Objectivity of classical quantum stochastic processes

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We investigate what can be concluded about the quantum system when the sequential quantum measurements of its observable—a prominent example of the so-called quantum stochastic process—fulfill the Kolmogorov consistency condition, and thus, appear to an observer as a sampling of classical trajectory. We identify a set of physical conditions imposed on the system dynamics, that when satisfied lead to the aforementioned trajectory interpretation of the measurement results. Then, we show that when another quantum system is coupled to the observable, the operator representing it can be replaced by an external noise. Crucially, the realizations of this surrogate (classical) stochastic process are following the same trajectories as those measured by the observer. Therefore, it can be said that the trajectory interpretation suggested by the Kolmogorov consistent measurements also applies in contexts other than sequential measurements.

1 Introduction

The quantum mechanics is a flagship example of non-classical physical theory. To human users, who are firmly ingrained in the classical realm, many aspects of the theory often appear highly counter-intuitive and sometimes even paradoxical; this is true both for the elements of the mathematical formalism as well as the interpretative rules that correlate the abstract mathematics with user's experiences. Hence, it is only natural to look for instances when the quantum mechanics puts on a familiar classical appearance. In this vein, there was a recent resurgence of efforts in the search for the signatures of classicality manifesting in the statistics of the sequential quantum measurements [1, 2, 3, 4, 5, 6].

In the language of quantum theory, the result of an experiment consisting of n consecutive measurement events is depicted as a sequence of stochastic variables described by a joint probability distribution P_n—the *quantum stochastic process* [7]. The classical intuition would suggest that the sequential measurement should allow the observer to uncover a trajectory that was traced by the evolving physical quantity (the observable); after all, this is how measurements are supposed to work in classical physics. Of course, quantum observables often behave in a way that contradicts this naive intuition. The theory of probability ascertains that the measured sequence can be interpreted as a sampling of an underlying trajectory—i.e., a trajectory that was traced over time independently of the measurement events—only if the joint probability distributions satisfy the *Kolmogorov consistency* (KC) condition. In formal terms: let $\Omega(F)$ be the set of all possible results of a single measurement of observable F, then $P_n(f_n,t_n;\ldots;f_1,t_1)$ is the probability of obtaining the sequence of results $f_n,\ldots,f_1$ [where each $f_i\in\Omega(F)$] in consecutive measurements performed at the corresponding times $0<t_1<\cdots<t_n$; the family $\{P_n\}_{n=1}^{\infty}$ satisfies KC condition (or 'is consistent', for short) when

$$\mathbf{KC:}\quad \begin{aligned}&\forall(n>1)\forall(0<t_1<\cdots<t_n)\forall(1\leq i\leq n)\forall(f_1,\ldots,\cancel{f_i},\ldots,f_n\in\Omega(F)):\\ &\sum_{f_i\in\Omega(F)}P_n(f_n,t_n;\ldots;f_i,t_i;\ldots;f_1,t_1)=P_{n-1}(f_n,t_n;\ldots;\cancel{f_i,t_i};\ldots;f_1,t_1).\end{aligned}\tag{1}$$

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By the Kolmogorov extension theorem [8], any family of consistent joint probability distributions uniquely defines a stochastic process. This makes the sequence of measured results effectively indistinguishable from the sampling of a randomly chosen trajectory (the realization of the stochastic process), and thus, the quantum stochastic process is transmuted into the *classical quantum process*. However, typically this is not the case because probability distributions P_n describing quantum measurements violate KC in general [9, 10], and fulfill it only under specific conditions [1, 2, 3, 4, 5, 6].

Therefore, in the special event when $\{P_n\}_{n=1}^{\infty}$ is consistent, it follows that the sequence of results witnessed by the classical observer can be modeled with a stochastic process, i.e., an observer performing the sequential quantum measurement of the observable represented by Hermitian operator $\hat{F}$ cannot distinguish their results from the sequential sampling of trajectory $f(t)$ that realizes the stochastic process $F(t)$. Given that, one is compelled to pose the following question: can this trajectory picture be applied in contexts other than measurements performed by the classical observer? Can those trajectories be considered as *objective* entities? To put it differently, if the classical observer perceives the observable $\hat{F}$ as a stochastic process, can we say that also a *non-classical observer*—i.e., a quantum system coupled to $\hat{F}$—evolves as if $\hat{F}$ was replaced with an external field represented by the process $F(t)$? If this question is answered in affirmative, then one is allowed to say that the quantum measurement of the observable $\hat{F}$ achieves the classical ideal where it becomes possible to observe the 'actual, objective state (the trajectory) of a physical quantity'. This is because, if there is a symmetry between classical and non-classical perceptions, then the same fundamental description of the measurement result (the family $\{P_n\}_{n=1}^{\infty}$) also describes the evolution of other quantum systems coupled to $\hat{F}$. Here, we define the conditions the dynamics of the observable $\hat{F}$ has to satisfy to achieve this objectivity of its trajectory picture and we discuss the physical implications it brings about. These conditions turn out to be formally equivalent with the decoherent histories criterion of the consistent histories framework [11, 12, 13, 14]. However, the fact that they guarantee the above-defined objectivity of the trajectory picture has apparently gone unnoticed until now.

2 Classical observer

The classical observer experiences the dynamics of a quantum system through readouts of the dedicated *measuring apparatus*. As per standard interpretation of the quantum mechanics, the apparatus itself is treated as a primitive notion: it is assumed that the device can be built, but the details of its inner workings and the particularities of its interactions with the observer as well as with the measured system are not specified. The one thing that has to be specified about every apparatus is the assignment of the Hilbert space partitioning $\{\hat{E}(n)\}_n$, such that $\hat{E}(n) \geqslant 0$ and $\sum_n \hat{E}(n) = \hat{1}$. The partitionings into mutually orthogonal subspaces are of particular interest because they correspond to the direct measurements of physical quantities (the observables). For example, the observable F is represented by the Hermitian operator $\hat{F} = \hat{F}^\dagger$; this operator has spectral decomposition,

$$\hat{F} = \sum_{f\in\Omega(F)} f\hat{P}(f), \tag{2}$$

where the set $\Omega(F)$ contains all unique eigenvalues of $\hat{F}$ and the operators $\hat{P}(f)$ are the projectors onto the corresponding orthogonal eigenspaces, i.e., $\sum_f \hat{P}(f) = \hat{1}$ and $\hat{P}(f)\hat{P}(f') = \delta_{f,f'}\hat{P}(f)$. (We will suppress the range of sums whenever it is clear from context that the variable belongs to $\Omega(F)$.) Therefore, $\{\hat{P}(f)\}_{f\in\Omega(F)}$ defines an orthogonal partitioning of the Hilbert space, and that partitioning can then be assigned to the apparatus measuring the physical quantity F.

The *Born rule* defines how the partitionings assigned to measuring apparatuses correlate with the readouts perceived by the classical observer; it states the following: given the unitary evolution operator $\hat{U}(t) = \exp(-it\hat{H})$ describing the dynamical law in the measured system S and the initial density matrix $\hat{\rho}$, the probability of reading out the result f_1 at the time $t_1 > 0$, and the result f_2 at $t_2 > t_1$, ..., and finally the result f_n at $t_n > t_{n-1}$, in the sequence of n measurements, is calculated according to the formula

$$P_n(\boldsymbol{f}_n,\boldsymbol{t}_n) = P_n(f_n,t_n;\ldots;f_1,t_1) = \mathrm{tr}\left[\left(\prod_{i=n}^{1}\hat{P}(f_i,t_i)\right)\hat{\rho}\left(\prod_{i=1}^{n}\hat{P}(f_i,t_i)\right)\right]. \tag{3}$$

Here, the symbol $\prod_{i=n}^{1}\hat{A}_i$ ($\prod_{i=1}^{n}\hat{A}_i$) indicates an ordered composition $\hat{A}_n\cdots\hat{A}_1$ ($\hat{A}_1\cdots\hat{A}_n$) and $\hat{P}(f,t) = \hat{U}^\dagger(t)\hat{P}(f)\hat{U}(t)$ is the Heisenberg picture of the projector.

By default, the *Born distributions* P_n are *causal* [9],

$$\forall(n\geqslant 1)\forall\boldsymbol{f}_n\,\forall(0<t_1<\cdots<t_n<t): \sum_f P_{n+1}(f,t;f_n,t_n;\ldots;f_1,t_1) = P_n(\boldsymbol{f}_n,\boldsymbol{t}_n). \tag{4}$$

Because of this relation it is possible to rewrite the joint probability distributions P_n as a product of conditional probabilities,

$$P(f|\hat{\rho}_t) = \mathrm{tr}[\hat{P}(f)\hat{\rho}_t], \tag{5}$$

with the condition in a form of density matrix $\hat{\rho}_t$,

$$\begin{aligned} P_n(\boldsymbol{f}_n,\boldsymbol{t}_n) &= \frac{P_n(\boldsymbol{f}_n,\boldsymbol{t}_n)}{P_{n-1}(\boldsymbol{f}_{n-1},\boldsymbol{t}_{n-1})}\frac{P_{n-1}(\boldsymbol{f}_{n-1},\boldsymbol{t}_{n-1})}{P_{n-2}(\boldsymbol{f}_{n-2},\boldsymbol{t}_{n-2})}\cdots\frac{P_2(\boldsymbol{f}_2,\boldsymbol{t}_2)}{P_1(f_1,t_1)}P_1(f_1,t_1) \\ &= P(f_1|\hat{U}(t_1)\hat{\rho}\hat{U}^\dagger(t_1))\prod_{k=1}^{n-1}P(f_{k+1}|\hat{\rho}_{t_{k+1}|\boldsymbol{f}_k,\boldsymbol{t}_k}). \end{aligned} \tag{6}$$

where the density matrices conditioned on the history of previous measurement results are given by

$$\hat{\rho}_{t_{k+1}|\boldsymbol{f}_k,\boldsymbol{t}_k} = \hat{U}(t_{k+1})\frac{\left(\prod_{i=k}^{1}\hat{P}(f_i,t_i)\right)\hat{\rho}\left(\prod_{i=1}^{k}\hat{P}(f_i,t_i)\right)}{P_k(\boldsymbol{f}_k,\boldsymbol{t}_k)}\hat{U}^\dagger(t_{k+1}), \tag{7}$$

and the relation (4) ensures their proper normalization. Typically, distribution of form (5) is interpreted as describing a single measurement performed on the current *state of the system* $\hat{\rho}_t$ [15]; therefore, the reformulation (6) suggests a reading that the sequential measurement is a composition of independent measurement events where in each step the apparatus measures the state as it is at the corresponding instant. However, it looks as if the very act of observation seemingly changes the state in between the steps. Indeed, we see in Eq. (6) that the state at the time of the next event, $\hat{\rho}_{t_{k+1}|\boldsymbol{f}_k,\boldsymbol{t}_k}$, appears as the state from the previous event, $\hat{\rho}_{t_k|\boldsymbol{f}_{k-1},\boldsymbol{t}_{k-1}}$, that was *collapsed* onto the subspace corresponding to the measured result,

$$\hat{\rho}_{t_{k+1}|\boldsymbol{f}_k,\boldsymbol{t}_k} = \hat{U}(t_{k+1}-t_k)\frac{\hat{P}(f_k)\hat{\rho}_{t_k|\boldsymbol{f}_{k-1},\boldsymbol{t}_{k-1}}\hat{P}(f_k)}{P(f_k|\hat{\rho}_{t_k|\boldsymbol{f}_{k-1},\boldsymbol{t}_{k-1}})}\hat{U}^\dagger(t_{k+1}-t_k). \tag{8}$$

Thus, in this reading, the statistical dependence between results in a sequential measurement is shifted to the state subjected to collapse events coinciding with each observation.

