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Ω-theorems for the Riemann zeta-function

by

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Abstract. For $s = \sigma + it$ the familiar formula

$$\zeta^k(s) = \sum_{j\geqslant 1} \frac{d_k(j)}{j^s} \tag{0.1}$$

is valid for $\sigma > 1$ and integer $k \geqslant 1$. Here Ω results for $\zeta(c+it)$, $\frac{1}{2} \leqslant c \leqslant 1$ will be obtained in an elementary way from (0.1) by choosing an appropriate integer n, $n = n(k, c)$. The size of the single term $d_k(n)$ will give Ω results that are slightly better than the existing results of Littlewood for $c = 1$ and Titchmarsh for $\frac{1}{2} \leqslant c < 1$. Their results are based on the use of $\sum d_k^2(j)/j^s$, $\sigma > 1$, which involves Legendre functions. The method used here also applies to $1/\zeta(1+it)$ as will be shown.

1. Let γ be the Euler constant.

THEOREM 1. *There exist arbitrarily large t for which*

$$|\zeta(1+it)| \geqslant e^{\gamma} \log\log t + O(1).$$

This improves slightly the result of Littlewood ([2]; [5], Theorem 8.9A).

THEOREM 2. *Let $\frac{1}{2} \leqslant c < 1$. Then there exists a positive constant B independent of c such that for sufficiently large T*

$$\max_{1\leqslant |t| \leqslant T} |\zeta(c+it)| \geqslant e^{B(\log T)^{1-c}/\log\log T}.$$

This improves slightly the result of Titchmarsh ([4]; [5], Theorem 8.12). The following improves slightly the result of Littlewood ([3]; [5], Theorem 8.9B).

THEOREM 3. *There exist arbitrarily large values of t for which*

$$\frac{1}{|\zeta(1+it)|} \geqslant \frac{6e^{\gamma}}{\pi^2} (\log\log t - \log\log\log t) + O(1).$$

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The proofs of Theorems 1 and 2 are entirely elementary except for the use of the fact that the integral of an analytic function around a rectangle is zero. For Theorem 3 we use a weak upper bound for $1/\zeta(s)$ for $\sigma \geqslant 1$ such as $O(\log^7 t)$, ([5], (3.6.5)).

The simple idea involved in choosing n appropriately is revealed in the Remark following the statement of Lemma 2.1. There are actually many choices of n that will work. However I was unable to find a large enough number of them which when combined substantially improve the Theorems.

Professor P. Turán has pointed out to me that under R. H., applying the inequality of Hadamard–Borel–Carathèodory to $\log\zeta(s)$, leads to Ω-results for $1/\zeta(s)$, $\frac{1}{2} < c < 1$, of the same kind as in Theorem 2 since otherwise Theorem 2 would be contradicted.

2. In what follows p will denote a prime number and k an integer exceeding 2. As usual let $[x]$ denote the integer part of x.

LEMMA 2.1. *Let c satisfy $\frac{1}{2} \leqslant c \leqslant 1$. Let the integer $m(=m_p)$ be defined by*

$$m = \left[\frac{k}{p^c-1}\right]. \tag{2.1}$$

Then there exists a constant A_1 such that

$$\left|\log\frac{d_k(p^m)}{p^{mc}} - k\log\frac{p^c}{p^c-1} + \frac{1}{2}\log m\right| \leqslant A_1.$$

Remark. For the moment let the integer m merely be positive. Then from consideration of the factor $(1-1/p^s)^{-k}$ in the product representation of $\zeta^k(s)$ it follows easily that

$$\frac{d_k(p^m)}{p^{mc}} = \frac{(k-1+m)!}{(k-1)!\,m!\,p^{mc}}. \tag{2.2}$$

Clearly this increases with m so long as

$$\frac{k-1-m}{mp^c} \geqslant 1$$

or so long as $k-1 \geqslant m(p^c-1)$. This motivates the choice of m in (2.1) as the way to maximize $d_k(p^m)/p^{mc}$. Since $d_k(n)/n^c$ is a multiplicative function, the above maximization leads easily to the choice of n which will essentially maximize $d_k(n)/n^c$.

