

Objectivity of strain measures in the geometrically exact three-dimensional beam theory and its finite-element implementation

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Received 31 October 1997; revised 14 July 1998; accepted 17 August 1998

The paper discusses the issue of discretization of the strain–configuration relationships in the geometrically exact theory of three-dimensional (3D) beams, which has been at the heart of most recent nonlinear finite-element formulations. It is demonstrated that the usual discretization procedures for implementing these strain measures within a finite-element framework violate the important property of objectivity: invariance to rigid-body rotations. A method is proposed for overcoming this limitation, which paves the way for an objective finite-element formulation of the geometrically exact 3D beam theory. For a two-noded element, this method involves obtaining the relative rotation matrix that rotates one nodal triad onto the other and then interpolating the resulting rotation vector.

Keywords: nonlinear beam theory; large (3D) rotations; strain invariance; finite-element method; interpolation of rotations

1. Notation

s, L	arc-length parameter of the centroid axis, initial length of the beam
\mathcal{R}, \mathcal{N}	sets of real numbers and positive integers
$SO(3)$	three-parametric Lie group of 3D rotations
δ_{ij}, e_{ijk}	Kronecker symbol and permutation symbol
\mathbf{E}_i	orthonormal base vectors of inertial coordinate system ($i = 1, 2, 3$)
\mathbf{G}_i	orthonormal base vectors attached at cross-sections
N, I_i	number of nodes on finite element, shape functions of order $N - 1$
$(\bullet), (\bullet)'$	quantity (\bullet) acted upon by a constant motion and $d(\bullet)/ds$
$\widehat{(\bullet)}, (\bullet)^h$	cross-product operator $(\bullet) \times$ and approximated quantity (\bullet)
$(\bullet)_R, (\bullet)_0$	constant quantity (\bullet) , initial quantity (\bullet)
$(\bullet)_a, (\bullet)_d$	average (mean) and difference for two given quantities
\mathcal{C}	configuration of the beam
\mathbf{r}, \mathbf{A}	position vector of centroid axis and rotation matrix from \mathbf{E}_i to \mathbf{G}_i
$\boldsymbol{\psi}, \psi$	total rotational vector and its norm
$\boldsymbol{\gamma}, \boldsymbol{\kappa}$	vectors of translational and rotational strain measures
I, J, c	fixed parameters in objective formulations

\mathbf{A}_r, ϕ_{IJ}	reference triad and relative local rotation in objective formulations
Ψ^l, Θ^l	total and incremental local rotation in objective formulations

2. Introduction

Many engineering structures can be modelled with sufficient accuracy using beam models, and extensive work on the basic theory has spanned almost three centuries. Most of the proposed theories assume the Bernoulli hypothesis of the cross-sections remaining undeformed in the deformed state of the beam and differ in the way in which the relationships between the displacements and rotations and corresponding strain measures are defined. Thus the ‘Euler’s elastica’ (Euler 1744) denotes a planar beam with no axial and shear strains and with the bending stress-couple being proportional to the curvature, while the ‘Kirchhoff’s beam’ (Kirchhoff 1859) is a spatial generalization of Euler’s elastica with the torsional stress-couple being proportional to the torsional curvature. When some small axial straining is added, we talk about the ‘Kirchhoff–Love beam’ theory (Love 1944). Due to the absence of shear strains at centroids of cross-sections, in all of these theories the cross-sections always remain orthogonal to the centroidal axis of the beam.

In contrast, the presence of shear strains introduces a change of the angle between a cross-section and the centroidal axis, so that the deformed configuration of a beam is defined by (i) the deformed centroidal axis and (ii) the set of orientations of cross-sections with respect to their undeformed positions. Such a beam, in which the fields of lateral displacements and rotations are considered as independent, is generally referred to as the ‘Timoshenko beam’.