Can this picture of system state being collapsed by the measurement be objectified? What is the standard one would apply to decide whether Eq. (6) and the narration we have built around it, is a sufficient reason to consider the state collapse as an objective event? Consider that the very idea of collapse, understood as a sudden, non-unitary (i.e., not following the dynamical law of the system) change of state that coincide with the measurement event, came to be only because of a particular result of algebraic transformations to the formula defining P_n. Given that, the most that can be said at this point is that the collapse is something subjectively perceived by a classical observer performing sequential measurements. To even entertain the possibility of its objectification, it should be possible to show this kind of state change appearing in, at least, one context other than the sequential measurements. The more diverse the contexts the picture of collapsing states can be adopted in, the more justified the claim of its objectivity—this is the standard we are setting for our present investigations. To our best knowledge, collapse has not been demonstrated in any other context of the quantum theory, and so, the state collapse picture does not clear our standard. Therefore, the collapse is, at most, inter-subjective among classical observers (i.e., classical observers agree that they perceive the same picture). Our aim is to show that the trajectory picture suggested by the Kolmogorovian consistency of Born distributions turns out to be more than inter-subjective when assessed against the same standard—we will demonstrate that the trajectory picture can be applied in contexts beyond sequential measurements.

3 Consistent Measurements

The causality relation (4) applies only to the latest measurement result in the sequence; an analogous relations between Born distributions that would involve results mid sequence are not readily apparent in the general case. Nevertheless, the quantum mechanics does not prohibit such relations as a matter of principle. An important example—and the subject of this paper—is the Kolmogorov consistency (1) (KC), a relation between Born distributions inspired by the fundamental laws of the classical theory of stochastic processes.

In the classical theory, physical quantities are represented by trajectories traced over time in accordance with the system's dynamical laws. These trajectories are considered as objective entities in the sense that the same fundamental description of the trajectory accounts for both the results of measurements, as well as any other non-measurement interaction with the physical quantity in question: 'when the (classical) tree falls, it makes a sound even if there is no one around to hear it.' The Kolmogorovian consistency is the manifestation of this fundamental postulate of classical physics; let us explain why by reviewing some fundamentals of the theory of stochastic processes.

Formally, a classical stochastic process $X(t)$ representing a physical quantity is defined as a map from the set of outcomes of random event (the sample space) to *trajectories* $x(t)$—the real-valued functions of time $t \in \mathbb{R}$. Every process $X(t)$ is assigned with a probability distribution functional $P_X[x]$ for its trajectories $x(t)$; as a probability distribution this functional is non-negative, $P_X[x] \geqslant 0$, and normalized,

$$\int P_X[x][Dx] = 1, \tag{9}$$

where $\int \cdots [Dx]$ indicates the functional integration. Given the probability distribution $P_X[x]$ one can calculate the expectation value of arbitrary functionals of $X(t)$,

$$\overline{W[X]} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} W[x_i] = \int P_X[x]W[x][Dx], \tag{10}$$

where $\{x_i(t)\}_{i=1}^{N}$ is an ensemble of independently sampled trajectories. The *moments* of the process,

$$\overline{X(t_n)\cdots X(t_1)} = \int P_X[x] \left(\prod_{i=1}^{n} x(t_i) \right) [Dx], \tag{11}$$

are important example of expectation value. Since any regular functional $W[X]$ has some form of series expansion into a combination of products of $X(t)$, the problem of computing $\overline{W[X]}$ can be broken down into manageable steps each solved with the use of an appropriate moment. However, the functional integral form of stochastic average (11) is difficult to work with. In practical applications the formal definition utilizing the functional $P_X[x]$ is almost always rewritten in the language of *joint probability distributions* $\{P_X^{(n)}\}_{n=1}^{\infty}$. Each joint distribution is a standard function of n pairs of arguments: a real value x_i and the time t_i. The functions $P_X^{(n)}$ are proper multi-varied probability distributions that are non-negative and normalized,

$$\begin{aligned}&\forall(n \geqslant 1)\forall(x_1,\ldots,x_n)\forall(0 < t_1 < \cdots < t_n) :\\ &P_X^{(n)}(x_n,t_n;\ldots;x_1,t_1) \geqslant 0; \qquad \sum_{x_n,\ldots,x_1} P_X^{(n)}(x_n,t_n;\ldots;x_1,t_1) = 1.\end{aligned} \tag{12}$$

(For simplicity we are assuming here that process $X(t)$ is discrete, hence the sums over x_i rather than integrals; although, $t_i \in \mathbb{R}$.) The joint distribution $P^{(n)}_X(x_n, t_n; \ldots; x_1, t_1)$ is interpreted as the probability that a randomly chosen trajectory $x(t)$ passed through all the consecutive values x_i at the corresponding points in time t_i (assuming $t_1 < \cdots < t_n$); in formal terms,

$$P^{(n)}_X(x_n, t_n; \ldots; x_1, t_1) = \int P_X[x] \left(\prod_{i=1}^{n} \delta(x(t_i) - x_i) \right) [Dx]. \tag{13}$$

This means that moments (and thus, any expectation value) can be rewritten to utilize the joint distributions instead of the unwieldy functional integration,

$$\int P_X[x] \left(\prod_{i=1}^{n} x(t_i) \right) [Dx] = \sum_{x_n, \ldots, x_1} P^{(n)}_X(x_n, t_n; \ldots; x_1, t_1) \left(\prod_{i=1}^{n} x_i \right). \tag{14}$$

However, as the joint probability distributions are meant to describe the course of objective trajectories, they have to be *consistent*. If $P^{(n)}_X(\ldots; x_i, t_i; \ldots)$ is the probability that trajectory has passed through x_i at time t_i, then $P^{(n)}_X(\ldots; x_i, t_i; \ldots) + P^{(n)}_X(\ldots, x'_i, t_i; \ldots)$ is the probability that $x(t)$ has passed through either x_i or x'_i, and thus, $\sum_{x_i} P^{(n)}_X(\ldots; x_i, t_i; \ldots)$ is the probability that it has passed through any value. Since trajectories are functions defined on the whole real line, the probability for the occurrence of all possible alternatives at time t_i (i.e., trajectory passing through any value) must be equal to the probability without any constraints imposed on the trajectory at this point in time. But the probability for the trajectory to pass through the sequence of values $x_n, \ldots, x_1$ minus the constraint at t_i is given by the joint distribution of order $n-1$ where the ith pair of arguments is skipped, $P^{(n-1)}_X(x_n, t_n; \ldots; \cancel{x_i, t_i}; \ldots; x_1, t_1)$—i.e., the Kolmogorov consistency. Hence, in the context of classical theory, the fact that stochastic processes are realized as trajectories is manifested as consistency of joint probability distributions. The reciprocal is also true: the extension theorem states that any infinite family of multi-varied probability distributions that is consistent, uniquely defines a stochastic process with trajectories as its realizations.

Coming back to the issue of quantum observables, when the system's properties cause the Born distributions to satisfy KC condition, the extension theorem implies that $\{P_n\}_{n=1}^{\infty}$ defines stochastic process $F(t)$. Distributions P_n then play the role of joint probabilities $P^{(n)}_F$, and, as for all stochastic processes, they combine into functional distribution $P_F[f]$ for process trajectories $f(t)$. Consequently, each value in a sequence $f_n, \ldots, f_1$ [with probability distribution $P_n(f_n, t_n; \ldots; f_1, t_1)$] is equivalent to the sample of a trajectory, $f_i = f(t_i)$, where $f(t)$ has distribution $P_F[f]$. On a surface level, we conclude from these observations that the results of the sequential measurements of KC-satisfying quantum observable $\hat{F}$ can be simulated with the sampling of classical stochastic process $F(t)$ without the loss of any information. On a higher level, the KC condition (1) itself suggests the interpretation that $\hat{F}$ appears to classical observer as a trajectory traced over time independently of the action of measuring apparatus and the individual measurement events only uncover the already determined value. To explain how we are coming to this conclusion, first consider the following. In the theory of probability, the sum over ith argument on the LHS of KC condition (1) is typically meant to indicate that the measurement was performed at time t_i but the observer has discarded, or has forgotten, the result. On the other hand, the Born distribution that skips the argument at t_i describes the situation where there was no measurement at that time. Hence, the KC condition could be understood to mean that the influence of the measuring apparatus on the course of the observable dynamics is insignificant (in the sense that it does not affect the statistics of the following measurements), because simply forgetting the result of a measurement that was performed, is indistinguishable from the situation when the measurement never happened in the first place.

In summary, we have argued here that when the Born distributions $\{P_n\}_{n=1}^{\infty}$ satisfy the Kolmogorov consistency condition, the classical observers perceive the measured observable as a trajectory akin to realizations of classical stochastic process. In such an event, we say that the *trajectory picture* (of observable F) applies in the context of sequential measurements. Therefore, when tested against our standard of objectivity, at this point we can only affirm that the trajectory picture is, at least, inter-subjective among classical observers. However, in what follows we will show that the trajectory picture is more than inter-subjective because—unlike the previously discussed state collapse picture—it can be applied in context that do not involve measurements or classical observers. To this end, first we shall define the notion of *non-classical observer* and introduce the formalism that allows to speak of its perceptions of quantum observables.

4 Non-classical observer

We define the non-classical observer (of the observable F) as any other quantum system O, with its own dynamical law $\hat{U}_o(t) = \exp(-it\hat{H}_o)$, that is brought into contact with the original system S. For O to 'observe' the physical quantity F (in an analogy to classical observer performing measurements of F directly on S) the two systems need to interact according to the law that involves operator $\hat{F}$, i.e., the total OS Hamiltonian of the form

$$\hat{H}_{os} = \hat{H}_o \otimes \hat{1} + \hat{1} \otimes \hat{H} + \lambda \hat{G}_o \otimes \hat{F}, \tag{15}$$

is a minimal model of such an 'observing' interaction. The unitary evolution generated by this minimal model reads

$$\hat{U}_{os}(t) = e^{-it\hat{H}_{os}} = \hat{U}_o(t) \otimes \hat{U}(t)\hat{V}_{os}(t,0), \tag{16}$$

where the interaction picture evolution operator is given by the standard time-ordered exp,

$$\hat{V}_{os}(t,s) = Te^{-i\lambda\int_s^t \hat{G}_o(\tau)\otimes\hat{F}(\tau)d\tau} = \sum_{k=0}^{\infty}(-i\lambda)^k \int_s^t d\tau_k \cdots \int_s^{\tau_2} d\tau_1 \prod_{i=k}^{1} \hat{G}_o(\tau_i) \otimes \hat{F}(\tau_i), \tag{17}$$

and the interaction pictures of coupling operators is given by $\hat{G}_o(\tau) = \hat{U}_o^\dagger(\tau)\hat{G}_o\hat{U}_o(\tau)$ and $\hat{F}(\tau) = \hat{U}^\dagger(\tau)\hat{F}\hat{U}(\tau)$. The system O is, of course, changed by the interaction with $\hat{F}$ which manifests as the deviations from the system's free evolution. The most basic way to quantify these changes is to examine the dynamics of the reduced state of O, $\hat{\rho}_o(t) = \mathrm{tr}_s[\hat{U}_{os}(t)\hat{\rho}_o \otimes \hat{\rho}\,\hat{U}_{os}^\dagger(t)]$, or more precisely, its interaction picture $\hat{\varrho}_o(t) = \hat{U}_o^\dagger(t)\hat{\rho}_o(t)\hat{U}_o(t)$. Note that, formally, O is an open quantum system and S plays here the role of the environment.