Proof of Lemma 2.1. Let $p^c \leqslant k$ so that $m \geqslant 1$ and let

$$R = \log\frac{d_k(p^m)}{p^{mc}}$$

with m as in (2.1). Since $\log j! = (j+\frac{1}{2})\log j - j + O(1)$ for $j \geqslant 1$ it follows from (2.2) that

$$\begin{aligned} R &= (k-\tfrac{1}{2}+m)\log(k-1+m) - (k-\tfrac{1}{2})\log(k-1) - (m+\tfrac{1}{2})\log m - \\ &\qquad - cm\log p + O(1) \\ &= I + H - \tfrac{1}{2}\log m + O(1) \end{aligned} \tag{2.3}$$

where

$$I = \left(k-\frac{1}{2}\right)\log\left(1+\frac{m}{k-1}\right), \quad H = m\log\frac{k-1+m}{mp^c}.$$

From the definition of m

$$m = \frac{k}{p^c-1} - \alpha, \quad 0 \leqslant \alpha < 1 \tag{2.4}$$

and so

$$I = \left(k-\frac{1}{2}\right)\log\left(1+\frac{k}{k-1}\cdot\frac{1}{p^c-1} - \frac{\alpha}{k-1}\right) = \left(k-\frac{1}{2}\right)\log\left(1+\frac{1}{p^c-1}+O\left(\frac{1}{k}\right)\right)$$
$$= \left(k-\frac{1}{2}\right)\log\left(\frac{p^c}{p^c-1}\right) + O(1) = k\log\frac{p^c}{p^c-1} + O(1).$$

Using (2.4) again

$$H = m\log\left(1+\frac{k-1-mp^c+m}{mp^c}\right) = m\log\left(1+\frac{\alpha(p^c-1)-1}{mp^c}\right).$$

Because $m \geqslant 1$, $0 \leqslant \alpha < 1$, $p^c \geqslant 2^{1/2}$

$$-\frac{1}{m2^{1/2}} \leqslant \frac{\alpha(p^c-1)-1}{mp^c} \leqslant \frac{1}{m}$$

and so $H = O(1)$. Hence from (2.3) follows

$$R = k\log\frac{p^c}{p^c-1} - \frac{1}{2}\log m + O(1)$$

which proves the lemma.

LEMMA 2.2. (The case $c = 1$). *Let m $(=m_p)$ be defined as in 2.1. Let*

$$n = \prod_{p\leqslant k} p^m. \tag{2.5}$$

Then as k increases

$$\left(\frac{d_k(n)}{n}\right)^{1/k} = e^{\gamma}\log k + O(1). \tag{2.6}$$

Moreover

$$\log n = k\log k + O(k). \tag{2.7}$$

Proof. Let $\pi(x)$ represent the number of primes not exceeding x. Then $\pi(x) = O(x/\log x)$. By Lemma 2.1 with $c = 1$

$$(2.8)\qquad \log\frac{d_k(n)}{n} = -k\sum_{p\leqslant k}\log\left(1-\frac{1}{p}\right)-\frac{1}{2}\sum\log\left[\frac{k}{p-1}\right]+O(\pi(k)).$$

For $p\leqslant k$

$$0\leqslant\log\left[\frac{k}{p-1}\right]\leqslant\log\frac{k}{p-1}\leqslant\log\frac{k}{p}+1.$$

Let

$$I=\sum\log\left[\frac{k}{p-1}\right]\leqslant\sum\log\frac{k}{p}+\pi(k).$$

Then with the sums for $p\leqslant k$

$$(2.9)\qquad \sum\log\frac{k}{p}=\sum\int_p^k\frac{dv}{v}=\int_2^k\frac{dv}{v}\sum_{p\leqslant v}1=\int_2^k\frac{\pi(v)}{v}\,dv=O\left(\frac{k}{\log k}\right).$$

