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OBJECTIVE FUNCTION APPROXIMATIONS IN MATHEMATICAL PROGRAMMING*

Arthur M. GEOFFRION
University of California at Los Angeles, CA, U.S.A.

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Mathematical programming applications often require an objective function to be approximated by one of simpler form so that an available computational approach can be used. An a priori bound is derived on the amount of error (suitably defined) which such an approximation can induce. This leads to a natural criterion for selecting the "best" approximation from any given class. We show that this criterion is equivalent for all practical purposes to the familiar Chebyshev approximation criterion. This gains access to the rich legacy on Chebyshev approximation techniques, to which we add some new methods for cases of particular interest in mathematical programming. Some results relating to post-computational bounds are also obtained.

Keywords: Approximation, Errorbounds, Modeling.

Most applications of mathematical programming require the modeler to exercise some discretion in estimating or approximating the objective function to be optimized. We give a simple a priori bound relating the amount of objective function approximation error to the amount of error thereby induced in the solution of the corresponding optimization problem. This furnishes a natural criterion to guide the choice of an estimated or approximate objective function. The criterion can often be applied via simple graphical constructions that we develop for the case of linear separability, and we show that it is generally equivalent to the familiar Chebyshev criterion – which thereby provides direct access to a powerful array of established results and techniques for the general case.

In addition to a priori error bounds, which facilitate the design of an objective function *before* doing any optimization, we also discuss the tighter error bounds available *after* an optimization has been performed. This leads to a natural subjective tie-breaking rule for use in conjunction with the primary criterion and also to the notion of "retrofit" objective functions: improved hybrids between the approximation actually used and the true unapproximated objective function. Since using a retrofit objective function in place of the approximate one would

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not destroy the optimality of the solution to the approximating problem, the analyst has the option of interpreting the optimization results as though they had been obtained using the retrofit objective function.

The results of Section 1 are applied in a related paper [4] to obtain new aggregation results in a specific applications context.

1. Basic results

Let the following two optimization problems be given:

(P) Minimize $f(x)$, subject to $x \in X$

($\tilde{\text{P}}$) Minimize $\tilde{f}(x)$, subject to $x \in X$,

where X is an arbitrary non-empty set and f and $\tilde{f}$ are both real-valued functions bounded below on X. Interpret (P) as the "true" problem and ($\tilde{\text{P}}$) as the "approximating" problem in the sense that an approximate objective function $\tilde{f}$ is used in place of f. What can be said about the relationship between (P) and ($\tilde{\text{P}}$) when the difference between f and $\tilde{f}$ can be bounded on X?

In the absence of further assumptions guaranteeing the existence of optimal solutions, it is necessary to phrase the answer to this question in terms of epsilon-optimal solutions, that is, in terms of feasible solutions having an objective function value known only to be within epsilon of the true infimal value. Let $v(\text{P})$ denote the infimal value of (P) and similarly for $v(\tilde{\text{P}})$.

Theorem 1 (*Objective Function Approximation*). *Let $\underline{\epsilon}$ and $\bar{\epsilon}$ be scalars (not necessarily nonnegative) satisfying*

$$-\underline{\epsilon} \leq \tilde{f}(x) - f(x) \leq \bar{\epsilon} \quad \text{for all } x \in X. \tag{1}$$

Then

$$-\underline{\epsilon} \leq v(\tilde{\text{P}}) - v(\text{P}) \leq \bar{\epsilon} \tag{2}$$

and, for any $\epsilon \geq 0$, any ϵ-optimal solution $\tilde{x}$ of ($\tilde{\text{P}}$) will necessarily be $(\epsilon + \underline{\epsilon} + \bar{\epsilon})$-optimal in (P).

Proof. Writing the first inequality of (1) as

$$f(x) \leq \tilde{f}(x) + \underline{\epsilon} \quad \text{for all } x \in X,$$

it is evident that

$$\operatorname*{Inf}_{x \in X} f(x) \leq \operatorname*{Inf}_{x \in X} \tilde{f}(x) + \underline{\epsilon}.$$

This is the first inequality of (2). The second is proved similarly. Now let $\tilde{x}$ be an

ϵ-optimal solution of $(\tilde{\mathrm{P}})$ for some $\epsilon \geq 0$. We have

$$\begin{aligned} f(\tilde{x}) - \underline{\epsilon} &\leq \tilde{f}(\tilde{x}) && \text{by the first inequality of (1)} \\ &\leq \operatorname*{Inf}_{x \in X} \tilde{f}(x) + \epsilon && \text{by the definition of } \tilde{x} \\ &\leq \operatorname*{Inf}_{x \in X} f(x) + \epsilon + \bar{\epsilon} && \text{by the second inequality of (2).} \end{aligned}$$

Thus $f(\tilde{x}) \leq v(\mathrm{P}) + (\epsilon + \underline{\epsilon} + \bar{\epsilon})$. Since $\tilde{x} \in X$, $\tilde{x}$ satisfies the definition of $(\epsilon + \underline{\epsilon} + \bar{\epsilon})$-optimality in (P).

This result can hardly be considered new (cf. [2]), but its importance for modeling in mathematical programming seems largely to have been overlooked. The writer has been unable to find the result in any textbook, even in the context of separable programming where, as shown in Section 2, it is especially easy and natural to apply.

It would be difficult to overstate the possible usefulness of being able to place a bound on the solution error of $(\tilde{\mathrm{P}})$ *before* solving it. Such a priori bounds can be used to design objective function approximations within a given tractable class that guarantee as much accuracy as possible. This will be illustrated in Section 2.