In appendix A, we show that the influence from the observable F exerted onto $\hat{\varrho}_o(t)$ can be parameterized with *q-average* over the two-component *q-stochastic* process $(F_+(\tau), F_-(\tau))$,

$$\hat{\varrho}_o(t) = \mathrm{tr}_s\left[\hat{V}_{os}(t,0)\hat{\rho}_o \otimes \hat{\rho}\,\hat{V}_{os}(0,t)\right] = \left\langle \hat{V}_o[F_+](t,0)\hat{\rho}_o\hat{V}_o[F_-](0,t)\right\rangle, \tag{18}$$

where we have introduced an auxiliary O-only operator functional, derived from the OS interaction operator by replacing $\hat{F}(\tau)$ with a function,

$$\hat{V}_o[\varphi](t,s) \equiv Te^{-i\lambda\int_s^t \varphi(\tau)\hat{G}_o(\tau)d\tau} = \sum_{k=0}^{\infty}(-i\lambda)^k \int_s^t d\tau_k \cdots \int_s^{\tau_2} d\tau_1 \prod_{i=k}^{1} \varphi(\tau_i)\hat{G}_o(\tau_i). \tag{19}$$

The q-average itself is defined as a linear operation formally resembling the stochastic average (10), but instead of probability distribution functional it utilizes the complex-valued *q-probability* functional

$$\langle W[F_+, F_-]\rangle \equiv \iint Q_F[f_+, f_-]W[f_+, f_-][Df_+][Df_-]. \tag{20}$$

The q-probability functional $Q_F[f_+, f_-]$ plays then the role of the distribution (but not probability!) for the two real-valued trajectories of q-stochastic process representing observable F. Analogously to the stochastic expectation values, in practical calculations any q-average can be solved with moments computed using the family of *joint q-probability distributions* $\{Q_F^{(n)}\}_{n=1}^{\infty}$ [16, 17] (compare with Eq. (13) and see appendix A),

$$Q_F^{(n)}(\boldsymbol{f}_n, \boldsymbol{f}_{-n}, \boldsymbol{t}_n) = Q_F^{(n)}(f_n, f_{-n}, t_n; \ldots; f_1, f_{-1}, t_1)$$

$$\equiv \iint Q_F[f_+,f_-]\left(\prod_{i=1}^{n}\delta(f_+(t_i)-f_i)\delta(f_-(t_i)-f_{-i})\right)[Df_+][Df_-]$$

$$= \mathrm{tr}\left[\left(\prod_{i=n}^{1}\hat{P}(f_i,t_i)\right)\hat{\rho}\left(\prod_{i=1}^{n}\hat{P}(f_{-i},t_i)\right)\right]. \tag{21}$$

We are using here the same notation convention as in Sec. 2. Each joint q-probability $Q_F^{(n)}$ is a complex-valued function of n argument triplets: the time point t_i, and a pair of real eigenvalues $f_{\pm i}$. Using the decomposition of identity, $\sum_f \hat{P}(f,t) = \hat{1}$, it is straightforward to verify though direct calculation that q-probabilities satisfy the relation analogous to Kolmogorovian consistency,

$$\forall(n>1)\forall(0<t_1<\cdots<t_n)\forall(1\leq i\leq n)\forall(f_1,f_{-1};\ldots;\cancel{f_i,f_{-i}};\ldots;f_n,f_{-n}):$$
$$\sum_{f_i,f_{-i}} Q_F^{(n)}(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n) = Q_F^{(n-1)}(f_n,f_{-n},t_n;\ldots;\cancel{f_i,f_{-i},t_i};\ldots;f_1,f_{-1},t_1), \tag{22}$$

which explains how $\{Q_F^{(n)}\}_{n=1}^{\infty}$ combines into the functional distribution $Q_F[f_+,f_-]$. Alternatively, the consistency of q-probabilities can be explained as a special case of so-called *generalized extension theorem* [9]. Also note that Eq. (21) formally resembles the formula for so-called *decoherence functional* found in the consistent histories formulation of quantum mechanics [11, 12, 13, 14]; we defer a more detailed discussion on this link to the end of this section.

The key point is that the family of joint q-probabilities can be considered as a stand-alone object, abstracted from the specific context of dynamics of the reduced state of O. In such an approach, q-probabilities can be thought of as a way to quantify the dynamics of a given observable, like F; the Eq. (21) is treated then as definition: using this formula one can compute q-probability $Q_F^{(n)}$ associated with the chosen observable given its spectral decomposition $\{\hat{P}(f)\}_f$, the dynamical law $\hat{U}(t)$ for the system the observable lives in, and the initial condition $\hat{\rho}$. Appendix B showcases a number of examples where $\{Q_F^{(n)}\}_{n=1}^{\infty}$ have been computed from first principles for a collection of concrete quantum systems. As a stand-alone object, the q-probability family contains the description of the classical observer's perceptions; indeed, note that the 'diagonal' part of $Q_F^{(n)}$ equals the Born distribution for the sequential measurement of observable F, $Q_F^{(n)}(\boldsymbol{f}_n,\boldsymbol{f}_n,\boldsymbol{t}_n) = P_n(\boldsymbol{f}_n,\boldsymbol{t}_n)$. However, the 'classical' joint probability is only a part of the whole distribution; hence, it is convenient to define the decomposition of q-probability into its diagonal and the off-diagonal 'non-classical' parts,

$$Q_F^{(n)}(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n) = \delta_{\boldsymbol{f}_n,\boldsymbol{f}_{-n}}P_n(\boldsymbol{f}_n,\boldsymbol{t}_n) + \Phi_n(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n). \tag{23}$$

We shall refer to Φ_n as the *interference term*. The relation (18) demonstrates that both parts of q-probabilities come into play in the context of the dynamics in system O where they fully encapsulate the influence exerted by the observable F. Moreover, this result generalizes to any context involving physical quantities (observables) in O: the influence from F onto q-probabilities associated with an arbitrary O-only observable are fully described by $\{Q_F^{(n)}\}_{n=1}^{\infty}$ as well. For example, the q-probabilities associated with observable represented by $\hat{X}_o = \sum_x x\hat{P}_o(x)$ can also be written in the terms of a q-average (see appendix A),

$$Q_{X_o}^{(n)}(\boldsymbol{x}_n,\boldsymbol{x}_{-n},\boldsymbol{t}_n)$$
$$= \mathrm{tr}\left\langle\left(\prod_{i=n}^{1}\hat{V}_o[F_+](0,t_i)\hat{P}_o(x_i,t_i)\hat{V}_o[F_+](t_i,0)\right)\hat{\rho}_o\left(\prod_{i=1}^{n}\hat{V}_o[F_-](0,t_i)\hat{P}_o(x_{-i},t_i)\hat{V}_o[F_-](t_i,0)\right)\right\rangle. \tag{24}$$

Based on these observations we conclude that, as Born distributions $\{P_n\}_{n=1}^{\infty}$ describe how classical observers perceive the observable F, then by analogy q-probabilities $\{Q_F^{(n)}\}_{n=1}^{\infty}$ describe the perceptions of F for non-classical observers. The plural 'observers' is no mistake here because $Q_F^{(n)}$'s are determined exclusively by the properties of the observed system S, and thus, one family of q-probabilities suffices to describe the perceptions of S-only observable F for any and all non-classical observers. Therefore, we can say that the appearance of the quantum observable described by $\{Q_F^{(n)}\}_{n=1}^{\infty}$ is inter-subjective among non-classical observers. Finally, since P_n's are a part of $Q_F^{(n)}$'s, the perceptions of classical and non-classical observers can be meaningfully compared. The most obvious observation that can be made here is that, in general, their perceptions are not the same as they differ by the interference terms Φ_n—in other words, the appearance of quantum observable F is generally *not* inter-subjective between classical and non-classical observers.

In closing we wish to point out some notable connections between q-probability parameterization and other established theoretical approaches found in the literature. The inspection of Eq. (18) suggests a possible intuitive interpretation for the components of q-stochastic process $(F_+(t), F_-(t))$. The $F_+(t)$-component appears to play the role of an external field driving the 'branch' of evolution going forwards in time, while $F_-(t)$ drives the backwards-in-time evolution branch. Then, the non-classical off-diagonal part of q-probabilities [the interference terms $\Phi_n(\boldsymbol{f}_n, \boldsymbol{f}_{-n}, \boldsymbol{t}_n)$] would be responsible for the quantum interference between the forwards and backwards branches. The classical diagonal part $\delta_{\boldsymbol{f}_n,\boldsymbol{f}_{-n}} P_n(\boldsymbol{f}_n, \boldsymbol{t}_n)$ comes into play only when $F_-(t)$ overlaps with $F_+(t)$ and both components merge into a single field. Such a structure of forwards and backwards fields is, of course, familiar from other approaches to open quantum systems, e.g., Feynman-Vernon influence functional [18] or Keldysh field theoretical approach [19, 20, 21]. It should however be noted that the components $F_+(t)$ and $F_-(t)$—the trajectories traced through the domain of the eigenvalues of operator $\hat{F}$—are fundamentally different objects than the paths of Feynman-Vernon, and fermionic or bosonic fields in Keldysh formalism, see [4].

In contrast, the q-probability parameterization of dynamics of observable F can be directly linked with the *consistent histories* formulation of quantum mechanics [11, 12, 13, 14]. The link is straightforward: the central object of the histories formalism—the *decoherence functional*—has the same form as the q-probability, provided that the events constituting a history are identified with eigenspaces of observable F. The principal difference between the two lies in their respective origins. In the histories formalism, the decoherence functional is postulated as an improvement on the Born rule of the standard quantum theory; as such, analogously to the Born rule, the decoherence functional has the status of primitive notion. Then, the diagonal part of the functional is used as a Born distribution while the off-diagonal part does not have an explicit utility. The q-probabilities, on the other hand, are not postulated but rather are identified as a facet of the standard formalism for describing the dynamics of quantum observable F. They appear naturally in the description of a quantum system O coupled to S through F: Eqs. (18) and (24) show that the reduced state of O, as well as any multi-time correlation functions of its observables, all can be expressed in terms of q-average over the totality of q-probability $Q_F[f_+, f_-]$, not excluding its off-diagonal part (in contrast to the consistent histories framework and the off-diagonal part of the decoherence functional). It could even be argued here that, through the link with q-probabilities, the decoherence functional (or Born distribution, for that matter) can be seen as an emergent quantity derived from general formalism of standard quantum theory rather than a primitive notion as it is postulated in the original formulation of the consistent histories formalism.