Hence in (2.8)

$$(2.10)\qquad \frac{1}{k}\log\frac{d_k(n)}{n}=-\sum\log\left(1-\frac{1}{p}\right)+O\left(\frac{1}{\log k}\right).$$

It is an elementary result ([1], (22.7.4)) that

$$\sum_{p\leqslant x}\frac{1}{p}=\log\log x+B_1+O\left(\frac{1}{\log x}\right)$$

where B_1 is a constant and that ([1], (22.8.1))

$$\sum_{p<\infty}\left\{\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\}=B_1-\gamma$$

and so

$$\sum_{p\leqslant x}\left\{\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\}=B_1-\gamma+O\left(\frac{1}{x}\right).$$

Hence

$$(2.11)\qquad \sum_{p\leqslant x}\log\left(1-\frac{1}{p}\right)=-\log\log x-\gamma+O\left(\frac{1}{\log x}\right).$$

Setting $x=k$ above, (2.10) becomes

$$\frac{1}{k}\log\frac{d_k(n)}{n}=\log\log k+\gamma+O\left(\frac{1}{\log k}\right)$$

which proves (2.6).

From (2.5) with the sum for $p\leqslant k$

$$\log n=\sum\left[\frac{k}{p-1}\right]\log p.$$

Since $\sum\log p=O(k)$,

$$\log n=k\sum\frac{\log p}{p-1}+O(k).$$

Since $\sum\log p/p(p-1)=O(1)$,

$$\log n=k\sum\frac{\log p}{p}+O(k)=k\log k+O(k)$$

by a well known elementary result which proves (2.7).

3. Here the proof of Theorem 1 will be given. It will be useful to recall the elementary formula ([5], (2.1.4))

$$\zeta(s)=\frac{s}{s-1}-s\int_1^\infty\frac{x-[x]}{x^{s+1}}\,dx$$

valid for $\sigma>0$. For $\sigma\geqslant\frac{1}{2}$ and t large

$$(3.1)\qquad |\zeta(s)|\leqslant 2+|s|\int_1^\infty\frac{dx}{x^{\sigma+1}}<3t,\qquad \sigma\geqslant\frac{1}{2}.$$

Proof of Theorem 1. With n as in (2.5), $a>0$ and $b>0$, let

$$(3.2)\qquad J=\frac{\pi^{-1/2}}{ib}\int_{1+a-i\infty}^{1+a+i\infty}\zeta^k(s)n^s e^{(s-1-a)^2/b^2}\,ds.$$

Then by (0.1)

$$J=\sum\frac{d_k(j)}{j^{1+a}}n^{1+a}\frac{\pi^{-1/2}}{b}\int_{-\infty}^{\infty}e^{it\log n/j-t^2/b^2}\,dt=\sum\frac{d_k(j)}{j^{1+a}}n^{1+a}\exp\left(-\frac{b^2}{4}\log^2\frac{n}{j}\right)$$

by a familiar integral formula. Since $d_k(j)>0$, it follows that

$$(3.3)\qquad J>d_k(n).$$

Divide J in (3.2) into three parts J_1, J_2 and J_3 with respective ranges of integration $(1+a-i\infty,1+a-i)$, $(1+a-i,1+a+i)$ and $(1+a+i,1+a+i\infty)$. Since the only singularity of $\zeta(s)$ is the pole at $s=1$, for some A,

$$(3.4)\qquad |\zeta(s)|\leqslant\frac{1}{|s-1|}+A,\qquad |s-1|\leqslant 2,$$

and so it follows easily that

$$|J_2| \leqslant \frac{\pi^{-1/2}}{b}\int_{-1}^{1}\left(\frac{1}{a}+A\right)^k n^{1+a}\,dt.$$

This is minimized essentially by choosing $a = k/\log n$. For this choice

$$(3.5)\qquad |J_2| \leqslant \frac{2ne^k}{b}\left(\frac{\log n}{k}+A\right)^k.$$

Since by (2.7) $\log n = k\log k + O(k)$

$$|J_2| \leqslant \frac{2n}{b}\, e^k(\log k)^k\left(1+O\left(\frac{1}{\log k}\right)\right)^k.$$