Some finite-element procedures start with the continuum equilibrium equations (Bathe & Bolourchi 1979; Dvorkin *et al.* 1988) and are known as degenerate-continuum formulations. In contrast, in the present work, we will concentrate on models that are derived directly from the resultant forms of the differential equations of equilibrium. In the present context, the equivalent strain–configuration relationships involve three ‘direct strains’ (including two shear strains) and three ‘curvatures’. These strain measures are considered to stem from a ‘geometrically exact beam theory’ when the relationships between the configuration and the strain measures are consistent with the virtual work principle and the equilibrium equations at a deformed state *regardless of the magnitude of displacements, rotations and strains*. The theory is often referred to as the ‘geometrically exact *finite-strain* beam theory’ (Reissner 1972, 1973, 1981; Simo 1985; Simo & Vu-Quoc 1986; Jelenić & Saje 1995). However, for a genuine finite-strain formulation, we should also adopt a consistent constitutive model and it is not immediately obvious how this could be implemented in conjunction with a beam theory based on the non-deformability of cross-sections. Indeed, none of the previous papers on the finite-element implementation of the geometrically exact beam theory (Simo 1985; Simo & Vu-Quoc 1986; Cardona & Gérardin 1988; Ibrahimbegović *et al.* 1995; Jelenić & Saje 1995) has considered a genuine finite-strain constitutive model. For this reason, in the following developments we will drop the term ‘finite strain’ from the name of the theory.

The present work can be considered to stem from the strain measures defined by Simo (1985). They are sometimes referred to as belonging to ‘Reissner’s beam theory’ (Reissner 1972, 1973, 1981), although a simplification in the parametrization of the rotation matrix in space was made by Reissner (1981), which facilitated the

derivation of the rotational strain measures, but also spoiled the geometric exactness of the theory. A different approach in the geometrically exact beam theory was presented by Antman (1974) and was used by Simo (1985) to propose a parametrization of the rotation matrix in space which furnished a full geometric exactness of the theory.

Among others, Simo's strain–configuration relations have been used by Simo & Vu-Quoc (1986), Cardona & Gérardin (1988), Ibrahimbegović *et al.* (1995), Jelenić & Saje (1995) and many others to provide most of the modern new finite-element formulations for static analysis of geometrically exact three-dimensional (3D) beams. Finite-element developers have followed the conventional approach for the displacements and have directly interpolated the rotation parameters or their incremental/iterative changes. In so doing, they have ignored the fact that rotational vectors are physically non-additive quantities. In the present paper, we will show that this approach leads to a non-objectivity of the *interpolated* strain measures, even though the strain measures themselves are objective (Simo 1985; Ibrahimbegović 1994; a more general discussion of the issue of frame-indifference in theories arising from degenerating 3D continuum theories was given by Ciarlet (1997)). We will also describe a new interpolation procedure which is *specifically designed* to preserve this objectivity. This procedure should pave the way for an objective finite-element formulation using the geometrically exact 3D beam theory (Jelenić & Crisfield 1999). Because beams are now being used to model highly complex problems with phenomena such as ‘localized writhing’ (Thompson & Champneys 1996), we believe that it is important to develop finite-element formulations that are built on such a firm basis. It is worth noting that, as pointed out by McRobie & Lasenby (1999), the geometrically exact 3D beam theory as given by Simo & Vu-Quoc (1986) may be recast by using geometric (Clifford) algebra notions of bivectors and rotors rather than the rotational pseudo-vectors and rotation matrices. We believe, however, that the problems of interpolation addressed in this paper also extend to the Clifford algebra approach.

The parametrization of the rotation matrix using the rotational (pseudo)-vector has proved to be particularly convenient in dynamic analyses (see Simo & Vu-Quoc 1988; Cardona & Gérardin 1988; Simo *et al.* 1995; Jelenić & Crisfield 1998, 1999), where the definition of the kinetic energy is easily performed using the definition of the beam kinematics with respect to the inertial frame. Recent work on dynamics of 3D beams has concentrated on energy/momentum conserving algorithms aimed at ensuring numerical stability (Simo *et al.* 1995). To construct these algorithms, it is found to be necessary to interpolate the *incremental* (tangent-scaled) rotations (Simo *et al.* 1995). Again, the standard approach leads to non-objectivity (which is even more severe due to the tangent-scaling) and hence, in the present paper, we also outline a new *incremental* interpolation procedure that should provide the basis for a future work on deriving objective energy/momentum conserving algorithms.