5 When the classical and non-classical observers are in agreement

By combining the decomposition of q-probabilities into diagonal and off-diagonal parts (23) and their consistency (22), we establish the general relation between Born distributions mediated by the interference term,

$$\sum_{f_i} P_n(\boldsymbol{f}_n, \boldsymbol{t}_n) - P_{n-1}(f_n, t_n; \ldots; \cancel{f_i, t_i}; \ldots; f_1, t_1) = \left(\prod_{j\neq i} \delta_{f_j, f_{-j}}\right) \sum_{f_i \neq f_{-i}} \Phi_n(\boldsymbol{f}_n, \boldsymbol{f}_{-n}, \boldsymbol{t}_n). \quad (25)$$

In the context of Kolmogorovian consistency (KC), the above equation shows that $\{P_n\}_{n=1}^{\infty}$ becomes consistent when the observable F satisfies the *consistent measurements* (CM) condition:

$$\mathbf{CM:}\quad \begin{aligned}&\forall(n\geqslant 1)\,\forall(0<t_1<\cdots<t_n)\,\forall(1\leqslant i\leqslant n)\,\forall(f_1,\ldots,\cancel{f_i},\ldots,f_n):\\ &0=\Bigg(\prod_{j\neq i}\delta_{f_j,f_{-j}}\Bigg)\sum_{f_i\neq f_{-i}}\Phi_n(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n)\\ &\quad=\sum_{f_i\neq f_{-i}}\Phi_n(f_n,f_n,t_n;\ldots;f_i,f_{-i},t_i;\ldots;f_1,f_1,t_1)\\ &\quad=\Bigg(\prod_{j\neq i}\delta_{f_j,f_{-j}}\Bigg)\sum_{f_i>f_{-i}}2\,\mathrm{Re}\{\Phi_n(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n)\}.\end{aligned} \tag{26}$$

Thus, the fulfillment of CM (which is mathematically equivalent to KC) guarantees that the trajectory picture of F applies to the observations made by the classical observer. However, not much can be said about the implications for contexts other than sequential measurements; in particular, it is unclear how it would affect the perceptions of non-classical observers. To improve on this issue, we propose to consider a simpler, but also stricter, condition—we say that the observable F satisfies the *surrogate field* (SF) condition when

$$\mathbf{SF:}\quad \forall(n\geqslant 1)\,\forall(\boldsymbol{f}_n,\boldsymbol{f}_{-n})\,\forall(0<t_1<\cdots<t_n):\ \Phi_n(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n)=0. \tag{27}$$

The SF (the reason for the name will soon become apparent) clearly implies CM&KC, and more importantly, its broader physical implications for other contexts—and non-classical observers in particular—are significantly more transparent. When the interference terms Φ_n all vanish individually, then q-probabilities reduce to proper probabilities, $Q_F^{(n)}(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n)\to\delta_{\boldsymbol{f}_n,\boldsymbol{f}_{-n}}P_n(\boldsymbol{f}_n,\boldsymbol{t}_n)$ (and $Q_F[f_+,f_-]\to\delta[f_+-f_-]P_F[f_+]$ in the terms of functional distributions); of course, the remaining Born distributions are then consistent because CM&KC are already implied. This means that, effectively, the two components of the q-stochastic process $(F_+(t),F_-(t))$ in any q-average [e.g., like in (18)] are merged together into a single proper stochastic process $F(t)$ defined by now consistent family $\{P_n\}_{n=1}^{\infty}$—in short, the q-averages reduce to the standard stochastic averages. Consequently, the non-classical observer's reduce density matrix simplifies to,

$$\hat{\varrho}_o(t)\xrightarrow{\Phi_n=0}\overline{\hat{V}_o[F](t,0)\hat{\rho}_o\hat{V}_o[F](0,t)}, \tag{28}$$

and thus, the dynamics in O appear as if the system was driven by an external noise $F(t)$ instead of being coupled to actual quantum system S. In other words, as far as the dynamics in O are concerned, the coupling to quantum observable $\hat{F}$ in SO Hamiltonian (15) has been effectively replaced by the stochastic process $F(t)$,

$$\hat{H}_{os}\to\hat{H}_o[F](t)=\hat{H}_o+\lambda F(t)\hat{G}_o, \tag{29}$$

and this stochastic Hamiltonian generates the evolution (28). In the context of open quantum system theory (i.e., when O is considered open to S that is then treated as the environment), when the surrogate field condition is satisfied—and the stochastic simulation of the dynamics in O is enabled—one says [16] that the *surrogate field representation* of F is valid and the process $F(t)$ is called the surrogate field, as it is a surrogate for the coupling operator $\hat{F}$.

To summarize, we have identified here the surrogate field condition (27) under which the sequential measurement of $\hat{F}$ and the dynamics of a quantum system coupled to $\hat{F}$ are both indistinguishable from their respective stochastic simulations. Moreover, and this is key, the stochastic process the simulations are carried out with in both contexts, is the same surrogate field $F(t)$—the process defined by the family of q-probabilities $\{Q_F^{(n)}\}_{n=1}^{\infty}$ satisfying surrogate field condition, and thus, reduced to consistent Born distributions $\{P_n\}_{n=1}^{\infty}$. In other words, when the observable $\hat{F}$ meets the surrogate field condition, then the perceptions of this physical quantity by the classical and non-classical observers are in agreement. It must be underlined that this symmetry between classical and non-classical perceptions is not only a philosophical curiosity, but also a physical effect with tangible consequences. For example, consider the scenario where S plays the role of the environment and one is interested in predicting the evolution of the reduced state of O coupled to $\hat{F}$, $\hat{\varrho}_o(t)$. When the surrogate field condition is satisfied and the surrogate field representation of $\hat{F}$ is valid, the explicit computation of the stochastic average in (28) is the main challenge in executing the simulation of $\hat{\varrho}_o(t)$. This problem can be circumvented completely if one also has access to the results of sequential measurements of $\hat{F}$, which, in this case, are equivalent to the trajectory sampling of the surrogate field $F(t)$. Indeed, since the measurement results and the surrogate field driving O are both described by the same probability distributions $\{P_n\}_{n=1}^{\infty}$, the measured samples can be used to approximate the stochastic average with the sample average [4],

$$\begin{aligned}\overline{\hat{V}_o[F](t,0)\hat{\rho}_o\hat{V}_o[F](0,t)} &= \int P_F[f]\hat{V}_o[f](t,0)\hat{\rho}_o\hat{V}_o[f](0,t)[Df] \\ &\approx \frac{1}{N}\sum_{j=1}^{N}\hat{V}_o[f_j](t,0)\hat{\rho}_o\hat{V}_o[f_j](0,t),\end{aligned} \tag{30}$$

where $\{f_j(t)\}_{j=1}^{N}$ is an ensemble of trajectories interpolated from the measured sequences [compare with Eq. (10)]. The Eq. (30) is the most explicit demonstration that the trajectory picture emerging for SF-satisfying observable applies equally in both measurement and non-measurement contexts—essentially, it shows that the trajectories are interchangeable between the contexts. Therefore, when assessed with respect to our objectivity standard, the trajectory picture is upgraded from inter-subjectivity among classical observers only to inter-subjectivity between classical and non-classical observers. However, since there is a categorical difference between the classical and non-classical observers, the extension of inter-subjectivity to encompass both categories is not merely quantitative either; typically, such cross-category extension is considered a sufficient condition for upgrading from inter-subjectivity to *objectivity*.

Coming back to the specific question of Kolmogorov consistency of Born distributions, we make a final note on the relation between the consistent measurements (CM) condition and the surrogate field (SF) condition treated only as its enablers. From purely mathematical point of view, the CM and SF conditions are obviously not equivalent, at least when the interference terms Φ_n are considered in 'vacuum'. That is, if $\{\Phi_n\}_{n=1}^{\infty}$ was a set of arbitrary functions, then the only thing that could be said is that CM is a weaker condition than SF (or SF is stronger than CM) because SF implies CM but not the other way around. However, Φ_n's are not arbitrary; to the contrary, each interference term has a complex internal structure emerging from interactions between the dynamical law $\hat{U}$, the partitioning $\{\hat{P}(f)\}_f$, and the initial condition $\hat{\rho}$. How exactly this internal complexity changes the relations between SF and CM is, unsurprisingly, an extremely difficult problem to crack. In a situation like this, when the mathematically strict answer is not forthcoming, the relative significance of conditions can be estimated based on the plurality of cases. If one cannot give an example of a physical system for which the formally weaker CM is satisfied and, at the same time, the formally stronger SF is violated, then the probability that CM is spurious (i.e., the weaker condition is only satisfied by first satisfying the stronger one) increases. In fact, to our best knowledge, this seems to be the case. On the one hand, a number of types of physical systems satisfying SF is readily available: we discuss some of the examples in appendix B, and also see [14] for additional important class of example. On the other hand, we are not aware of even a single instance of a system that would satisfy only the consistent measurements condition, which strongly suggests that CM alone might be uninteresting from the physical point of view. At the same time, the surrogate field condition seems to be well worth further consideration, given its far reaching consequences in contexts beyond the problem of Kolmogorovian consistency. Surrogate field representations, the non-classical observer perceptions, and open systems in general, all count among those other contexts; aside the examples discussed here, we note that the consistent histories theory should also be included on the list as the SF condition is formally equivalent to the *decoherent histories* condition [22, 14], provided that one identifies the decoherence functional as the q-probability.

6 Conclusions

We have identified the condition—the surrogate field (SF) criterion (27)—under which a quantum observable $\hat{F}$ appears to both classical and non-classical observers as the stochastic process $F(t)$.

On the side of the classical observer this manifests as consistent measurements of $\hat{F}$, i.e. sequential measurements described by the Born probability distributions that satisfy the Kolmogorov consistency (KC) criterion (1). Consequently, a result of a measurement performed by the observer is equivalent to the sampling of a trajectory $f(t)$ that realizes the process $F(t)$. On the side of non-classical observer—that is, a quantum system coupled to observable $\hat{F}$—the dynamics of the system is indistinguishable from stochastic simulation where the quantum operator $\hat{F}$ is replaced with the process $F(t)$. The key point is that this surrogate field that mimics the coupling with actual quantum system, is the same stochastic process sampled by the classical observer. Therefore, it can be said that when the observable $\hat{F}$ satisfies the SF condition, then not only the quantum stochastic process becomes classical, but it can also be considered as an *objective* entity.

Apart from the SF condition, we have also found a weaker (than SF) consistent measurements (CM) condition (26) for Kolmogorovian consistency of $\{P_n\}_{n=1}^{\infty}$. In fact, CM and KC conditions are equivalent—it is a direct consequence of relation (25) between Born distributions, a new and interesting formal result in itself. However, we have argued that the stronger SF condition is physically more substantial: when it is satisfied, the CM&KC are, of course, implied, but more importantly, it also brings about the symmetry between the perceptions of classical and non-classical observers. Hence, it seems more vital—even from practical point of view because of stochastic simulations, see Eq. (30)—to verify whether SF is satisfied over just checking for CM&KC. If one accepts this assessment, then the only real value of CM&KC would come from its potential ability to determine the status of SF. Even though is it probable that CM is actually a spurious condition (i.e., it can only be satisfied when SF is satisfied), we cannot dismiss the possibility that Kolmogorovian consistency could be satisfied without SF; therefore, at this time, the consistency (or rather its violation) can serve only as a witness of surrogate field violation.

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A Derivation of q-average forms

A.1 Reduced state

The objective is to show that the interaction picture of the reduced state of O,

$$\hat{\varrho}_o(t) = \mathrm{tr}_s\left[\hat{V}_{os}(t,0)\hat{\rho}_o \otimes \hat{\rho}\hat{V}_{os}(0,t)\right], \tag{31}$$

where

$$\hat{V}_{os}(t,s) = Te^{-i\lambda\int_s^t \hat{G}_o(\tau)\otimes\hat{F}(\tau)d\tau};\ \hat{G}_o(\tau) = \hat{U}_o^\dagger(\tau)\hat{G}_o\hat{U}_o(\tau);\ \hat{F}(\tau) = \hat{U}^\dagger(\tau)\hat{F}\hat{U}(\tau), \tag{32}$$

can be rewritten in the q-average form (18).