If $b = e^k$ it follows from (2.6) that for large k

$$(3.6)\qquad |J_2| \leqslant 4d_k(n)\,e^{-\gamma k}.$$

To appraise J_3 we deform the integral to a sum of two, J_{31} and J_{32}, with the help of (3.1), where J_{31} is for $s = \sigma + i$, $1 < \sigma < 1+a$, and J_{32} is for $s = 1+it$, $1 < t$. On J_{31}, $|\zeta(\sigma+it)| \leqslant 1+A$ by (3.4). Hence

$$|J_{31}| \leqslant \frac{n^{1+a}(1+A)^k}{b}\,\frac{e^{(a^2-1)/b^2}}{\pi^{1/2}}\int_1^{1+a} d\sigma.$$

Since a is small for large k

$$(3.7)\qquad |J_{31}| \leqslant \frac{n^{1+a}(1+A)^k}{b} = n(1+A)^k.$$

For J_{32} we divide the range of integration into two parts $1 \leqslant t \leqslant T$ and $T < t$. Hence since b is large

$$(3.8)\qquad |J_{32}| \leqslant \frac{n}{b}\int_1^T |\zeta(1+it)|^k e^{-t^2/b^2}\,dt + J'$$

where using (3.1)

$$J' = \frac{3^k n}{b}\int_T^\infty t^k e^{-t^2/b^2}\,dt.$$

Let $T = 2kb$. Since

$$\frac{d}{dt}\left(k\log t - \frac{t^2}{b^2}\right) = \frac{k}{t} - \frac{2t}{b^2} \leqslant -\frac{1}{b}$$

for $t \geqslant 2kb = T$ it follows that

$$(3.9)\qquad J' \leqslant T^k e^{-T^2/b^2}\,\frac{3^k n}{b}\int_T^\infty e^{-(t-T)/b}\,dt = T^k e^{-4k^2}3^k n.$$

Since $\log T = k + \log 2k$

$$J' \leqslant n\exp\{k(k+6\log 6k) - 4k^2\} \leqslant n$$

for large k. Thus if $M_T = \max|\zeta(1+it)|$ for $1 \leqslant |t| \leqslant T$ then

$$|J_{32}| \leqslant nM_T^k\,\frac{1}{b}\int_0^\infty e^{-t^2/b^2}\,dt + n \leqslant nM_T^k + n.$$

Using (3.7)

$$|J_3| \leqslant nM_T^k + n(1+A)^k + n \leqslant nM_T^k + e^{-\gamma k}d_k(n).$$

Clearly $|J_1|$ has the same dominant. Using this and (3.6) in (3.3)

$$(1e - 6^{-\gamma k})\,\frac{d_k(n)}{n} \leqslant 2M_T^k.$$

By (2.6)

$$(1-6e^{-\gamma k})^{1/k}\big(e^\gamma \log k + O(1)\big) \leqslant 2^{1/k}M_T$$

or

$$e^\gamma \log k + O(1) \leqslant M_T.$$

Since $\log T = k + \log 2k$,

$$\log\log T = \log k + O\left(\frac{\log k}{k}\right)$$

and so

$$e^\gamma \log\log T + O(1) \leqslant M_T$$

which proves Theorem 1.

4. For Theorem 2 the following lemma is required.

LEMMA 4.1. *For $\frac12 \leqslant c < 1$ there is a constant $A_2 > 2$ such that if*

$$(4.1)\qquad y = \left(\frac{k}{A_2}\right)^{1/c},$$

if m $(= m_p)$ is given by (2.1) and

$$n = \prod_{p\leqslant y} p^m,$$

then there exists a constant $A_3 > 0$ such that

$$(4.2)\qquad \frac{1}{k}\log\frac{d_k(n)}{n^c} \geqslant A_3\,\frac{(\log n)^{1-c}}{\log\log n}$$