Notice that Theorem 1 draws no conclusion whatever regarding the "distance" between optimal solutions of (P) and $(\tilde{\mathrm{P}})$. It is this writer's opinion that this is not a drawback in most practical applications, since bounds relating to the value of f seem to address more directly a decision-maker's concerns when using $(\tilde{\mathrm{P}})$ as a surrogate for (P). Bounds in decision space rather than payoff space appear to require additional structure on X and a Lipschitz condition on f.

Theorem 1 provides a natural criterion for the choice of $\tilde{f}$ when alternatives are available, as when a linear objective function is desired in the presence of known nonlinearities. Let $\tilde{f}$ be drawn from some class of functions indexed by the parameter vector p constrained to a set P; the notation $\tilde{f}(\cdot\,; p)$ refers to a particular choice. Then the *natural criterion* for the choice of p is to select that member of P which leads to the smallest permissible value of $\underline{\epsilon} + \bar{\epsilon}$.

If $\underline{\epsilon}$ and $\bar{\epsilon}$ are selected to be as tight as possible for each allowable choice of p, the natural criterion is embodied by the optimization problem

$$\operatorname*{Minimize}_{p \in P} \, [\operatorname*{Sup}_{x \in X} \{f(x) - \tilde{f}(x; p)\} + \operatorname*{Sup}_{x \in X} \{\tilde{f}(x; p) - f(x)\}]. \tag{3}$$

The first (second) term within brackets is the tightest possible value for $\underline{\epsilon}(\bar{\epsilon})$ for a given p. One would interpret $\underline{\epsilon}(\bar{\epsilon})$ as the maximum amount by which $\tilde{f}$ can undershoot (overshoot) f on X. Note that neither $\underline{\epsilon}$ nor $\bar{\epsilon}$ need be nonnegative.

An alternative way of expressing (3) is

$$\begin{aligned} \operatorname*{Minimize}_{p, \underline{\epsilon}, \bar{\epsilon}} \quad & \underline{\epsilon} + \bar{\epsilon}, && \qquad (4) \\ \text{subject to} \quad & f(x) - \tilde{f}(x; p) \leq \underline{\epsilon} && \text{for all } x \in X, \\ & \tilde{f}(x; p) - f(x) \leq \bar{\epsilon} && \text{for all } x \in X, \\ & p \in P. \end{aligned}$$

This has particular appeal when $\tilde{f}$ is to be chosen based on knowledge of f at only a finite number of points.

How can one solve (3) or (4)? The following theoretical result can be useful.

Theorem 2. *Let P be a convex set and let $\tilde{f}(x;\cdot)$ be linear on P for every fixed x in X. Then* (3), *the problem of finding the p in P which is best by the natural criterion, is a convex programming problem.*

Proof. The first term within brackets in (3) is the supremum of a collection (indexed by x) of functions that are assumed linear in p on P, and hence it is a convex function of p on P. The same is true of the second term within the brackets. Thus (3) has a convex objective function defined over a convex set.

It is to be stressed that $\tilde{f}$ can be decidedly nonlinear (even discontinuous) in x for fixed p; linearity in p for fixed x is not as restrictive an assumption as one might think at first glance.

Theorem 2 can simplify the task of seeking best approximations. For instance, it gives conditions under which any locally optimal choice for p must be globally optimal (this fact is used in Section 2 to help justify a simple graphical construction for finding best linear approximations in one dimension). It also indicates when a direct computational attack on (3) by convex programming methods is possible.

Our main result toward characterizing and finding best approximations is the demonstration that the natural criterion and the familiar Chebyshev (minimax) criterion are equivalent in a very strong sense. This means that the extensive theoretical and algorithmic legacy of Chebyshev approximation (see, e.g., [3]) can be employed to determine approximations that are best by the natural criterion. For future reference, we remind the reader that the Chebyshev criterion is embodied by the optimization problem

$$\underset{p \in P}{\text{Minimize}} \, [\sup_{x \in X} |f(x) - \tilde{f}(x;p)|]. \tag{5}$$

The essential equivalence between (3) and (5) requires an assumption that is entirely innocuous in the context of (P) and ($\tilde{\text{P}}$). We shall say that $\tilde{f}$ is drawn from a class of approximation functions that includes a *simple translation parameter*, say p_0, if $\tilde{f}$ is of the form $\tilde{f}(x;p)+p_0$, where $p \in P$ and $p_0 \in P_0 \subseteq R^1$ and p_0 does not enter into P or the functional form of $\tilde{f}$ in any way. Certainly we may always make this assumption, as p_0 could be ignored when solving ($\tilde{\text{P}}$) and then used to translate the optimal value afterward if desired (it can have no influence on the optimal solution).

Lemma 1. *Assume that $\tilde{f}$ is drawn from a class that includes a simple translation parameter. The value of this parameter is a matter of indifference so far as the natural criterion is concerned.*

Proof. In this case, (3) can be written as

$$\underset{\substack{p \in P \\ p_0 \in P_0}}{\text{Minimize}} \, [\underset{x \in X}{\text{Sup}} \, \{f(x) - \tilde{f}(x; p) - p_0\} + \underset{x \in X}{\text{Sup}} \, \{\tilde{f}(x; p) + p_0 - f(x)\}] \tag{6}$$

Clearly, p_0 cancels out of the objective function entirely.

This invariance property is not satisfied by the Chebyshev criterion. It is the key to the relationship between the two criteria.

Theorem 3 (Criterion Equivalence). *Assume (as one can without loss of generality in mathematical programming) that $\tilde{f}$ is drawn from a class of approximation functions that includes an unconstrained simple translation parameter. If a choice is optimal by the Chebyshev criterion, then it is also optimal by the natural criterion. If a choice is optimal by the natural criterion, then it can be translated to become optimal by the Chebyshev criterion. Moreover, the optimal value of* (3) *is twice the optimal value of* (5).