We begin by switching to the super-operator language (i.e., linear operators that act on operators):

$$\hat{\varrho}_o(t) \equiv \Lambda_t\hat{\rho}_o, \tag{33}$$

where the dynamical map Λ_t is a super-operator acting on operators in O and is defined as

$$\begin{aligned}\Lambda_t &= \mathrm{tr}_s\left[\hat{V}_{os}(t,0)(\bullet\otimes\hat{\rho})\hat{V}_{os}(0,t)\right] = \mathrm{tr}_s\left[Te^{-i\lambda\int_0^t[\hat{G}_o(\tau)\otimes\hat{F}(\tau),\bullet]d\tau}(\bullet\otimes\hat{\rho})\right]\\ &= \sum_{k=0}^{\infty}(-i\lambda)^k\int_0^t d\tau_k\cdots\int_0^{\tau_2}d\tau_1\,\mathrm{tr}_s\left([\hat{G}_o(\tau_k)\otimes\hat{F}(\tau_k),\cdots[\hat{G}_o(\tau_1)\otimes\hat{F}(\tau_1),\bullet\otimes\hat{\rho}]\cdots]\right)\\ &= \sum_{k=0}^{\infty}(-i\lambda)^k\int_0^t d\tau_k\cdots\int_0^{\tau_2}d\tau_1\,\mathrm{tr}_s\left([\hat{G}_o(\tau_k)\otimes\hat{F}(\tau_k),\bullet]\cdots[\hat{G}_o(\tau_1)\otimes\hat{F}(\tau_1),\bullet](\bullet\otimes\hat{\rho})\right),\end{aligned} \tag{34}$$

where the symbol $\bullet$ is understood as a 'placeholder' for the argument of the super-operator acting on operators to its right, e.g., $[\hat{A},\bullet]\hat{B} = [\hat{A},\hat{B}]$, or $\bullet\hat{A} = \hat{A}$, etc. To arrive at this form we have made use of a well known identity for exp (time-ordered and otherwise),

$$Te^{-i\int_s^t \hat{A}(\tau)d\tau} \bullet \left(Te^{-i\int_s^t \hat{A}(\tau)d\tau}\right)^\dagger = Te^{-i\int_s^t [\hat{A}(\tau),\bullet]d\tau}. \tag{35}$$

Next, we aim to rewrite the map as a moment series. First, we use the spectral decomposition of observable $\hat{F}$,

$$\hat{F}(\tau) = \hat{U}^\dagger(\tau)\left(\sum_f f\hat{P}(f)\right)\hat{U}(\tau) = \sum_f f\hat{P}(f,\tau), \tag{36}$$

and we substitute it into the formula,

$$\Lambda_t = \sum_{k=0}^{\infty}(-i\lambda)^k \int_0^t d\tau_k \cdots \int_0^{\tau_2} d\tau_1$$
$$\times \operatorname{tr}_s\left(\sum_{f_k} f_k[\hat{G}_o(\tau_k)\otimes\hat{P}(f_k,\tau_k),\bullet]\cdots\sum_{f_1} f_1[\hat{G}_o(\tau_1)\otimes\hat{P}(f_1,\tau_1),\bullet](\bullet\otimes\hat{\rho})\right). \tag{37}$$

Second, we show that the S and O parts of the above expression are actually separable; to see this, consider one of the super-operators $\sum_f f[\hat{G}_o(\tau)\otimes\hat{P}(f,\tau),\bullet]$ and act with it onto an operator in an outer product form $\hat{O}\otimes\hat{S}$,

$$\begin{aligned}
&\sum_f f[\hat{G}_o(\tau)\otimes\hat{P}(f,\tau),\bullet]\hat{O}\otimes\hat{S}\\
&= \sum_{f_+} f_+\big(\hat{G}_o(\tau)\hat{O}\big)\otimes\big(\hat{P}(f_+,\tau)\hat{S}\big) - \sum_{f_-} f_-\big(\hat{O}\hat{G}_o(\tau)\big)\otimes\big(\hat{S}\hat{P}(f_-,\tau)\big)\\
&= \sum_{f_+,f_-} f_+\big(\hat{G}_o(\tau)\hat{O}\big)\otimes\big(\hat{P}(f_+,\tau)\hat{S}\hat{P}(f_-,\tau)\big) - \sum_{f_+,f_-} f_-\big(\hat{O}\hat{G}_o(\tau)\big)\otimes\big(\hat{P}(f_+,\tau)\hat{S}\hat{P}(f_-,\tau)\big)\\
&= \sum_{f_+,f_-} \big(f_+\hat{G}_o\hat{O} - \hat{O}\hat{G}_o(\tau)f_-\big)\otimes\big(\hat{P}(f_+,\tau)\hat{S}\hat{P}(f_-,\tau)\big)\\
&= \sum_{f_+,f_-} \big(f_+\hat{G}_o(\tau)\bullet - \bullet\hat{G}_o(\tau)f_-\big)\otimes\big(\hat{P}(f_+,\tau)\bullet\hat{P}(f_-,\tau)\big)(\hat{O}\otimes\hat{S})\\
&\equiv \sum_{f_+,f_-} \mathcal{G}_\tau(f_+,f_-)\otimes\mathcal{P}_\tau(f_+,f_-)(\hat{O}\otimes\hat{S}),
\end{aligned} \tag{38}$$

which can then be easily generalized to the super-operator equality,

$$\sum_f f[\hat{G}_o(\tau)\otimes\hat{P}(f,\tau),\bullet] = \sum_{f_+,f_-} \mathcal{G}_\tau(f_+,f_-)\otimes\mathcal{P}_\tau(f_+,f_-). \tag{39}$$

To see this, take a product basis in the $O+S$ operator space: if $\{\hat{O}_i\}_i$ is an orthonormal basis in O-operator space, and $\{\hat{S}_\alpha\}_\alpha$ is a basis in S, then $\{\hat{O}_i\otimes\hat{S}_\alpha\}_{i,\alpha}$ is a basis in $O+S$-operator space. With this basis, any arbitrary operator $\hat{A}_{os}$ can be decomposed into linear combination of separable products, $\hat{A}_{os} = \sum_{i,\alpha} A_{i\alpha}\hat{O}_i\otimes\hat{S}_\alpha$; the general result (39), then follows directly from the special result (38).

Thanks to the separability, we arrive at the form where the partial trace over S can be computed explicitly,

$$\Lambda_t = \sum_{k=0}^{\infty}(-i\lambda)^k \int_0^t d\tau_k \cdots \int_0^{\tau_2} d\tau_1 \sum_{f_k,f_{-k}} \cdots \sum_{f_1,f_{-1}}$$

$$\times\,\mathrm{tr}_s\left[\mathcal{P}_{\tau_k}(f_k,f_{-k})\cdots\mathcal{P}_{\tau_1}(f_1,f_{-1})\hat{\rho}\right]\mathcal{G}_{\tau_k}(f_k,f_{-k})\cdots\mathcal{G}_{\tau_1}(f_1,f_{-1})$$

$$=\sum_{k=0}^{\infty}(-i\lambda)^k\int_0^t d\tau_k\cdots\int_0^{\tau_2}d\tau_1\sum_{f_k,f_{-k}}\cdots\sum_{f_1,f_{-1}}$$

$$\times\,\mathrm{tr}_s\left[\hat{P}(f_k,\tau_k)\cdots\hat{P}(f_1,\tau_1)\hat{\rho}\hat{P}(f_{-1},\tau_1)\cdots\hat{P}(f_{-k},\tau_k)\right]\mathcal{G}_{\tau_k}(f_k,f_{-k})\cdots\mathcal{G}_{\tau_1}(f_1,f_{-1})$$

$$=\sum_{k=0}^{\infty}(-i\lambda)^k\int_0^t d\tau_k\cdots\int_0^{\tau_2}d\tau_1\sum_{f_k,f_{-k}}\cdots\sum_{f_1,f_{-1}}Q_F^{(k)}(\boldsymbol{f}_k,\boldsymbol{f}_{-k},\boldsymbol{\tau}_k)\mathcal{G}_{\tau_k}(f_k,f_{-k})\cdots\mathcal{G}_{\tau_1}(f_1,f_{-1}).\quad(40)$$

We have recognized in the traced factor the formula for q-probability (21). We are also recognizing here a q-average in each term of the series,

$$\sum_{f_k,f_{-k}}\cdots\sum_{f_1,f_{-1}}Q_F^{(k)}(\boldsymbol{f}_k,\boldsymbol{f}_{-k},\boldsymbol{\tau}_k)\mathcal{G}_{\tau_k}(f_k,f_{-k})\cdots\mathcal{G}_{\tau_1}(f_1,f_{-1})$$

$$=\sum_{f_k,f_{-k}}\cdots\sum_{f_1,f_{-1}}\iint[Df_+][Df_-]Q_F[f_+,f_-]\left(\prod_{i=1}^{k}\delta(f_i-f_+(\tau_i))\delta(f_{-i}-f_-(\tau_i))\right)$$

$$\times\,\mathcal{G}_{\tau_k}(f_+(\tau_k),f_-(\tau_k))\cdots\mathcal{G}_{\tau_1}(f_+(\tau_1),f_-(\tau_1))$$

$$=\iint Q_F[f_+,f_-]\mathcal{G}_{\tau_k}(f_+(\tau_k),f_-(\tau_k))\cdots\mathcal{G}_{\tau_1}(f_+(\tau_1),f_-(\tau_1))[Df_+][Df_-]$$

$$=\langle\mathcal{G}_{\tau_k}(F_+(\tau_k),F_-(\tau_k))\cdots\mathcal{G}_{\tau_1}(F_+(\tau_1),F_-(\tau_1))\rangle\,.\quad(41)$$

As a result, we can take the whole series under the sign of q-average,

$$\Lambda_t=\sum_{k=0}^{\infty}(-i\lambda)^k\int_0^t d\tau_k\cdots\int_0^{\tau_2}d\tau_1\,\langle\mathcal{G}_{\tau_k}(F_+(\tau_k),F_-(\tau_k))\cdots\mathcal{G}_{\tau_1}(F_+(\tau_1),F_-(\tau_1))\rangle$$

$$=\left\langle\sum_{k=0}^{\infty}(-i\lambda)^k\int_0^t d\tau_k\cdots\int_0^{\tau_2}d\tau_1\mathcal{G}_{\tau_k}(F_+(\tau_k),F_-(\tau_k))\cdots\mathcal{G}_{\tau_1}(F_+(\tau_1),F_-(\tau_1))\right\rangle$$

$$=\left\langle Te^{-i\lambda\int_0^t\mathcal{G}_\tau(F_+(\tau),F_-(\tau))d\tau}\right\rangle.\quad(42)$$

The last step is to demonstrate that the series under the q-average evaluates to the form presented in Eq. (18), i.e., we must prove that

$$Te^{-i\lambda\int_0^t\mathcal{G}_\tau(F_+(\tau),F_-(\tau))d\tau}=\hat{\mathcal{V}}_o[F_+](t,0)\bullet\hat{\mathcal{V}}_o[F_-](0,t).\quad(43)$$

The simplest way to show it exploits the fact that the two super-operators constituting $\mathcal{G}_\tau$,

$$\mathcal{G}_\tau(F_+(\tau),F_-(\tau))=F_+(\tau)\hat{G}_o(\tau)\bullet+(-F_-(\tau)\bullet\hat{G}_o(\tau))\equiv\mathcal{G}_+(\tau)+\mathcal{G}_-(\tau),\quad(44)$$

commute with each other,

$$\forall(\tau,\tau'):\mathcal{G}_+(\tau)\mathcal{G}_-(\tau')-\mathcal{G}_-(\tau')\mathcal{G}_+(\tau)=0.\quad(45)$$