for large k. Moreover $\log n = O(k^3)$ and

$$(4.3)\qquad \log n \geqslant \frac{k^{1/c}}{4A_2}.$$

Proof of Lemma 4.1. By Lemma 2.1

$$\log\frac{d_k(n)}{n^c} \geqslant \sum_{p\leqslant y} H(p,c,k) \tag{4.4}$$

where

$$\begin{aligned} H &= k\log\frac{p^c}{p^c-1} - \frac{1}{2}\log\frac{k}{p^c-1} - A_1 \\ &= \left(k-\frac{1}{2}\right)\log\frac{p^c}{p^c-1} - \frac{1}{2}\log\frac{k}{p^c} - A_1 \\ &\geqslant \left(k-\frac{1}{2}\right)\frac{1}{p^c} - \frac{1}{2}\log\frac{k}{p^c} - A_1 \geqslant \frac{k}{p^c} - \frac{1}{2}\log\frac{k}{p^c} - \frac{A_2}{4} \end{aligned} \tag{4.5}$$

where $A_2 = 4(A_1+\frac{1}{2}) > 2$.

It will now be shown that

$$x - \frac{1}{2}\log x - \frac{A_2}{4} \geqslant \frac{1}{2}x, \quad x \geqslant A_2. \tag{4.6}$$

Indeed $x-\log x-\frac{1}{2}A_2$ is an increasing function of x for $x>1$ and it is positive for $x = A_2 > 2$ since $\frac{1}{2}v-\log v$ is increasing for $v\geqslant 2$ and is positive for $v=2$. By (4.6) it follows from (4.5) that

$$H \geqslant \frac{1}{2}\frac{k}{p^c} \quad \text{for} \quad \frac{k}{p^c} \geqslant A_2.$$

Hence, if $\sum$ is for $p\leqslant y$, (4.4) implies

$$\frac{1}{k}\log\frac{d_k(n)}{n^c} \geqslant \frac{1}{2}\sum\frac{1}{p^c}. \tag{4.7}$$

From the definition of n

$$\begin{aligned} \log n &= \sum m\log p \leqslant k\sum\frac{\log p}{p^c-1} < k\sum\frac{\log p}{p^c} + 4k\sum\frac{\log p}{p^{2c}} \\ &\leqslant k\sum\frac{\log p}{p^c} + 4k\sum\frac{\log p}{p} \leqslant k\log y\sum\frac{1}{p^c} + O(k\log y). \end{aligned} \tag{4.8}$$

This implies that

$$\sum\frac{1}{p^c} \geqslant \frac{\log n}{k\log y} + O(1)$$

and so by (4.7)

$$\frac{1}{k}\log\frac{d_k(n)}{n^c} \geqslant \frac{\log n}{2k\log y} + O(1) = \frac{c\log n}{2k(\log k-\log A_2)} + O(1). \tag{4.9}$$

Since $p^c \leqslant k/A_2$

$$m = \left[\frac{k}{p^c-1}\right] \geqslant \frac{k}{p^c-1} - 1 \geqslant \frac{k-p^c}{p^c-1} \geqslant \frac{k(1-1/A_2)}{p^c-1} \geqslant \frac{k}{2p^c}$$

and so

$$\log n = \sum m\log p \geqslant \frac{k}{2}\sum\frac{\log p}{p^c} \geqslant \frac{k}{2y^c}\sum\log p \geqslant \frac{ky}{4y^c}$$

for large k, and hence for large y. Using (4.1) this implies that

$$\log n \geqslant \frac{k^{1/c}}{4A_2^{1/c-1}} \geqslant \frac{k^{1/c}}{4A_2}$$

and so

$$k \leqslant (4A_2\log n)^c$$

which proves (4.3) and in (4.9) implies that for large k

$$\frac{1}{k}\log\frac{d_k(n)}{n^c} \geqslant \frac{A_3\log n}{(\log n)^c\log\log n}$$

for some A_3 which is (4.2).

From the top line of (4.8)

$$\log n \leqslant k\sum\frac{\log p}{p^c-1} \leqslant \frac{k}{2^{1/2}-1}\sum\log p = kO(y) = O(k^3)$$

by (4.1).

