$

and, upon extending c_{even} and c_{odd} as above, we set

$$\vec{c} = \vec{c}_{even} + \vec{c}_{odd}.$$

We turn finally to the solution of the periodic interpolation problem, Eq. A.1,

$$\mathfrak{M}\, c = (2n + 1)!\, f \equiv g,$$

where, as in Section 7, $\mathfrak{M} = \mathcal{T}_1 \mathcal{T}_2 \cdots \mathcal{T}_n$, each $\mathcal{T}_k = \{1, r_k^{(n)}\}$, $r_k^{(n)} > 2$ for $1 \le k \le n$. Forming g_{even} as above, we solve, successively, the n problems

$$\vec{v}^{\,(0)} \equiv \vec{g}_{even};$$

$$\mathcal{T}_k^* \vec{v}^{\,(k)} = \vec{v}^{\,(k-1)}; \quad k = 1, 2, \ldots, n;$$

$$\vec{c}_{even} \equiv \vec{v}^{\,(n)}.$$

Finding $\vec{c}_{odd}$ from g_{odd} in a similar fashion, we set

$$\vec{c} = \vec{c}_{even} + \vec{c}_{odd}$$

to complete the process.

As for finding the spline matching f and its first n odd (or even) derivatives at the ends, we first set

$$g \equiv (2n + 1)!\, (f - P)$$

where P is the Polya polynomial of degree 2n + 1 interpolating f and its n odd (or even) derivatives at the ends. By the existence argument of Section 2, g is to be extended by even (odd) reflection in zero and interpolated by a spline (of degree 2n + 1 and of period two) on [-1, 1]. It should now be clear that the numerical procedure for finding this spline s - P will be to solve

$$\mathcal{T}_1^* \mathcal{T}_2^* \cdots \mathcal{T}_n^* \vec{c} = \vec{g}$$

where $\mathcal{T}_k^*$ is now the N + 1 by N + 1 analogue of Eq. A.2 (or Eq. A.3 for the even derivative boundary conditions), $\vec{c} = (c_0, c_1, \ldots, c_N)^T$, and $\vec{g} = (g_0, g_1, \ldots, g_N)^T$.

We have computed the $r_k^{(n)}$ for $1 \le n \le 7$. They are given in Table AI.

Table AI. $r_k^{(n)}$.

Notation: 2.34 + 5 means 2.34 x 10^5.

n\k	1	2	3
1	4.0 +0		
2	2.7530 49234 04039 +0	2.3266 95076 59596 +1	
3	2.4034 60066 11789 +0	8.2821 82046 85356 +0	1.0931 43578 87028 +2
4	2.2527 41294 01893 +0	5.1583 67231 97466 +0	2.3179 26260 62979 +1
5	2.1735 16654 60274 +0	3.9462 14873 06176 +0	1.1230 62923 71162 +1
6	2.1266 10293 31673 +0	3.3351 43517 76947 +0	7.3382 67460 97167 +0
7	2.0965 06956 15866 +0	2.9790 42678 09236 +0	5.5478 72341 75666 +0

n\k	4	5	6
4	4.7140 96288 67708 +2		
5	6.0006 01300 19854 +1	1.9586 43626 23323 +3	
6	2.3183 93408 99078 +1	1.4841 80377 67637 +2	7.9935 98006 87039 +3
7	1.3249 82041 18882 +1	4.5994 39711 72399 +1	3.5699 88459 56139 +2

n\k	7
7	3.2325 13351 45386 +4

Appendix B. A Local Basis for Odd Derivative Polya Interpolation.

(June, 1968). Eqs. B.6 and B.7 below define $n + 1$ polynomials, P_j, of degree $2n + 1$, such that

$$\left.\begin{array}{l} P_o(0) = 1,\ P_o(1) = 0; \\ P_o^{(2\ell-1)}(0) = P_o^{(2\ell-1)}(1) = 0,\ 1 \le \ell \le n \end{array}\right\} \quad (B.1)$$

and, for $1 \le j \le n$

$$\left.\begin{array}{l} P_j(0) = P_j(1) = 0; \\ P_j^{(2\ell-1)}(0) = \delta_{j\ell},\ P^{(2\ell-1)}(1) = 0,\ 1 \le \ell \le n. \end{array}\right\} \quad (B.2)$$

We note in passing that these polynomials lead immediately to the construction of the polynomial, P, required in the existence proof of Section 2 and the next to last paragraph of Appendix A. They also are identical with the ψ_{oj} used in the proof of Lemma 4 for the Polya conditions given by Eq. 2.3.

We now define, for $0 \le k \le N$ and $1 \le j \le n$

$$\phi_{ok}(x) = \begin{cases} P_o(|x-x_k|/h), & |x-x_k| \le h \\ 0 & ,\text{ otherwise} \end{cases} \quad (B.3)$$

$$\phi_{jk}(x) = h^{2j-1} \begin{cases} P_j[(x-x_k)/h], & x_k \le x \le x_k + h \\ -P_j[(x_k-x)/h], & x_k-h \le x \le x_k \\ 0 & ,\text{ otherwise}. \end{cases} \quad (B.4)$$

These $(n + 1)(N + 1)$ functions form a basis for the vector space of all piecewise polynomials, $\Pi(x)$, of degree $\le 2n + 1$ having joints $x_o, \ldots, x_k = k/N, \ldots, x_N$ such that H and $H^{(2j-1)}$, $1 \le j \le n$ are continuous. This basis is a local basis in the sense that each element vanishes identically outside an interval of width (at most) 2h. Furthermore, Polya interpolation, H, of f and its first n odd derivatives at the joints, is given by

$$H(x) = \sum_{k=0}^{N} \left[f(x_k)\phi_{ok}(x) + \sum_{j=1}^{n} f^{(2j-1)}(x_k)\phi_{jk}(x) \right]. \quad (B.5)$$