If so, then the exp in (42) simply factorizes into composition of two (commuting) exps,

$$Te^{-i\lambda\int_0^t\mathcal{G}_\tau(F_+(\tau),F_-(\tau))d\tau}=\left(Te^{-i\lambda\int_0^t\mathcal{G}_+(\tau)d\tau}\right)\left(Te^{-i\lambda\int_0^t\mathcal{G}_-(\tau)d\tau}\right)$$

$$=Te^{\left(-i\lambda\int_0^t F_+(\tau)\hat{G}_o(\tau)d\tau\right)\bullet}\,Te^{\bullet\left(+i\lambda\int_0^t F_-(\tau)\hat{G}_o(\tau)d\tau\right)}$$

$$=\left(\sum_{k=0}^{\infty}(-i\lambda)^k\int_0^t d\tau_k\cdots\int_0^{\tau_2}d\tau_1F_+(\tau_k)\hat{G}_o(\tau_k)\cdots F_+(\tau_1)\hat{G}_o(\tau_1)\right)\bullet$$

$$\times \bullet \left(\sum_{k=0}^{\infty} (i\lambda)^k \int_0^t d\tau_k \cdots \int_0^{\tau_2} d\tau_1 F_-(\tau_1)\hat{G}_o(\tau_1) \cdots F_-(\tau_k)\hat{G}_o(\tau_k) \right)$$
$$= \hat{V}_o[F_+](t,0) \bullet \big(\hat{V}_o[F_-](t,0)\big)^\dagger = \hat{V}_o[F_+](t,0) \bullet \hat{V}_o[F_-](0,t). \tag{46}$$

A.2 Q-probabilites

Here, the goal is to show that the general formula (21) for q-probability associated with the O-only observable, say $\hat{X}_o = \sum_x x\hat{P}_o(x)$,

$$Q^{(n)}_{X_o}(\boldsymbol{x}_n, \boldsymbol{x}_{-n}, \boldsymbol{t}_n) = \mathrm{tr}\left[\left(\prod_{i=n}^{1} \hat{U}^\dagger_{os}(t_i)\hat{P}_o(x_i) \otimes \hat{1}\hat{U}_{os}(t_i) \right) \hat{\rho}_o \otimes \hat{\rho} \left(\prod_{i=1}^{n} \hat{U}^\dagger_{os}(t_i)\hat{P}_o(x_{-i}) \otimes \hat{1}\hat{U}_{os}(t_i) \right) \right], \tag{47}$$

can also be written in the q-average form.

First, we translate the general definition of q-probability to the language of super-operators,

$$\begin{aligned}
&Q^{(n)}_{X_o}(\boldsymbol{x}_n, \boldsymbol{x}_{-n}, \boldsymbol{t}_n)\\
&\quad = \mathrm{tr}\left[\left(\prod_{i=n}^{1} \hat{V}_{os}(0,t_i)\big(\hat{U}^\dagger_o(t_i)\hat{P}_o(x_i)\hat{U}_o(t_i)\big) \otimes \big(\hat{U}^\dagger(t_i)\hat{U}(t_i)\big)\hat{V}_{os}(t_i,0) \right) \hat{\rho}_o \otimes \hat{\rho} \Big(\cdots \Big) \right]\\
&\quad = \mathrm{tr}\left[\left(\prod_{i=n}^{1} \big(\hat{P}_o(x_i,t_i) \otimes \hat{1}\big)\hat{V}_{os}(t_i,t_{i-1}) \right) \hat{\rho}_o \otimes \hat{\rho} \left(\prod_{i=1}^{n} \hat{V}_{os}(t_{i-1},t_i)\big(\hat{P}_o(x_{-i},t_i) \otimes \hat{1}\big) \right) \right]\\
&\quad = \mathrm{tr}\Big[\big(\hat{P}_o(x_n,t_n) \otimes \hat{1} \bullet \hat{P}_o(x_{-n},t_n) \otimes \hat{1}\big)\big(\hat{V}_{os}(t_n,t_{n-1}) \bullet \hat{V}_{os}(t_{n-1},t_n)\big) \cdots\\
&\qquad \cdots \big(\hat{P}_o(x_1,t_1) \otimes \hat{1} \bullet \hat{P}_o(x_{-1},t_1) \otimes \hat{1}\big)\big(\hat{V}_{os}(t_1,0) \bullet \hat{V}_{os}(0,t_1)\big)\hat{\rho}_o \otimes \hat{\rho} \Big]\\
&\quad \equiv \mathrm{tr}\left[\big(\mathcal{P}_{t_n}(x_n,x_{-n}) \otimes \bullet\big)\Omega_{os}(t_n,t_{n-1}) \cdots \big(\mathcal{P}_{t_1}(x_1,x_{-1}) \otimes \bullet\big)\Omega_{os}(t_1,0)\hat{\rho}_o \otimes \hat{\rho} \right],
\end{aligned} \tag{48}$$

where $t_0 = 0$ and we have used the definition of the interaction picture of evolution operator (16).

Second, using methods (and notation) from previous section, we rewrite the propagators Ω's in the terms of generators $\mathcal{G}$'s,

$$\Omega_{os}(t,s) = \sum_{k=0}^{\infty} (-i\lambda)^k \int_s^t d\tau_k \cdots \int_s^{\tau_2} d\tau_1 \sum_{\boldsymbol{f}_k, \boldsymbol{f}_{-k}} \left(\prod_{i=k}^{1} \mathcal{G}_{\tau_k}(f_i, f_{-i}) \right) \otimes \left(\prod_{i=k}^{1} \mathcal{P}_{\tau_k}(f_i, f_{-i}) \right). \tag{49}$$

When we substitute this form into $Q^{(n)}_{X_o}$ we can, again, compute the trace over S explicitly,

$$\begin{aligned}
&Q^{(n)}_{X_o}(\boldsymbol{x}_n, \boldsymbol{x}_{-n}, \boldsymbol{t}_n)\\
&= \sum_{k_n=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} (-i\lambda)^{\sum_{i=1}^n k_i} \left(\int_{t_{n-1}}^{t_n} d\tau^n_{k_n} \cdots \int_{t_{n-1}}^{\tau^n_2} d\tau^n_1 \right) \cdots \left(\int_0^{t_1} d\tau^1_{k_1} \cdots \int_0^{\tau^1_2} d\tau^1_1 \right) \sum_{\boldsymbol{f}^n_{k_n}} \sum_{\boldsymbol{f}^n_{-k_n}} \cdots \sum_{\boldsymbol{f}^1_{k_1}} \sum_{\boldsymbol{f}^1_{-k_1}}\\
&\quad \times \mathrm{tr}_s\left[\mathcal{P}_{\tau^n_{k_n}}(f^n_{k_n}, f^n_{-k_n}) \cdots \mathcal{P}_{\tau^1_1}(f^1_1, f^1_{-1})\hat{\rho} \right]\\
&\quad \times \mathrm{tr}_o\Big[\mathcal{P}_{t_n}(x_n, x_{-n})\mathcal{G}_{\tau^n_{k_n}}(f^n_{k_n}, f^n_{-k_n}) \cdots \mathcal{G}_{\tau^n_1}(f^n_1, f^n_{-1}) \cdots\\
&\qquad \cdots \mathcal{P}_{t_1}(x_1, x_{-1})\mathcal{G}_{\tau^1_{k_1}}(f^1_{k_1}, f^1_{-k_1}) \cdots \mathcal{G}_{\tau^1_1}(f^1_1, f^1_{-1})\hat{\rho}_o \Big]\\
&= \sum_{k_n=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} (-i\lambda)^{\sum_{i=1}^n k_i} \left(\int_{t_{n-1}}^{t_n} d\tau^n_{k_n} \cdots \int_{t_{n-1}}^{\tau^n_2} d\tau^n_1 \right) \cdots \left(\int_0^{t_1} d\tau^1_{k_1} \cdots \int_0^{\tau^1_2} d\tau^1_1 \right) \sum_{\boldsymbol{f}^n_{k_n}} \sum_{\boldsymbol{f}^n_{-k_n}} \cdots \sum_{\boldsymbol{f}^1_{k_1}} \sum_{\boldsymbol{f}^1_{-k_1}}\\
&\quad \times Q^{(k_n+\cdots+k_1)}_F(f^n_{k_n}, f^n_{-k_n}, \tau^n_{k_n}; \ldots; f^1_1, f^1_{-1}, \tau^1_1)
\end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{tr}_o\left[\mathcal{P}_{t_n}(x_n,x_{-n})\mathcal{G}_{\tau^n_{k_n}}(f^n_{k_n},f^n_{-k_n})\cdots\mathcal{G}_{\tau^1_1}(f^1_1,f^1_{-1})\hat{\rho}_o\right]\\
&=\mathrm{tr}_o\left[\left\langle\mathcal{P}_{t_n}(x_n,x_{-n})\left(\sum_{k_n=0}^{\infty}(-i\lambda)^{k_n}\int\limits_{t_{n-1}}^{t_n}d\tau^n_{k_n}\cdots\int\limits_{t_{n-1}}^{\tau^n_2}d\tau^n_1\prod_{i=k_n}^{1}\mathcal{G}_{\tau^n_i}(F_+(\tau^n_i),F_-(\tau^n_i))\right)\right.\right.\\
&\quad\left.\left.\cdots\mathcal{P}_{t_1}(x_1,x_{-1})\left(\sum_{k_1=0}^{\infty}(-i\lambda)^{k_1}\int\limits_0^{t_1}d\tau^1_{k_1}\cdots\int\limits_0^{\tau^1_2}d\tau^1_1\prod_{i=k_1}^{1}\mathcal{G}_{\tau^1_i}(F_+(\tau^1_i),F_-(\tau^1_i))\right)\right\rangle\hat{\rho}_o\right]\\
&=\mathrm{tr}\left[\left\langle\mathcal{P}_{t_n}(x_n,x_{-n})Te^{-i\lambda\int_{t_{n-1}}^{t_n}\mathcal{G}_\tau(F_+(\tau),F_-(\tau))d\tau}\cdots\mathcal{P}_{t_1}(x_1,x_{-1})Te^{-i\lambda\int_0^{t_1}\mathcal{G}_\tau(F_+(\tau),F_-(\tau))d\tau}\right\rangle\hat{\rho}_o\right].
\end{aligned}
\tag{50}
$$

Where we have again recognized the definition of $Q_F^{(n)}$ in the term traced over S. The final step is to use the identity (46),

$$
\begin{aligned}
&Q^{(n)}_{X_o}(\boldsymbol{x}_n,\boldsymbol{x}_{-n},\boldsymbol{t}_n)\\
&\quad=\mathrm{tr}\Big[\Big\langle\big(\hat{P}_o(x_n,t_n)\bullet\hat{P}_o(x_{-n},t_n)\big)\big(\hat{V}_o[F_+](t_n,t_{n-1})\bullet\hat{V}_o[F_-](t_{n-1},t_n)\big)\cdots\\
&\qquad\cdots\big(\hat{P}_o(x_1,t_1)\bullet\hat{P}_o(x_{-1},t_1)\big)\big(\hat{V}_o[F_+](t_1,0)\bullet\hat{V}_o[F_-](0,t_1)\big)\Big\rangle\hat{\rho}_o\Big]\\
&\quad=\mathrm{tr}\left\langle\left(\prod_{i=n}^{1}\hat{P}_o(x_i,t_i)\hat{V}_o[F_+](t_i,t_{i-1})\right)\hat{\rho}_o\left(\prod_{i=1}^{n}\hat{V}_o[F_-](t_{i-1},t_i)\hat{P}_o(x_{-i},t_i)\right)\right\rangle\\
&\quad=\mathrm{tr}\left\langle\left(\prod_{i=n}^{1}\hat{V}_o[F_+](0,t_i)\hat{P}_o(x_i,t_i)\hat{V}_o[F_+](t_i,0)\right)\hat{\rho}_o\left(\prod_{i=n}^{1}\hat{V}_o[F_-](0,t_i)\hat{P}_o(x_{-i},t_i)\hat{V}_o[F_-](t_i,0)\right)\right\rangle.
\end{aligned}
\tag{51}
$$

B Examples of q-probabilities

B.1 Quasi-static observable

As the first example consider the case when

$$
[\hat{F},\hat{H}]=0,
\tag{52}
$$

so that the Heisenberg picture of the observable is static, $\hat{F}(t)=\hat{F}$ [and thus, $\hat{P}(f,s)=\hat{P}(f)$]. Then, the q-probabilities have a very simple form,

$$
\begin{aligned}
Q_F^{(n)}(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n)&=\mathrm{tr}\left[\left(\prod_{i=n}^{1}\hat{P}(f_i,t_i)\right)\hat{\rho}\left(\prod_{i=1}^{n}\hat{P}(f_{-i},t_i)\right)\right]=\mathrm{tr}\left[\left(\prod_{i=n}^{1}\hat{P}(f_i)\right)\hat{\rho}\left(\prod_{i=1}^{n}\hat{P}(f_{-i})\right)\right]\\
&=\delta_{f_n,f_{-n}}\left(\prod_{i=1}^{n-1}\delta_{f_n,f_i}\delta_{f_{-n},f_{-i}}\right)\mathrm{tr}\left[\hat{P}(f_n)\hat{\rho}\right]=\delta_{\boldsymbol{f}_n,\boldsymbol{f}_{-n}}\left(\prod_{i=1}^{n-1}\delta_{f_n,f_i}\right)\mathrm{tr}\left[\hat{P}(f_n)\hat{\rho}\right].
\end{aligned}
\tag{53}
$$

As a result, the interference terms Φ_n vanish automatically, and thus, the q-stochastic process $(F_+(t),F_-(t))$ is reduced to a single-component time-independent stochastic variable F with probability distribution $P_F(f)=\mathrm{tr}[\hat{P}(f)\hat{\rho}]$. Therefore, the system satisfies the surrogate field (SF) condition (27) with a constant surrogate field.