Proof of Theorem 2. Here n is defined as in Lemma 4.1. Then with that n and J given by (3.2) there follows exactly as in (3.3)

$$J > d_k(n). \tag{4.10}$$

With $a = k/\log n$ and J divided into J_1, J_2 and J_3 as below (3.3) there follows (3.5) which since $\log n = O(k^3)$ gives for large k

$$J_2 = O\left(\frac{n}{b}e^{3k\log k}\right).$$

Let $b = n$. By (4.3)

$$\log k \leqslant \log\log n$$

for large k. Hence

$$J_2 = O(e^{3k\log\log n}). \tag{4.11}$$

Here, where $c<1$, J_1 and J_3 are indented to the line $\sigma = c$ instead of $\sigma = 1$. Then as in the first part of (3.7)

$$|J_{31}| \leqslant \frac{n^{1+a}(1+A)^k}{b} \leqslant e^k(1+A)^k \tag{4.12}$$

since here $b = n$ and as before $a = k/\log n$. In place of (3.8)

$$|J_{32}| \leqslant \frac{n^c}{b}\int_1^T |\zeta(c+it)|^k e^{-t^2/b^2}\,dt + J' \tag{4.13}$$

where now

$$J' = \frac{3^k n^c}{b}\int_T^\infty t^k e^{-t^2/b^2}\,dt.$$

Let $T = k^3 b$ so that much as before

$$J' \leqslant 3^k n^c T^k e^{-T^2/b^2}\frac{1}{b}\int_T^\infty e^{-(t-T)b} = n^c \exp\{-k^6 + k(3\log k + \log 3 + \log n)\}$$

and since $\log n = O(k^3)$

$$J' = O(n^c e^{-k^6/2}).$$

Hence if $M_T = \max|\zeta(c+it)|$ for $1 \leqslant t \leqslant T$ then

$$|J_3| \leqslant n^c M_T^k + n^c O(e^{-k^6/2}) + e^k(1+A)^k.$$

With (4.11) and (4.10)

$$\frac{d_k(n)}{n^c} \leqslant 2M_T^k + O(e^{3k\log\log n}).$$

Since $(\alpha+\beta)^{1/k} \leqslant \alpha^{1/k} + \beta^{1/k}$ for $\alpha, \beta \geqslant 0$,

$$\left(\frac{d_k(n)}{n^c}\right)^{1/k} \leqslant 2^{1/k} M_T + O\big((\log n)^3\big).$$

By (4.2) for any fixed c, $\frac{1}{2} \leqslant c < 1$ and sufficiently large k, and hence n, this implies

$$M_T \geqslant \exp\left(\frac{A_3}{2}\frac{(\log n)^{1-c}}{\log\log n}\right). \tag{4.14}$$

Since $T = k^3 n$ it follows that $\log n = \log T - 3\log k$ and that

$$\log\log n = \log\log T + O\left(\frac{\log k}{\log T}\right).$$

Hence

$$\frac{(\log n)^{1-c}}{\log\log n} = \frac{(\log T)^{1-c}}{\log\log T}\left(1 - O\left(\frac{\log k}{\log T}\right)\right).$$

Since $\log T > \log n \geqslant k^{1/c}/4A_2$

$$\frac{\log k}{\log T} = O\left(\frac{\log k}{k^{1/c}}\right)$$

and so for large k

$$\frac{(\log n)^{1-c}}{\log\log n} \geqslant \frac{1}{2}\frac{(\log T)^{1-c}}{\log\log T}$$

which in (4.14) proves Theorem 2.