Proof. The Chebyshev problem (5) becomes

$$\underset{\substack{p \in P \\ p_0}}{\text{Minimize}} \, [\underset{x \in X}{\text{Sup}} \, |f(x) - \tilde{f}(x; p) - p_0|]. \tag{7}$$

For any given p, one may readily verify that the optimal choice for p_0 in (7) is that which yields

$$\underset{x \in X}{\text{Sup}} \, |f(x) - \tilde{f}(x; p) - p_0| = \underset{x \in X}{\text{Sup}} \, \{f(x) - \tilde{f}(x; p) - p_0\}$$

$$= \underset{x \in X}{\text{Sup}} \, \{\tilde{f}(x; p) + p_0 - f(x)\}, \tag{8}$$

namely

$$p_0 = \tfrac{1}{2}[\underset{x \in X}{\text{Sup}} \, \{f(x) - \tilde{f}(x; p)\} - \underset{x \in X}{\text{Sup}} \, \{\tilde{f}(x; p) - f(x)\}]. \tag{9}$$

Relations (8) and (9) can be used to eliminate p_0 in (7) and express it equivalently as:

$$\underset{p \in P}{\text{Minimize}} \, \tfrac{1}{2}[\underset{x \in X}{\text{Sup}} \, \{f(x) - \tilde{f}(x; p)\} + \underset{x \in X}{\text{Sup}} \, \{\tilde{f}(x; p) - f(x)\}]. \tag{10}$$

Thus we have shown that (p^*, p_0^*) is optimal by the Chebyshev criterion if and only if p^* is optimal in (10) and (p^*, p_0^*) satisfies (9), and that the optimal value of (5) equals that of (10). We also know from Lemma 1 that (p^{**}, p_0^{**}) is optimal by the natural criterion if and only if p^{**} is optimal in (6) – the value of p_0 is immaterial. But (10) is equivalent to (6) except for the factor of $\tfrac{1}{2}$. The conclusions of the theorem are now at hand.

2. Constructing best fits in one dimension

Some particularly simple constructions are available to determine best fits of functions of a single variable. The simplicity of these constructions and the Criterion Equivalence Theorem suggest that they would be identical to standard manual methods for determining Chebyshev fits, but this does not appear to be the case. This may be due partly to simplifications obtained by exploiting the arbitrariness of the pure translation parameter for the natural criterion, and partly to the fact that most of our constructions involve assumptions that tend to be peculiar to the mathematical programming context. Whatever the reason, the Chebyshev legacy of approximation techniques is still a very rich one for cases other than those discussed here.

The one-dimensional cases treated below are especially applicable to mathematical programming applications in which the approximate problem ($\tilde{\mathrm{P}}$) will be solved by linear or mixed integer linear programming, or by a minimum cost flow network technique. Think, for instance, of a network problem with some nonlinear flow costs, or of an LP with some nonlinear activity costs. Each nonlinear univariate function requires a suitable approximation which, depending on the circumstances, may be purely linear, fixed plus variable, piecewise-linear convex, etc.

The standard procedure we envisage is that the modeler will design a suitable approximation to each nonlinear univariate function independently, as it is usually impractical to design the approximations with full regard for the joint dependencies caused by the coupling constraints of X. Suppose that X implies $0 \le x_j \le u_j$ for all j, and that $f(x)$ is of the form $\Sigma_j f_j(x_j)$. Let each f_j be approximated by $\tilde{f}_j$, with undershoot and overshoot bounds $\underline{\epsilon}_j$ and $\bar{\epsilon}_j$ satisfying

$$-\underline{\epsilon}_j \le \tilde{f}_j(x_j) - f_j(x_j) \le \bar{\epsilon}_j \quad \text{for all } 0 \le x_j \le u_j.$$

It follows that (1) holds with $\underline{\epsilon} = \Sigma_j \underline{\epsilon}_j$ and $\bar{\epsilon} = \Sigma_j \bar{\epsilon}_j$, and hence that Theorem 1 applies. One attempts to select $\tilde{f}_j$ so as to make $\underline{\epsilon}_j + \bar{\epsilon}_j$ as small as possible – that is, one applies the natural fitting criterion to f_j over the interval $[0, u_j]$. This is a practical alternative to applying the natural criterion directly to $\Sigma_j f_j$ over X.

For simplicity of notation, we shall drop the variable subscript j throughout the rest of this section. The function f to be approximated is assumed to be real-valued and bounded. For expository convenience, separate treatment will not be accorded the discrete data case; it is obvious how to adapt each fitting procedure to deal with f when it is defined on some discrete subset of $[0, u]$.

2.1. *Linear fits*

We seek the best approximation to f over $[0, u]$ of the form

$$\tilde{f}(x; p_0, p_1) = p_0 + p_1 x, \tag{11}$$

where p_0 and p_1 are scalar parameters. Recall from Lemma 1 that the value of the simple translation parameter p_0 can be selected arbitrarily. It can therefore be selected so as to make $\tilde{f}$ exactly equal to f at any preselected "anchor" point.

The value $x = 0$ is an obvious choice for an anchor point, although any other point in the interval could be used.