B.2 Pointer observable

Consider a system S composed of two interacting subsystems, A and B,

$$
\hat{H}=\hat{H}_{ab}=\hat{H}_a\otimes\hat{1}+\hat{1}\otimes\hat{H}_b+\mu\hat{G}_a\otimes\hat{G}_b;\quad\hat{\rho}=\hat{\rho}_a\otimes\hat{\rho}_b,
\tag{54}
$$

but the observable F belongs only to A,

$$\hat{F} = \sum_f f\hat{P}_a(f)\otimes\hat{1}. \tag{55}$$

In words: A can be treated as a non-classical observer of $\hat{G}_b$, or, equivalently, B can be thought of as an environment to the open system A. In that case, we can use the result of appendix (A) to express the q-probabilities associated with $\hat{F}$ in terms of q-average over $Q_{G_b}[g_+,g_-]$—q-probabilities associated with observable $\hat{G}_b$ of subsystem B,

$$Q_F^{(n)}(\boldsymbol{f}_n,\boldsymbol{f}_{-n},\boldsymbol{t}_n) = \mathrm{tr}_a\left[\left\langle\prod_{i=n}^{1}\mathcal{P}_{t_i}(f_i,f_{-i})\Omega[G_+,G_-](t_i,t_{i-1})\right\rangle\hat{\rho}_a\right], \tag{56}$$

where $t_0=0$, the symbol $\prod_{i=n}^{1}\Lambda_i$ indicates an ordered composition of super-operators, $\Lambda_n\cdots\Lambda_1$, and

$$\mathcal{P}_t(f,\bar{f}) = \hat{P}_a(f,t)\bullet\hat{P}_a(\bar{f},t) = e^{it\hat{H}_a}\hat{P}_a(f)e^{-it\hat{H}_a}\bullet e^{it\hat{H}_a}\hat{P}_a(\bar{f})e^{-it\hat{H}_a}; \tag{57a}$$

$$\Omega[g_+,g_-](t,s) = Te^{-i\mu\int_s^t\mathcal{G}_\tau(g_+(\tau),g_-(\tau))d\tau}; \tag{57b}$$

$$\mathcal{G}_\tau(g_+,g_-) = g_+e^{i\tau\hat{H}_a}\hat{G}_ae^{-i\tau\hat{H}_a}\bullet - \bullet e^{i\tau\hat{H}_a}\hat{G}_ae^{-i\tau\hat{H}_a}g_-. \tag{57c}$$

Essentially, Eq. (56) is of the same form as Eq. (50) from appendix A.

Next, assume (i) the stationary initial state in B, $[\hat{\rho}_b,\hat{H}_b]=0$, and (ii) the weak AB coupling regime,

$$\mu\tau_b \ll 1, \tag{58}$$

where τ_b is the *correlation time* in B—the time scale on which the components of q-stochastic process $G_\pm(t)$ decorrelate; in more formal terms, if τ_b is finite, then it is defined as such a length of time that

$$\begin{array}{ccc}\langle\cdots G_+(t+\tau)G_+(t)\cdots\rangle & & \langle\cdots G_+(t+\tau)\rangle\langle G_+(t)\cdots\rangle\\ \langle\cdots G_+(t+\tau)G_-(t)\cdots\rangle & \xrightarrow{\tau\gg\tau_b} & \langle\cdots G_+(t+\tau)\rangle\langle G_-(t)\cdots\rangle\\ \langle\cdots G_-(t+\tau)G_+(t)\cdots\rangle & & \langle\cdots G_-(t+\tau)\rangle\langle G_+(t)\cdots\rangle\\ \langle\cdots G_-(t+\tau)G_-(t)\cdots\rangle & & \langle\cdots G_-(t+\tau)\rangle\langle G_-(t)\cdots\rangle\end{array},$$

i.e., when the distance between consecutive time arguments in a moment is larger than τ_b, the moment factorizes.

One of the effects of weak coupling is the decorrelation of propagators Ω in between the consecutive projectors $\mathcal{P}$,

$$\mathrm{tr}_a\left[\left\langle\prod_{i=n}^{1}\mathcal{P}_{t_i}(f_i,f_{-i})\Omega[G_+,G_-](t_i,t_{i-1})\right\rangle\hat{\rho}_a\right] \approx \mathrm{tr}_a\left[\left(\prod_{i=n}^{1}\mathcal{P}_{t_i}(f_i,f_{-i})\left\langle\Omega[G_+,G_-](t_i,t_{i-1})\right\rangle\right)\hat{\rho}_a\right]. \tag{59}$$

Indeed, if $t_i-t_{i-1}\sim\tau_b$, then $\Omega[G_+,G_-](t_i,t_{i-1})\approx\bullet$ (the super-operator identity) because $\mu(t_i-t_{i-1})\ll 1$, therefore, a non-negligible contribution comes only from propagators for which $t_i-t_{i-1}\gg\tau_b$. But this means that there is enough distance between any two non-negligible Ω's for $(G_+(t),G_-(t))$ to decorrelate, and so, the q-average factorizes as in Eq. (59).

The other effect is the significant simplification of propagators. Each q-averaged evolution segment in Eq. (63),

$$\langle\Omega[G_+,G_-](t_i,t_{i-1})\rangle = \left\langle Te^{-i\mu\int_{t_{i-1}}^{t_i}\mathcal{G}_\tau(G_+(\tau),G_-(\tau))d\tau}\right\rangle \tag{60}$$

formally resembles the dynamical map (42) that was 'shifted' in time from interval $(t,0)$ to (t_i,t_{i-1}). The weak coupling approximation to the dynamical map (also know as the Born-Markov approximation or Davies approximation) is one of the fundamental results of the theory of open systems (see [17, 23] for the derivation carried out in the language of q-probabilities). The formal resemblance to dynamical map allows us to apply the weak coupling/Born-Markov/Davies approximation to the non-negligible q-averaged propagators (60); as a result we obtain a map belonging to a semi-group (for simplicity we are setting here $\langle G_\pm(t)\rangle = 0$),

$$\langle \Omega[G_+, G_-](t_i, t_{i-1})\rangle \xrightarrow{\mu\tau_b \ll 1} e^{\mu^2 (t_i - t_{i-1})\mathcal{L}}, \tag{61}$$

generated by the GKLS form super-operator,

$$\mathcal{L} = -i\Big[\sum_\omega \mathrm{Im}\{\gamma_\omega\}\hat{G}_\omega^\dagger \hat{G}_\omega, \bullet\Big] + \sum_\omega 2\,\mathrm{Re}\{\gamma_\omega\}\left(\hat{G}_\omega \bullet \hat{G}_\omega^\dagger - \frac{1}{2}\{\hat{G}_\omega^\dagger \hat{G}_\omega, \bullet\}\right); \tag{62a}$$

$$\gamma_\omega = \int_0^\infty e^{-i\omega s}\langle G_+(s)G_+(0)\rangle ds \quad (\text{ and thus, } \mathrm{Re}\{\gamma_\omega\} \geqslant 0 \text{ }); \tag{62b}$$

$$\hat{G}_\omega = \sum_{\alpha,\alpha'} \delta(\omega - \epsilon_\alpha + \epsilon_{\alpha'})|\alpha\rangle\langle\alpha|\hat{G}_a|\alpha'\rangle\langle\alpha'| \quad (\text{ where } \hat{H}_a|\alpha\rangle = \epsilon_\alpha|\alpha\rangle \text{ }), \tag{62c}$$

which guarantees that the approximated map is completely positive. Plugging this form of the propagator back into Eq. (59) we arrive at the *quantum regression formula* (QRF) [24, 25],

$$\begin{aligned} Q_F^{(n)}(\boldsymbol{f}_n, \boldsymbol{f}_{-n}, \boldsymbol{t}_n) &\xrightarrow{\mu\tau_b \ll 1} \mathrm{tr}_a\left[\left(\prod_{i=n}^{1} \mathcal{P}_{t_i}(f_i, f_{-i}) e^{\mu^2(t_i - t_{i-1})\mathcal{L}}\right)\hat{\rho}_a\right] \\ &\equiv \mathrm{tr}_a\left[\left(\prod_{i=n}^{1}\mathcal{P}(f_i, f_{-i})\Lambda(t_i - t_{i-1})\right)\hat{\rho}_a\right], \end{aligned} \tag{63}$$

where $\mathcal{P}(f_+, f_-) = \hat{P}_a(f_+) \bullet \hat{P}_a(f_-) = \mathcal{P}_0(f_+, f_-)$ and

$$\begin{aligned} \Lambda(t-s) &= e^{(t-s)(-i[\hat{H}_a,\bullet] + \mu^2\mathcal{L})} \\ &= \left(e^{-it\hat{H}_a}\bullet e^{it\hat{H}_a}\right)e^{\mu^2(t-s)\mathcal{L}}\left(e^{is\hat{H}_a}\bullet e^{-is\hat{H}_a}\right) = \left(e^{-i(t-s)\hat{H}_a}\bullet e^{i(t-s)\hat{H}_a}\right)e^{\mu^2(t-s)\mathcal{L}}. \end{aligned} \tag{64}$$

The QRF form (63) does not satisfy the surrogate field (SF) condition (27) by default, but it does open new options for suppressing the interference terms. One such option is to set the map, or rather, its generator $-i[\hat{H}_a, \bullet] + \mu^2\mathcal{L}$, to be a 'lower block triangular', i.e.,

$$\forall (f, f_+ \neq f_-): \ \mathcal{P}(f, f)\big(-i[\hat{H}_a, \bullet] + \mu^2\mathcal{L}\big)\mathcal{P}(f_+, f_-) = 0, \tag{65}$$

which, of course, implies $\mathcal{P}(f, f)\Lambda(t)\mathcal{P}(f_+, f_-) = 0$ for all $t \geqslant 0$. Now, since each $\Phi_n(\boldsymbol{f}_n, \boldsymbol{f}_{-n}, \boldsymbol{t}_n)$ has to involve at least one case of such a super-operator composition (note that $f_{-n} = f_n$ is always true), we conclude that all interference terms vanish individually.