5. For $\sigma > 1$

$$\left(\frac{1}{\zeta(s)}\right)^k = \sum \frac{a_k(j)}{j^s} \tag{5.1}$$

where from $(1-1/p^s)^k$ follows easily

$$a_k(p^r) = (-1)^r \frac{k!}{r!(k-r)!}. \tag{5.2}$$

Maximizing $|a_k(p^r)/p^r|$ much as in the Remark preceding Lemma 1.1 leads to choosing $r = m$ where

$$m = \left[\frac{k}{p+1}\right]. \tag{5.3}$$

LEMMA 5.1. *With m $(= m_p)$ as above let*

$$n = \prod_{p<k} p^m. \tag{5.4}$$

Then

$$\frac{1}{k}\log\frac{|a_k(n)|}{n} = \sum_{p<k}\log\left(\frac{p+1}{p}\right) + O\left(\frac{1}{\log k}\right). \tag{5.5}$$

Moreover

$$\log n = k\log k + O(k). \tag{5.6}$$

Proof of Lemma 5.1. Since $a_k(n)/n$ is multiplicative it follows from (5.2) with $r = m$ $(= m_p)$ that, if $\sum$ is for $p < k$, then

$$R = \log\frac{|a_k(n)|}{n} = \sum\{\log k! - \log m! - \log(k-m)! - m\log p\}$$

$$= \sum\left\{\left(k+\frac{1}{2}\right)\log k - \left(m+\frac{1}{2}\right)\log m - \left(k-m+\frac{1}{2}\right)\times \right.$$

$$\left. \times\log(k-m) - m\log p\right\} + O\left(\frac{k}{\log k}\right)$$

$$= \sum\left\{\left(k+\frac{1}{2}\right)\log\frac{k}{k-m} + m\log\frac{k-m}{mp}\right\} + O\left(\frac{k}{\log k}\right)$$

where $\frac{1}{2}\sum\log m$ was appraised as in (2.9).

From (5.3), $m = k/(p+1)-a$ where $0 \leqslant a < 1$. Hence

$$\left(k+\frac{1}{2}\right)\log\frac{k}{k-m} = \left(k+\frac{1}{2}\right)\log\frac{p+1}{p+a(p+1)/k}$$
$$= \left(k+\frac{1}{2}\right)\log\frac{p+1}{p} - \left(k+\frac{1}{2}\right)\log\frac{p+a(p+1)/k}{p}$$
$$= \left(k+\frac{1}{2}\right)\log\frac{p+1}{p} + O(1)$$

and

$$m\log\frac{k-m}{mp} = m\log\left(1+\frac{k-m(p+1)}{mp}\right) = m\log\left(1+\frac{a(p+1)}{mp}\right) = O(1).$$

Therefore

$$R = \sum\left(k+\frac{1}{2}\right)\log\frac{p+1}{p} + O\left(\frac{k}{\log k}\right)$$
$$= k\sum\log\frac{p+1}{p} + \frac{1}{2}\sum\frac{1}{p} + O\left(\frac{k}{\log k}\right)$$

which easily yields (5.5). The proof of (5.6) is the same as for (2.7).

Proof of Theorem 3. With n as in (5.4) let

(5.7) $$J = \frac{\pi^{-1/2}}{bi}\int_{-i\infty+2}^{i\infty+2}(\zeta(s))^{-k}n^s e^{(s-2)^2/b^2}\,ds.$$

By (5.1)

$$J = \sum\frac{a_k(j)}{j^2}n^2\exp\left(-\frac{b^2}{4}\log^2\frac{n}{j}\right).$$

Hence

(5.8) $$|J| \geqslant |a_k(n)| - I$$

where, using $|\log n/j| > 1/2n$, $j \neq n$,

(5.9) $$I = \sum\frac{|a_k(j)|}{j^2}n^2e^{-b^2/16n^2}.$$

Since by (5.2)

$$|a_k(j)| \leqslant d_k(j), \qquad \sum\frac{|a_k(j)|}{j^2} \leqslant 2\sum\frac{d_k(j)}{j^2}\log j + 1.$$