3. Strain measures in the geometrically exact 3D beam theory

(a) *Basic kinematics. Initial and deformed configurations*

For a given parameter $s \in [0, L] \subset \mathcal{R}$, $L \in \mathcal{R}$, we define an initial (undeformed) configuration of the beam centroid axis by a space curve $s \rightarrow \mathbf{r}_0(s) \in \mathcal{R}^3$ in a

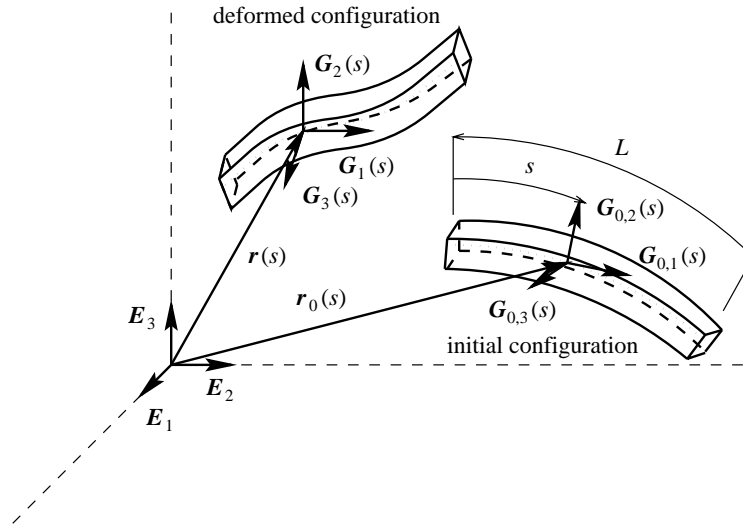


Figure 1. Kinematics of initial and deformed configurations of a curved beam.

three-dimensional ambient space \mathcal{R}^3 with a right-handed inertial Cartesian frame,

$$\mathbf{E}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{E}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \quad \mathbf{E}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \quad (3.1)$$

The parameter s is usually referred to as the arc-length of the initial centroid axis of the beam and L as the initial length of the beam (see figure 1). Cross-sections of the beam in its initial configuration are defined by a right-handed orthonormal triad of base vectors $s \rightarrow \mathbf{G}_{0,1}(s), \mathbf{G}_{0,2}(s), \mathbf{G}_{0,3}(s) \in \mathcal{R}^3 \mid \mathbf{G}_{0,i}\mathbf{G}_{0,j} = \delta_{ij}, \mathbf{G}_{0,k} = e_{ijk}\mathbf{G}_{0,i}\mathbf{G}_{0,j}$ with the base vector $\mathbf{G}_{0,1}(s)$ being directed along the initial centroid axis so that

$$\mathbf{G}_{0,1}(s) = \mathbf{r}'_0(s), \quad (3.2)$$

and with the base vectors $\mathbf{G}_{0,2}(s)$ and $\mathbf{G}_{0,3}(s)$ being directed along the principal axes of inertia of the cross-section at s . Also, δ_{ij} and e_{ijk} are the Kronecker and the permutation symbols and repeated indices are assumed as summational over the dimension of the space \mathcal{R}^3 . In (3.2), the prime ($'$) denotes a derivative with respect to arc-length parameter s .

The orthonormal bases $\mathbf{G}_{0,1}, \mathbf{G}_{0,2}, \mathbf{G}_{0,3}$ and $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ are related through a linear transformation $s \rightarrow \mathbf{A}_0(s) \in SO(3)$ as $\mathbf{G}_{0,i}(s) = \mathbf{A}_0(s)\mathbf{E}_i, i = 1, 2, 3$, where $SO(3)$ is the three-parametric Lie group of proper orthogonal transformations satisfying $\det \mathbf{A}(s) = 1$ and $\mathbf{A}(s)\mathbf{A}^t(s) = \mathbf{I} \forall s \in [0, L]$, with \mathbf{I} being a 3×3 unit matrix. Obviously, the initial position vector of the centroid axis of the beam and the orientation of the orthonormal frame attached to the cross-section at s fully define the initial configuration of the beam $s \rightarrow \mathcal{C}_0 = (\mathbf{r}_0, \mathbf{A}_0) \in \mathcal{R}^3 \times SO(3)$.

In a similar manner, we define the deformed configuration of the beam centroid axis by a space curve $s \rightarrow \mathbf{r}(s) \in \mathcal{R}^3$ and the cross-sections of the beam in the deformed configuration by a right-handed orthonormal triad of base vectors $s \rightarrow \mathbf{G}_1(s), \mathbf{G}_2(s), \mathbf{G}_3(s) \in \mathcal{R}^3 \mid \mathbf{G}_i\mathbf{G}_j = \delta_{ij}, \mathbf{G}_k = e_{ijk}\mathbf{G}_i\mathbf{G}_j$.