Another option is to have the 'upper block triangular' generator together with the block diagonal initial density matrix, i.e.,

$$\forall (f, f_+ \neq f_-): \ \mathcal{P}(f_+, f_-)\big(-i[\hat{H}_a, \bullet] + \mu^2\mathcal{L}\big)\mathcal{P}(f, f) = 0 \quad \text{and} \quad \sum_f \mathcal{P}(f, f)\hat{\rho}_a = \hat{\rho}_a\,. \tag{66}$$

Such a generator (and thus, the map) on its own would not guarantee that all interference terms are automatically zero. If there were no constraints placed on the density matrix, it would be possible to pass from non-block diagonal $\hat{\rho}_a$ to $\mathcal{P}(f_1, f_{-1})$ with impunity. Hence, the assumption $\sum_f \mathcal{P}(f, f)\hat{\rho}_a = \hat{\rho}_a$ is needed to close this loophole.

A simple example of a system that satisfies (65) can be found for a two-level A with

$$\hat{F} = \frac{1}{2}\hat{\sigma}_z \otimes \hat{1} = \sum_{f=\pm 1/2} f|f\rangle\langle f| \otimes \hat{1}; \quad \Lambda(t) = e^{-\frac{1}{2}\gamma t[\hat{\sigma}_x,[\hat{\sigma}_x,\bullet]]}, \tag{67}$$

resulting in a surrogate field $F(t)$ in the form of random telegraph noise [16, 17]—a stochastic process that switches randomly between two values ($\pm 1/2$ in this case) with the rate γ.

Previous papers investigating Kolmogorovian consistency in the context of sequential measurements [1, 2, 3] either assume QRF is in effect, or consider the case when P_n can be parameterized with QRF without establishing the relation between actual dynamics and maps appearing in the formula. Therefore, us having arrived at QRF form (63) as a result of weak coupling approximation gives us an opportunity to compare our results with the conclusions drawn in those previous works. In particular, QRF allows us to reformulate the consistent measurements (CM) condition (26) as a condition placed on the map (this mimics the approach of [1, 3]),

$$\begin{aligned}
&\forall(n \geqslant 1)\forall(0 < t_1 < \cdots < t_n)\forall(1 \leqslant i \leqslant n)\forall(f_1, \ldots, \not{f_i}, \ldots, f_n) : \\
&\left(\prod_{j\neq i} \delta_{f_j,f_{-j}}\right) \sum_{f_i \neq f_{-i}} \Phi_n(\boldsymbol{f}_n, \boldsymbol{f}_{-n}, \boldsymbol{t}_n) \\
&\quad = \mathrm{tr}_a \left[\cdots \mathcal{P}(f_{i+1}, f_{i+1})\Lambda(t_{i+1} - t_i) \left(\sum_{f_i \neq f_{-i}} \mathcal{P}(f_i, f_{-i})\right) \Lambda(t_i - t_{i-1})\mathcal{P}(f_{i-1}, f_{i-1}) \cdots\right] \\
&\quad = \mathrm{tr}_a \left[\cdots \mathcal{P}(f_{i+1}, f_{i+1})\Lambda(t_{i+1} - t_i) (\bullet - \Delta) \Lambda(t_i - t_{i-1})\mathcal{P}(f_{i-1}, f_{i-1}) \cdots\right] \\
&\quad = 0,
\end{aligned} \tag{68}$$

where $\Delta = \sum_f \mathcal{P}(f, f)$. If the density matrix is block diagonal, $\Delta\hat{\rho}_a = \hat{\rho}_a$, then the condition is met when the map satisfies

$$\forall(f, f')\forall(t > t' > 0) : \mathcal{P}(f, f)\Lambda(t)\mathcal{P}(f', f') = \mathcal{P}(f, f)\Lambda(t - t')\Delta\Lambda(t')\mathcal{P}(f', f'). \tag{69}$$

In comparison, it is shown in [1, 3] that the Born distributions parameterized with QRF are consistent when (i) the density matrix is block-diagonal, and (ii) the map satisfies the 'non coherence generating and detecting' (NCGD) condition,

$$\forall(t > t' > 0) : \Delta\Lambda(t)\Delta = \Delta\Lambda(t - t')\Delta\Lambda(t')\Delta. \tag{70}$$

Interestingly, the NCGD and CM conditions are, in this case, equivalent. Indeed, on the one hand, when (69) is satisfied it clearly implies NCGD. On the other hand, since $\mathcal{P}(f, f) = \Delta\mathcal{P}(f, f) = \mathcal{P}(f, f)\Delta$, assuming NCGD gives us

$$\begin{aligned}
\mathcal{P}(f, f)\Lambda(t)\mathcal{P}(f', f') &= \mathcal{P}(f, f)\Delta\Lambda(t)\Delta\mathcal{P}(f', f') = \mathcal{P}(f, f)\Delta\Lambda(t - t')\Delta\Lambda(t')\Delta\mathcal{P}(f', f') \\
&= \mathcal{P}(f, f)\Lambda(t - t')\Delta\Lambda(t')\mathcal{P}(f', f').
\end{aligned} \tag{71}$$

Finally, we find that the two types of surrogate field-satisfying maps we have listed above can be identified as examples of NCGD map subtypes. According to the classification introduced in [1], the lower block triangular map (65) belongs with the *coherence non-activating* maps that satisfy $\Delta\Lambda(t)\Delta = \Delta\Lambda(t)$, and the upper block triangular map (66) is one of the *coherence non-generating* maps that satisfy $\Delta\Lambda(t)\Delta = \Lambda(t)\Delta$.

B.3 Macro-scale action

Consider a system living in a Hilbert space with a continuous basis, $\{|x\rangle\}_{x=-\infty}^{\infty}$, so that the projectors onto eigenspaces of the observable $\hat{F}$ can be written as

$$\hat{P}(f) = \int_{\Gamma(f)} |x\rangle\langle x| dx, \tag{72}$$

where $\Gamma(f)$ is an interval of a real line corresponding to the eigenvalue f. Then, we can rewrite the q-probability associated with $\hat{F}$ in the terms of *probability amplitudes* $\phi_t(x|x') = \langle x|\exp(-it\hat{H})|x'\rangle$,

$$\begin{aligned}
Q_F^{(n)}(\boldsymbol{f}_n, \boldsymbol{f}_{-n}, \boldsymbol{t}_n) = &\left(\prod_{i=1}^{n} \int_{\Gamma(f_i)} dx_i \int_{\Gamma(f_{-i})} dx_{-i}\right) \int_{-\infty}^{\infty} dx_0 dx_{-0}\, \delta(x_n - x_{-n}) \\
&\times \langle x_0|\hat{\rho}|x_{-0}\rangle \left(\prod_{i=0}^{n-1} \phi_{t_{i+1}-t_i}(x_{i+1}|x_i)\, \phi_{t_{i+1}-t_i}(x_{-(i+1)}|x_{-i})^*\right),
\end{aligned} \tag{73}$$

where $t_0 = 0$.

We now assume that the system can be assigned with a macroscopic-scale *action* $S[q] \gg \hbar$, so that the amplitudes can be reformulated as Feynman path integrals [26] with the stationary phase approximation in effect,

$$\phi_{t-t'}(x|x') = \int_{q(t')=x'}^{q(t)=x} e^{\frac{i}{\hbar}S[q]}[Dq] \propto e^{\frac{i}{\hbar}S_{\rm cl}(x,t|x',t')}. \tag{74}$$

Here, $S_{\rm cl}(x,t|x',t') = S[q_{\rm cl}]$ is the action minimized by the path $q_{\rm cl}(t)$ that solves the classical equations of motion (with boundary conditions $q_{\rm cl}(t) = x$ and $q_{\rm cl}(t') = x'$) corresponding to the *least action principle* of the classical mechanics. In other words, $q_{\rm cl}(t)$ is the extremum of $S[q]$, and thus, in its vicinity the variation of the phase factor in Eq. (74) slows down considerably allowing for a brief window of constructive interference; this is the basic explanation why the stationary phase approximation could have been employed here. In the q-probability (73), aside the individual probability amplitudes themselves, the interference effects due to macro-scale values of the action also kick in at the point of integration over each interval $\Gamma(f_{\pm i})$, e.g.,

$$\int_{\Gamma(f_i)} e^{\frac{i}{\hbar}S_{\rm cl}(x_{i+1},t_{i+1}|x_i,t_i)+\frac{i}{\hbar}S_{\rm cl}(x_i,t_i|x_{i-1},t_{i-1})}dx_i. \tag{75}$$

As a result, such an integral vanishes due to destructive interference unless the interval contains a stationary point of the phase, $x_i^{\rm st} \in \Gamma(f_i)$. Of course, the point counts as stationary when

$$0 = \frac{\partial S_{\rm cl}(x_{i+1},t_{i+1}|x_i,t_i)}{\partial x_i}\Big|_{x_i=x_i^{\rm st}} + \frac{\partial S_{\rm cl}(x_i,t_i|x_{i-1},t_{i-1})]}{\partial x_i}\Big|_{x_i=x_i^{\rm st}} = -p^{t_i}_{i\to i+1} + p^{t_i}_{i-1\to i}, \tag{76}$$

where $p^{t_i}_{i\to i+1}$ ($p^{t_i}_{i-1\to i}$) is the initial (terminal) momentum of the path $i \to i+1$ ($i-1 \to i$) that starts at x_i, t_i (x_{i-1}, t_{i-1}) and ends at x_{i+1}, t_{i+1} (x_i, t_i). Therefore, the phase is stationary only if there is no jump in the momentum when switching from one path segment to the next. However, even though the segments end/start at the same point, they are otherwise independent, and thus, there is no reason why there should be no momentum discontinuity for an arbitrary value of x_i. The one case when the momentum would be continuous at every point is when x_i happens to be a part of the classical path $i-1 \to i+1$, i.e., there is no jump when the two paths $i-1 \to i$ and $i \to i+1$ are actually parts of a single path going directly from x_{i-1}, t_{i-1} to x_{i+1}, t_{i+1}. Applying the same argument to each integral we conclude that the q-probability survives the destructive interference only when the arguments $\boldsymbol{f}_n$ are chosen in such a way that the corresponding intervals $\Gamma(f_n), \ldots, \Gamma(f_1)$ intersect the path $q_{\rm cl}(t)$ that solves the classical equations of motion with the boundary conditions $q_{\rm cl}(0) = x_0$ and $q_{\rm cl}(t_n) = x_n$. Of course, the same goes for $\boldsymbol{f}_{-n}$ and the classical path with the boundary conditions $q_{\rm cl}(0) = x_{-0}$ and $q_{\rm cl}(t_n) = x_{-n}$. If the initial density matrix is diagonal, $\langle x_0|\hat\rho|x_{-0}\rangle = \delta(x_0 - x_{-0})\rho(x_0)$, then, it follows, that $Q_F^{(n)}$ is non-zero only when $\boldsymbol{f}_{-n} = \boldsymbol{f}_n$ because $x_{-n} = x_n$ by default and when also $x_{-0} = x_0$ the classical paths passing through $\Gamma(f_i)$'s and $\Gamma(f_{-i})$'s are overlapping. As a result, $Q_F^{(n)}(\boldsymbol{f}_n, \boldsymbol{f}_{-n}, \boldsymbol{t}_n) = \delta_{\boldsymbol{f}_n,\boldsymbol{f}_{-n}} P_n(\boldsymbol{f}_n, \boldsymbol{t}_n)$, so that $\Phi_n = 0$, and the surrogate field (SF) condition (27) is satisfied.

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