Finding the best approximation thus reduces to the problem of finding the best choice (according to (3)) for p_1 with p_0 fixed at $f(0)$. The best choice for p_1 can be found by a conceptually simple procedure performed directly on the graph of f. The basic idea is to start with $p_1 = \infty$, reduce it parametrically and track the loci of the x-values at which the maximum undershoot and maximum overshoot occur, and stop as soon as the point of maximum undershoot equals or exceeds the point of maximum overshoot. Possible nonexistence of a point of maximum undershoot or overshoot for some values of p_1 should be dealt with by using accumulation points of sequences tending to the corresponding supremal values of undershoot or overshoot. Possible ties for points of maximum undershoot or overshoot (whether exact or by accumulation) should be resolved by taking the smallest such point in the case of undershoot and largest such point in the case of overshoot (use the infimum or supremum in case the smallest or largest does not exist).

The validity of this procedure rests on the convexity of maximum undershoot plus maximum overshoot as a function of p_1 (apply Theorem 2) and on the fact that $\tilde{f}$ changes faster with p_1 relative to f for larger values of x than for smaller (think of $\tilde{f}$ as a straight line superimposed on the graph of f that pivots about the point $(0, \mathrm{f}(0))$ as p_1 is changed).

The procedure is easy to carry out for most functions likely to be encountered in practice, and can be reduced to a simple non-graphical method for many closed-form functions. Figure 1 illustrates the procedure for an economy-of-scale type function:

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 15, \\ 15 + \frac{1}{2}(x - 15) & \text{if } 15 \le x \le 25 = u. \end{cases}$$

As p_1 decreases from ∞, the point of maximum undershoot (overshoot) is $x = 0$ $(x = 25)$ until $p_1 = 1$. For $0.8 \le p_1 < 1.0$, the point of maximum undershoot (overshoot) is $x = 15$ $(x = 25)$. For $0.5 \le p_1 < 0.8$, the point of maximum un-

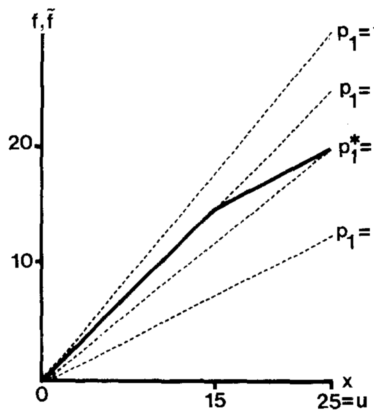

Fig. 1

dershoot (overshoot) is $x = 15$ ($x = 0$). Thus the optimal value of p_1 must be 0.8. Fig. 2 illustrates the procedure for an S-shaped curve. For $p_1 \geq 0.88$, the point of maximum undershoot clearly remains less than the point of maximum overshoot. For $p_1 < 0.88$, the undershoot at $x = 25$ increases rapidly until it becomes the point of maximum undershoot for all $p_1 < 0.8$. Thus $p_1^* = 0.8$.

Independent confirmation of the validity of the solutions obtained in Figs. 1 and 2 can be obtained by observing that an obvious translation of each solution satisfies the famous Chebyshev equioscillation property for a first degree polynomial (e.g., Section 7.6 of [3]). The translated solutions must therefore be optimal Chebyshev linear fits, from which validity under the natural criterion follows by Theorem 3. We point out that this validity argument applies only when fitting a continuous f since the equioscillation Theorem [3] assumes continuity, whereas our graphical procedure does not require f to be continuous.

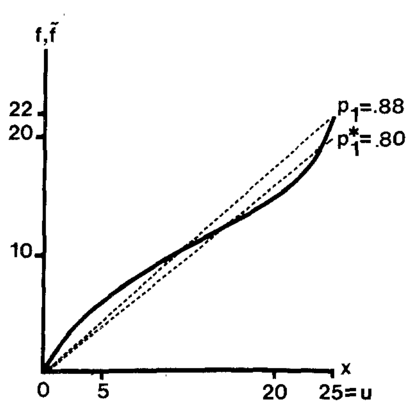

Fig. 2

2.2. *Fixed-plus-linear fits*

The application of integer programming frequently calls for best approximations of the form

$$\tilde{f}(x; p_1, p_2) = p_1 x + \begin{cases} 0 & \text{if } x = 0, \\ p_2 & \text{if } 0 < l \leq x \leq u \end{cases} \tag{12}$$

to f over $\{0\} \cup [l, u]$. Coefficient p_2 is usually interpreted as a "fixed charge" for the use of the activity measured by x. A lower limit l is given for the value of x when $x > 0$. We can assume without loss of generality that $f(0) = 0$.[1]

The optimal choices of p_1 and p_2 can be determined by applying the previously described linear fit procedure to f over $[l, u]$ with $x = l$ instead of $x = 0$ as the anchor point. Certainly this produces an optimal fit to f over $[l, u]$; as p_1 and p_2

[1] f can be translated to enforce this assumption, if necessary, without altering the optimal solution set of (P). Alternatively, f could be left untranslated and a simple translation parameter could be introduced into the class of approximating functions and used to make f and $\tilde{f}$ coincide at $x = 0$.

do not alter the fit to f at $x = 0$ (the fit is necessarily perfect), the fit must be optimal over $\{0\} \cup [l, u]$.