But

$$\sum\frac{d_k(j)}{j^2}\log j = \frac{1}{2\pi i}\int_{-i\infty+3/2}^{i\infty+3/2}\frac{\zeta^k(s)}{(s-2)^2}\,ds = O(e^{2k})$$

since $\zeta(3/2) < e^2$. Hence by (5.9)

$$I = O(e^{2k})n^2e^{-b^2/16n^2}.$$

Let $b = 8nk$. Then

$$I = O(e^{2k})n^2e^{-4k^2}.$$

Using (5.6) it follows easily that

$$I = O(e^{-k^2}).$$

Hence

(5.10) $$|J| \geqslant |a_k(n)| + O(e^{-k^2}).$$

Next using ([5], (3.6.5))

$$\frac{1}{|\zeta(s)|} = O\big((\log t)^7\big), \qquad \sigma \geqslant 1$$

for $|t| \geqslant 1$ in (5.7) and moving the contour to $\sigma = 1$ gives

$$|J| \leqslant \frac{n}{b}\int_{-\infty}^{\infty}|\zeta(1+it)|^{-k}e^{-t^2/b^2}\,dt$$

for large k, and hence large b. Let $T = k^2b$ and let

$$J_1 = \frac{n}{b}\int_0^T|\zeta(1+it)|^{-k}e^{-t^2/b^2}\,dt$$

and

$$J_2 = \frac{n}{b}\int_T^{\infty}|\zeta(1+it)|^{-k}e^{-t^2/b^2}\,dt.$$

Then

(5.11) $$|J| \leqslant 2J_1 + 2J_2.$$

Clearly

(5.12) $$J_1 \leqslant n\max_{|t|\leqslant T}|\zeta(1+it)|^{-k}$$

and since $(\log t)^7/t \to 0$ as $t \to \infty$,

$$J_2 \leqslant \frac{n}{b}\int_T^{\infty}t^ke^{-t^2/b^2}\,dt \leqslant T^ke^{-T^2/b^2}\frac{n}{b}\int_T^{\infty}e^{-(t-T)/b}\,dt \leqslant ne^{k\log T-k^4}$$

much as in (3.9). Evidently

$$\log T = \log k^2 + \log b = \log 8k^3 + \log n = O(k\log k)$$

by (5.6). Hence

(5.13) $$J_2 = O(ne^{-k^4/2}).$$

Using (5.10), (5.11), (5.12) and (5.13)

$$2\max_{|t|\leqslant T}|\zeta(1+it)|^{-k} + O(e^{-k^2}) \geqslant \left|\frac{a_k(n)}{n}\right|$$

and so by (5.5)

$$2^{1/k}\max_{|t|\leqslant T}|\zeta(1+it)|^{-1}+O(e^{-k})\geqslant\prod_{p<k}\left(1+\frac{1}{p}\right)\left(1+O\left(\frac{1}{\log k}\right)\right)$$

$$=\prod_{p<k}\left(\frac{1}{1-1/p}\right)\prod_{p<k}\left(1-\frac{1}{p^2}\right)\left(1+O\left(\frac{1}{\log k}\right)\right)$$

$$=\frac{e^{\gamma}\log k}{\zeta(2)}+O(1)$$

as in (2.11). Since $T = k^2b = 8k^3n$,

$$\log T = 3\log k+\log 8+k\log k+O(k) = k\log k+O(k).$$

Hence

$$\log\log T = \log k+\log\log k+O\left(\frac{1}{\log k}\right)$$

and so for large k, $\log\log T>\log k$ and so

$$\log k\geqslant\log\log T-\log\log\log T+O\left(\frac{1}{O(1)}\right).$$

Thus

$$\max_{|t|\leqslant T}\frac{1}{|\zeta(1+it)|}\geqslant\frac{e^{\gamma}6}{\pi^2}(\log\log T-\log\log\log T)+O(1)$$

which completes the proof.

References

[1] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford 1960.

[2] J. E. Littlewood, *On the Riemann zeta-function*, Proc. Lond. Math. Soc. (2) 24 (1925), pp. 175–201.

[3] — *On the function* $1/\zeta(1+it)$, Proc. Lond. Math. Soc. (2) 27 (1928), pp. 349–357.

[4] E. C. Titchmarsh, *On an inequality satisfied by the zeta-function of Riemann*, Proc. Lond. Math. Soc. (2) 28 (1928), pp. 70–80.

[5] — *The theory of the Riemann zeta-function*, Oxford 1951.

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