As an application we shall see that the integral of H yields the Euler-Maclaurin formula for the integral of f (the one which stops with the $2n - 1^{st}$ derivative of f). Maximum and $\mathcal{L}_2$ norm error estimates follow from Lemma 3 and Example 3.4 of this report and from Eqs. 3 and 4 of Swartz.[12]

The polynomials, P_j (Eqs. B.1 and B.2), are constructed from the Euler polynomials, $E_m(x)$, and the Bernoulli polynomials, $B_m(x)$. (See Ref. 29, Section 23 and its references). To construct these P_j's we first define a map

$$\mathscr{B}[f(x)] = g(x)$$

taking the function f onto the function g such that

$$g'' \equiv f \text{ in } [0,1],\ g(0) = g(1) = 0.$$

We note

$$\mathscr{B}[1] = \mathscr{B}[E_o(x)] = E_2(x)/2!$$

$$\mathscr{B}[E_{2k}(x)/(2k)!] = E_{2k+2}(x)/(2k+2)!,\ k \ge 0; \text{ and}$$

$$\mathscr{B}[B_{2k-1}(x)/(2k-1)!] = B_{2k+1}(x)/(2k+1)!,\ k \ge 1.$$

Turning to construct $P_o(x)$ (Eq. B.1) we assume $P_o^{(2n+1)}(x) \equiv a_o$. Then

$$P_o^{(2n-1)}(x) = \mathscr{B}[a_o] = a_o E_2(x)/2!$$

$$\vdots$$

$$P_o^{(1)}(x) = \mathscr{B}[P_o^{(3)}(x)] = a_o E_{2n}(x)/(2n)!$$

Hence

$$P_o(x) = a_o \int_1^x E_{2n}(t)dt/2n!$$

$$= a_o[E_{2n+1}(x) - E_{2n+1}(1)]/(2n+1)!$$

where a_o is picked so that $P_o(0) = 1$.

Thus, from Ref. 29, formula 23.1.20,

$$P_o(x) = [2 - (2n+2)E_{2n+1}(x)/(2^{2n+2}-1)/B_{2n+2}]/4 \quad (B.6)$$

where E_{2n+1} is the $2n + 1^{st}$ Euler polynomial and B_{2n+2} is the $2n + 2^{nd}$ Bernoulli number.

To find $P_j(x)$ (Eq. B.2) we assume

$$P_j^{(2n+1)}(x) \equiv a_j.$$

Then

$$P_j^{(2n+1-2)}(x) = \mathscr{I}[a_j] = a_j E_2(x)/2!$$

$$\vdots$$

$$P_j^{(2j+1)}(x) = \mathscr{I}[P^{(2j+3)}(x)] = a_j E_{2n-2j}(x)/(2n-2j)!$$

$$P_j^{(2j-1)}(x) = a_j E_{2n+2-2j}(x)/(2n+2-2j)! + (1-x)$$

$$= a_j E_{2n+2-2j}(x)/(2n+2-2j)! - B_1(x) + 1/2$$

$$P_j^{(2j-3)}(x) = \mathscr{I}[P^{(2j-1)}(x)]$$

$$= a_j E_{2n+4-2j}(x)/(2n+4-2j)! - B_3(x)/3!$$

$$+ E_2(x)/2(2!)$$

$$\vdots$$

$$P_j^{(1)}(x) = \mathscr{I}[P^{(3)}(x)]$$

$$= a_j E_{2n}(x)/2n! - B_{2j-1}(x)/(2j-1)!$$

$$+ E_{2j-2}(x)/2[2j-2)!]$$

$$P_j(x) = a_j\left[E_{2n+1}(x) - E_{2n+1}(0)\right]/(2n+1)!$$

$$- [B_{2j}(x) - B_{2j}(0)]/(2j)! \qquad \text{(B.7)}$$

$$+ [E_{2j-1}(x) - E_{2j-1}(0)]/2[(2j-1)!]$$

where a_j is picked so that $P_j(1) = 0$ (see Ref. 29, formula 23.1.20). Here $E_m(x)$ [$B_m(x)$] is the m^{th} Euler [Bernoulli] polynomial.

Finally, to verify that the integral of the odd derivative Polya interpolant of f, Eq. B.5, yields the Euler-Maclaurin formula (terminated with the $f^{(2n-1)}$ end corrections) we observe from Eqs. B.3, B.4, B.6, and B.7 that

$$\int_0^1 \phi_{oo} = \int_0^1 \phi_{oN} = h/2;$$

$$\int_0^1 \phi_{ok} = h, \ 1 \le k \le N-1;$$

$$\int_0^1 \phi_{jk} = 0, \ 1 \le k \le N-1, \ 1 \le j \le n;$$

$$\int_0^1 \phi_{jo} = -\int_0^1 \phi_{jN} = h^{2j} B_{2j}/(2j)!, \ 1 \le j \le n.$$

Thus, if H is given by Eq. B.5,

$$\int_0^1 H(x)dx = h(f_o/2 + f_1 + \dots + f_{N-1} + f_N/2)$$

$$- \sum_{j=1}^{n} h^{2j} B_{2j}(f_N^{(2j-1)} - f_o^{(2j-1)})/(2j)!.$$

For further discussion of some connections between splines, Bernoulli polynomials, the Euler-Maclaurin formula, and best quadrature formulas, we refer the reader to I. J. Schoenberg's "On Monosplines of Least Deviation and Best Quadrature Formulae," SIAM Journal on Numerical Analysis, Volume 2, 1965, pp. 144-169.