Figure 3 illustrates an optimal fit to the following function (note the discontinuity):

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 10 + \frac{7}{15}(x - 5) & \text{if } 5 \le x \le 20, \\ 19 + \frac{6}{5}(x - 20) & \text{if } 20 < x \le 25. \end{cases}$$

Apply the linear fit procedure to this function over [5, 25] with $x = 5$ as the anchor point. For $p_1 \ge (25 - 17)/(25 - 20) = \frac{8}{5}$, the point of maximum undershoot (overshoot) is $x = 5$ ($x = 25$). For $\frac{3}{4} \le p_1 < \frac{8}{5}$, the point of maximum undershoot (overshoot) is $x = 5$ ($x = 20$). Below $p_1 = 0.75$, the point of maximum undershoot jumps to $x = 25$ and the point of maximum overshoot stays at $x = 20$. Hence $p_1^* = 0.75$. The line through the point $(5, f(5))$ with slope 0.75 yields $p_2^* = \frac{25}{4}$ (in general, $p_2^* + p_1^* l = f(l)$ yields $p_2^* = f(l) - p_1^* l$). Thus the best fit to f in Figure 3 is given by $\tilde{f}$ equal to

$$0.75x + \begin{cases} 0 & \text{if } x = 0, \\ \frac{25}{4} & \text{if } 5 \le x \le 25. \end{cases}$$

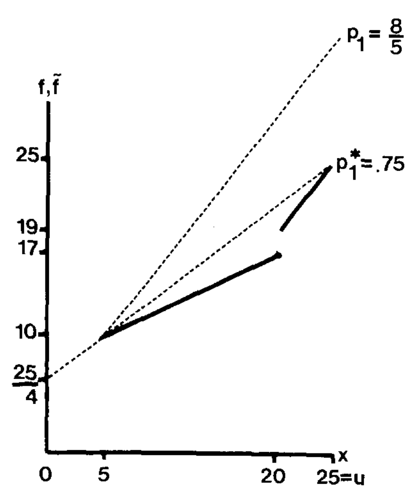

Fig. 3

2.3. *Piecewise-linear fits to convex and concave functions*

Economies of scale often lead to concave cost functions; diseconomies of scale and congestion effects often lead to convex cost functions. In this section we seek the best piecewise-linear continuous fit to continuous functions that are convex or concave. It suffices to treat the case where f is concave on $[0, u]$, as the results for the convex case are entirely analogous.

We shall require the following result stating conditions under which a fitting problem can be conditionally decomposed into independent subproblems.

Lemma 2. *Suppose that f is to be approximated on $[0, u]$ by $\tilde{f}$ drawn from a class of functions constructed independently on the subintervals $[0, t]$ and $[t, u]$, where t is a given point in $[0, u]$. The value of $\tilde{f}(t)$ is specified. Let $\tilde{f}_L$ denote the best choice on $[0, t]$ ignoring $[t, u]$, and let $\tilde{f}_R$ denote the best choice on $[t, u]$ ignoring $[0, t]$. Denote the maximum undershoot (overshoot) of $\tilde{f}_L$ on $[0, t]$ by $\underline{\epsilon}_L(\bar{\epsilon}_L)$, and define $\underline{\epsilon}_R$ and $\bar{\epsilon}_R$ similarly for $\tilde{f}_R$. Put $\underline{\epsilon} = \text{Max}\{\underline{\epsilon}_L, \underline{\epsilon}_R\}$ and $\bar{\epsilon} = \text{Max}\{\bar{\epsilon}_L, \bar{\epsilon}_R\}$. If $\underline{\epsilon}_L = \underline{\epsilon}$ and $\bar{\epsilon}_L = \bar{\epsilon}$, or if $\underline{\epsilon}_R = \underline{\epsilon}$ and $\bar{\epsilon}_R = \bar{\epsilon}$, then $\tilde{f}_L$ and $\tilde{f}_R$ together comprise the best choice of $\tilde{f}$ on $[0, u]$.*

Proof. Let $\tilde{f}'$ be any permissible approximation on $[0, u]$, and denote its maximum undershoot (overshoot) by $\underline{\epsilon}'(\bar{\epsilon}')$. Clearly $\underline{\epsilon}' + \bar{\epsilon}' \geq \underline{\epsilon}_L + \bar{\epsilon}_L$ by the definition of $\tilde{f}_L$, and similarly $\underline{\epsilon}' + \bar{\epsilon}' \geq \underline{\epsilon}_R + \bar{\epsilon}_R$. But by hypothesis we know $\underline{\epsilon} + \bar{\epsilon}$ equals $\underline{\epsilon}_L + \bar{\epsilon}_L$ or $\underline{\epsilon}_R + \bar{\epsilon}_R$, and so $\underline{\epsilon}' + \bar{\epsilon}' \geq \underline{\epsilon} + \bar{\epsilon}$. Thus $\tilde{f}'$ could be no better than the $\tilde{f}$ comprised of f_L and f_R.

Consider first the problem of finding a 2-piece continuous fit. That is, $\tilde{f}$ is of the form

$$\tilde{f}(x; p_1, p_2, p_3, p_4, p_5) = \begin{cases} p_1 + p_2 x & \text{if } 0 \leq x \leq p_5, \\ p_3 + p_4 x & \text{if } p_5 \leq x \leq u \end{cases}$$

where p_5 is interpreted as the "transition point" between the two linear pieces and $p_1 + p_2 p_5 = p_3 + p_4 p_5$. In view of Lemma 1, without loss of optimality one may restrict consideration to approximating functions that exactly equal f at the transition point (simple translation is accomplished by adding an identical constant to the values of p_1 and p_3, whereby one may obtain $p_1 + p_2 p_5 = p_3 + p_4 p_5 = f(p_5)$ for any given p_5). Given a trial transition point in the interval $[0, u]$, one may determine p_1 and p_2 so as to yield a best linear fit to f on $[0, p_5]$ with p_5 serving as anchor point, and p_3 and p_4 so as to yield a best linear fit to f on $[p_5, u]$ with p_5 again serving as anchor point. These determinations can be made via the graphical procedure of Section 2.1, which simplifies in an obvious way due to the assumed concavity of f: the resulting $\tilde{f}$ is uniquely specified by the two chords to f determined by the three points 0, p_5 and u. Now in general this would not necessarily yield the best possible *joint* choices for p_1, p_2, p_3 and p_4 for a given p_5. The concavity of f, however, implies via Lemma 2 (since $\bar{\epsilon}_L = \bar{\epsilon}_R = 0$) that these independent choices are also jointly best. To put it another way, fixing p_5 decomposes the 2-piece fitting problem into two independent linear fitting problems of a type easily solved.

It remains but to find the best choice of the transition point p_5. As p_5 moves from $x = 0$ toward $x = u$, the maximum undershoot of the "left" chord increases and that of the "right" chord decreases (this follows from the concavity of f). The best transition point is the one where the maximum undershoot of the left chord equals that of the right, for this obviously minimizes the maximum undershoot over the entire interval $[0, u]$ (remember that the maximum overshoot is 0 for all p_5).

To summarize the procedure for finding the best 2-piece continuous fit to a continuous concave function, then, one parametrically increases the transition point starting from 0 and monitors the maximum undershoot of the chord $\overline{AB}$ and of the chord $\overline{BC}$, where $A = (0, f(0))$, $B = (p_5, f(p_5))$, $C = (u, f(u))$. Stop when the maximum undershoot of $\overline{AB}$ becomes equal to that of $\overline{BC}$.

This procedure is illustrated in Fig. 4 for $f(x) = 5\sqrt{x}$ on [0, 25]. The maximum undershoot of the left chord is less than that of the right until $p_5 = 2.78$, after which it is greater. Thus $p_5^* = 2.78$ and the best fit is as shown; the maximum undershoot for each chord is 2.08, occurring at $x = 0.69$ for the left chord and at $x = 11.11$ for the right chord. Two other fits are shown for comparison: one for $p_5 = 0.69$ and one for $p_5 = 11.11$.

Now consider the problem of finding a best piecewise-linear continuous fit with 3 pieces. There are two transition points instead of one. By Lemma 1, consideration can be restricted to approximating functions that exactly equal f at the leftmost transition point, say. Let B be the point on the graph of f corresponding to a trial left transition point t. Construct the best one-piece linear fit to f on $[0, t]$ using B as an anchor point: we know that it is the chord $\overline{AB}$, where A is the left end-point of the graph of f. Construct the best two-piece continuous fit to f on $[t, u]$ using B as an anchor point: we know that it is given by the chords $\overline{BC}$ and $\overline{CD}$, where D is the right end-point of the graph of f and C is chosen to make the maximum undershoots of $\overline{BC}$ and $\overline{CD}$ equal. The concavity of f implies via Lemma 2 that the fit given by the three chords $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ is the (jointly) best 3-piece continuous linear fit to f on $[0, u]$ passing through B. It remains but to find the best value for t. As the left transition point increases, starting from 0, the maximum undershoot of the chord $\overline{AB}$ increases and that of $\overline{BC}$ and $\overline{CD}$ decrease. The best value for t is the one where the maximum undershoots of all three chords are equal since this minimizes the maximum undershoot on [0, u] (the maximum overshoot is 0 for all t).

One may proceed inductively in this fashion to obtain the following general necessary and sufficient conditions characterizing best piecewise-linear continuous fits.

Theorem 4 (Piecewise-Linear Fits). *Let f be a continuous concave function on an interval $[l, u]$, to which an approximation is desired from the class of all*

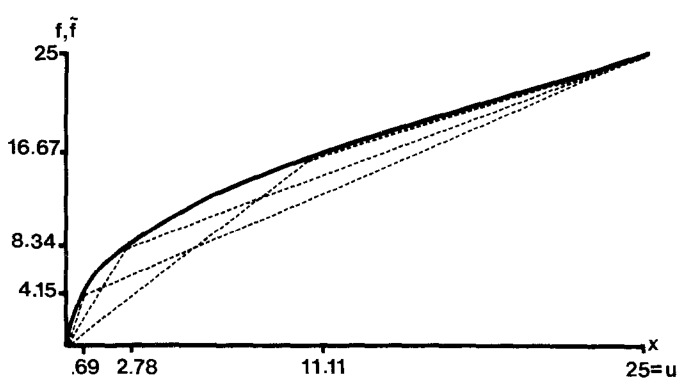

Fig. 4

piecewise-linear continuous functions with n pieces. Such an approximation is best in the sense of the natural criterion if and only if a simple translate of it coincides in value with f at both end points and at each transition point and the maximum undershoots associated with the n chords are all of equal value.

The same result holds for f convex except that overshoot plays the role of undershoot.

3. After the approximate problem is solved

Up to this point we have taken the viewpoint of a modeler who is designing an approximation ($\tilde{\mathrm{P}}$) to (P) prior to any computations. The bounds and associated fitting procedures developed do not presume any knowledge concerning an optimal solution of ($\tilde{\mathrm{P}}$), much less of (P). We now consider the situation *after* ($\tilde{\mathrm{P}}$) has been solved. Can the bounds of Theorem 1 be improved? Can the fits based on a priori analysis be improved?

3.1. *A posteriori bounds*

The bounds of Theorem 1 can indeed be improved as follows. See Schweitzer [5] for results similar in spirit but different in substance.

Theorem 5 (Post-Computational Error Bounds). *Let $\underline{\epsilon}$ and $\bar{\epsilon}$ be scalars (not necessarily nonnegative) satisfying* (1). *If $\tilde{x}$ is any ϵ-optimal solution of* ($\tilde{\mathrm{P}}$), *then*

$$-\operatorname{Min}\{\underline{\epsilon}, f(\tilde{x}) - \tilde{f}(\tilde{x}) + \epsilon\} \leq v(\tilde{\mathrm{P}}) - v(\mathrm{P}) \leq \bar{\epsilon} \tag{2A}$$

and the suboptimality of $\tilde{x}$ in (P) *is at most $\epsilon + \bar{\epsilon} + f(\tilde{x}) - \tilde{f}(\tilde{x})$.*

Proof. To prove (2A) it suffices to demonstrate

$$\operatorname{Max}\{-\underline{\epsilon}, \tilde{f}(\tilde{x}) - f(\tilde{x}) - \epsilon\} \leq \operatorname{Max}\{-\underline{\epsilon}, v(\tilde{\mathrm{P}}) - f(\tilde{x})\} \leq v(\tilde{\mathrm{P}}) - v(\mathrm{P}) \leq \bar{\epsilon}. \tag{13}$$

The first inequality of (13) follows from the definition of $\tilde{x}$, which asserts $\tilde{f}(\tilde{x}) \leq v(\tilde{\mathrm{P}}) + \epsilon$. The second inequality follows from the feasibility of $\tilde{x}$ in (P), which implies $v(\mathrm{P}) \leq f(\tilde{x})$, and the first inequality of (2). The third inequality of (13) is identical with the second of (2). It remains to show

$$f(\tilde{x}) \leq v(\mathrm{P}) + \epsilon + \bar{\epsilon} + f(\tilde{x}) - \tilde{f}(\tilde{x}).$$

This follows by the same line of argument used to prove the analogous part of Theorem 1 except for one change: the initial inequality $f(\tilde{x}) - \underline{\epsilon} \leq \tilde{f}(\tilde{x})$ is replaced by $f(\tilde{x}) + \tilde{f}(\tilde{x}) - f(\tilde{x}) = \tilde{f}(\tilde{x})$.

Observe that the lower bound in (2A) will be strictly improved if

$$f(\tilde{x}) - \tilde{f}(\tilde{x}) \leq \underline{\epsilon} - \epsilon \tag{14}$$

that is, if $\tilde{f}$ undershoots f at $\tilde{x}$ by *less* than the greatest possible amount ($\underline{\epsilon}$)

minus ϵ. Since

$$f(\tilde{x}) - \tilde{f}(\tilde{x}) \leq \underline{\epsilon}$$

by the definition of $\underline{\epsilon}$, (14) is highly likely if ϵ is relatively small.

3.2. *A secondary criterion*

The nature of post-computational bounds suggests an appealing approach to the a priori choice of approximations when the natural criterion admits non-trivial ties, i.e., when (3) admits alternative optimal solutions other than those obtainable by simple translation. Suppose for simplicity that ($\tilde{\text{P}}$) will be solved to optimality, so that $\epsilon = 0$. Given alternative possibilities for the choice of $\tilde{f}$, all of which are best by the natural criterion of minimizing the corresponding $\underline{\epsilon} + \bar{\epsilon}$, how should a selection be made? An obvious idea would be to make a selection according to one's subjective expectations regarding the post-computational bound $\bar{\epsilon} + f(\tilde{x}) - \tilde{f}(\tilde{x})$. According to Theorem 5, this quantity is the suboptimality bound on $\tilde{x}$ and also the width of the interval of uncertainty containing $v(\text{P})$ when $\epsilon = 0$.

We therefore propose this subjective secondary criterion: among $\tilde{f}$ satisfying the natural criterion, choose one which minimizes the subjective expected value of $\bar{\epsilon} + f(\tilde{x}) - \tilde{f}(\tilde{x})$ (bearing in mind, of course, that $\tilde{x}$ and $\bar{\epsilon}$ depend on $\tilde{f}$).

As an illustration, consider again the function $f(x) = 5\sqrt{x}$ on $[0, 25]$ with approximations of the fixed-plus-linear type:

$$\tilde{f}(x; p_1, p_2) = p_1 x + \begin{cases} 0 & \text{if } x = 0, \\ p_2 & \text{if } 0 < x \leq 25. \end{cases}$$

The procedure of Section 2.2 indicates that an optimal fit is given by $(p_1^*, p_2^*) = (1, 0)$ which has $\underline{\epsilon} = 6.25$ and $\bar{\epsilon} = 0$. This fit is not uniquely optimal. In fact, $(p_1, p_2) = (1, p_2)$ is optimal for any p_2 satisfying $0 \leq p_2 \leq 6.25$ ($\underline{\epsilon} = 6.25 - p_2$ and $\bar{\epsilon} = p_2$). The secondary criterion suggested above would select p_2 between 0 and 6.25 in an effort to minimize

$$\begin{aligned} \bar{\epsilon} + f(\tilde{x}) - \tilde{f}(\tilde{x}) &= p_2 + 5\sqrt{\tilde{x}} - \tilde{x} - \begin{cases} 0 & \text{if } \tilde{x} = 0, \\ p_2 & \text{if } \tilde{x} > 0, \end{cases} \\ &= \begin{cases} p_2 & \text{if } \tilde{x} = 0, \\ 5\sqrt{\tilde{x}} - \tilde{x} & \text{if } \tilde{x} > 0. \end{cases} \end{aligned} \tag{15}$$

It is not immediately obvious what choice of p_2 would minimize (15). Choosing p_2 small would make (15) small if $\tilde{x}$ equals 0, but the contrary case $\tilde{x} > 0$ becomes more likely as p_2 becomes smaller because then less of a penalty is imposed in ($\tilde{\text{P}}$) for $x > 0$. Notice also that, among choices of p_2 leading to $\tilde{x} > 0$, the value of (15) is constant.

A somewhat idealized analysis to minimize the subjective expected value of (15) is as follows. Estimate the subjective probability

$$\theta(p_2) \triangleq \text{Prob}\,[\tilde{x} = 0 \text{ in } (\tilde{\text{P}}) \text{ given } p_2]$$

and also the subjective conditional expected value

$$E \triangleq \operatorname{Exp}\,[5\sqrt{\bar{x}} - \bar{x} \mid \bar{x} > 0 \text{ in } (\tilde{\mathrm{P}})]$$

(as noted above, E must be independent of p_2). Then the subjective expected value of (15) is

$$\theta(p_2)p_2 + (1 - \theta(p_2))E, \quad \text{or} \quad E + \theta(p_2)(p_2 - E).$$

This expression is to be minimized over $0 \le p_2 \le 6.25$. Certainly $\theta(\cdot)$ will be a non-decreasing function. It follows that p_2 should not be selected larger than E (since the product $\theta(p_2)(p_2 - E)$ is increasing for $p_2 > \mathrm{E}$). Thus the problem of choosing p_2 reduces to

$$\begin{aligned} &\underset{p_2}{\text{Minimize}} \quad E + \theta(p_2)(p_2 - E) \\ &\text{subject to} \quad 0 \le p_2 \le \operatorname{Min}\{E, 6.25\}. \end{aligned} \tag{16}$$

This can be solved with the help of a simple manual construction directly on the graph of $\theta(\cdot)$ which locates the values of p_2 (if any) at which the slope of the objective function of (16) turns from negative to positive. The optimal choice of p_2 will be among these points and the end points 0 and $\operatorname{Min}\{E, 6.25\}$. The mechanics of the graphical construction are left as an exercise for the reader.

The usefulness of this procedure depends on the availability of subjective estimates for E and $\theta(\cdot)$, which depends in turn on having some insight into the system being modeled. In distribution system planning, to cite just one possible domain of application, it is the author's experience that simple calculations can lead to plausible rough guesses of E and $\theta(\cdot)$. And, of course, subjective judgments are likely to emerge from prior computational experience with variants of the model at hand.

The alert reader may have noticed that a fixed-plus-linear fit to $5\sqrt{x}$ on $[0, 25]$ has $\underline{\epsilon} + \bar{\epsilon} = 6.25$, whereas a 2-piece linear fit as in Fig. 4 has $\underline{\epsilon} + \bar{\epsilon} = 2.08$ (recall the discussion of Section 2.3). Both approximations require one new binary variable to permit solving $(\tilde{\mathrm{P}})$ by standard methods. Depending on a modeler's subjective analysis of the situation, it may well be wiser to use a 2-piece fit than a fixed-plus-linear fit.

3.3. *Retrofits*

An optimal solution $\tilde{x}$ of $(\tilde{\mathrm{P}})$ would remain optimal even if a nonnegative function $\emptyset$ were added to $\tilde{f}$ so long as $\emptyset(\tilde{x}) = 0$. This elementary observation leads to the notion of "retrofits", a concept of value when there is an opportunity to revise $\tilde{f}$ using knowledge of $\tilde{x}$ or when it is desired to present the results of an optimization to sponsors in the most favorable light.

A *retrofit* objective function, which can only be defined after an optimal solution $\tilde{x}$ has been found for $(\tilde{\mathrm{P}})$, is any function of the form $\tilde{f} + \emptyset$ with $\emptyset$ defined over X such that $\emptyset(x) \ge 0$ for all $x \in X$ and $\emptyset(\tilde{x}) = 0$. One selects $\emptyset$ to make $\tilde{f} + \emptyset$ a closer approximation to f than $\tilde{f}$.

To illustrate, suppose that f is linearly separable with respect to some

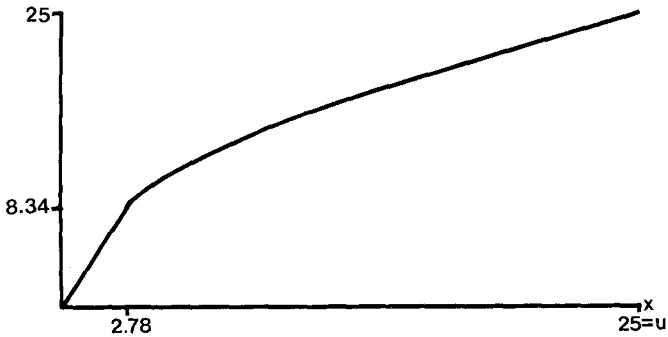

Fig. 5

variables and that one component is $5\sqrt{x_j}$ as in Fig. 4. Suppose further that the 2-piece approximation given in Fig. 4 has been used to define this component of $\tilde{f}$ and that $\tilde{x}$ *is less than 2.78*. Now define this component of $\emptyset$, $\emptyset_j$, to be 0 for $0 \leq x_j \leq 2.78$ and $5\sqrt{x_j} - \tilde{f}_j(x_j)$ for $2.78 \leq x_j \leq 25$. The graph of $\tilde{f}_j + \emptyset_j$ is shown in Fig. 5. Then $\tilde{x}$ would still be optimal in the approximating problem if $\tilde{f}_j + \emptyset_j$ were to replace $\tilde{f}_j$.

In this illustration, improving the approximation to f_j requires changing the part of $\tilde{f}_j$ where $x \leq 2.78$ if there is to be any effect on $\tilde{x}$ – a fact that every modeler should know instinctively. It might not be quite so clear that the results of the optimization could be presented as having employed the more realistic-looking approximation of Fig. 5 rather than the 2-piece approximation of Fig. 4.

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