In contrast to the initial configuration, the base vector $\mathbf{G}_1(s)$ need not be directed along the deformed centroid axis so that such a beam model is capable of representing the shear deformation. See Reissner (1972, 1973, 1981), Antman (1974) and Simo (1985) for details. The base vectors $\mathbf{G}_2(s)$ and $\mathbf{G}_3(s)$ are still directed along the principal axes of inertia of the cross-section at s and the orthonormal bases $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ and $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ are related through a linear transformation $s \rightarrow \mathbf{A}(s) \in SO(3)$ as

$$\mathbf{G}_i(s) = \mathbf{A}(s)\mathbf{E}_i, \quad i = 1, 2, 3. \quad (3.3)$$

The position vector of the deformed centroid axis and the orientation of the orthonormal frame \mathbf{A} attached to the cross-section at s fully define the deformed configuration $s \rightarrow \mathcal{C} = (\mathbf{r}, \mathbf{A}) \in \mathcal{R}^3 \times SO(3)$.

(b) *Strain measures and their relation to the deformed configuration*

The development of relationships between a deformed configuration of the beam and the adopted strain measures takes a prominent role in the geometrically exact beam theory. Reissner (1972, p. 795) defined the geometrically exact beam theory as one in which we use ‘a system of non-linear strain displacement relations which is consistent with exact one-dimensional equilibrium equations for forces and moments via what is considered to be an appropriate version of the principle of virtual work’. To derive such relations (which we will call the strain–*configuration* relations, rather than the strain displacement relations) ‘we (i) take as basic the differential equations of force and moment equilibrium for elements of the deformed curve. We then (ii) stipulate a form of the principle of virtual work, and (iii) use this principle so as to obtain a system of strain displacement relations, involving force strains and moment strains in association with the assumed cross-sectional forces and moments’ (Reissner 1973, p. 87; ordinal numbers in the above quote are inserted by the present authors). This three-step procedure is performed in Appendix A and leads to the strain–configuration relations of the type,

$$\boldsymbol{\gamma} = \mathbf{A}^t \mathbf{r}' - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad (3.4)$$

$$\hat{\boldsymbol{\kappa}} = \mathbf{A}^t \mathbf{A}' - \mathbf{A}_0^t \mathbf{A}'_0, \quad (3.5)$$

where $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ are (material) translational and rotational strain measures, which are energy-conjugate to the stress and stress-couple resultants acting at cross-sections as a consequence of applied loads that take the beam from the initial to the deformed configuration (see Reissner 1973, 1981; Simo 1985; Ibrahimbegović 1994; also see Appendix A for details). In (3.5) and throughout the text, the hat denotes a skew-symmetric matrix so that

$$\hat{\mathbf{v}} = -\hat{\mathbf{v}}^t = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}; \quad \mathbf{v} = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}. \quad (3.6)$$

(c) *Objectivity of strain measures in the geometrically exact 3D beam theory*

The objectivity of material strain measures at a particular configuration is understood as their inherent ability to remain unaffected by a constant motion of the

configuration. We also say that such strain measures are *invariant under a superposed rigid-body motion*. We define a configuration $\underline{\mathcal{C}} = (\underline{\mathbf{r}}, \underline{\mathbf{A}})$, obtained by superposing an arbitrary constant ‘rigid-body’ motion $\mathcal{C}_R = (\mathbf{r}_R, \mathbf{A}_R)$ onto the configuration $\mathcal{C} = (\mathbf{r}, \mathbf{A})$ as

$$\underline{\mathbf{r}} = \mathbf{A}_R(\mathbf{r}_R + \mathbf{r}), \quad (3.7)$$

$$\underline{\mathbf{A}} = \mathbf{A}_R \mathbf{A}. \quad (3.8)$$

The constant rotation may be defined as $\mathbf{A}_R = \exp \hat{\boldsymbol{\psi}}_R$ and understood as the rotation around axis $\boldsymbol{\psi}_R/\psi_R$ by the angle $\psi_R = \|\boldsymbol{\psi}_R\|$. For the sake of clarity, the axis of rotation may be viewed as passing through the origin of the inertial coordinate system with basis $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$. This is illustrated in figure 2.

The above description is intended to set the scene for the following important corollary of the geometrically exact beam theory.

Proposition 3.1. *The strain measures $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ at the configuration $\mathcal{C} = (\mathbf{r}, \mathbf{A})$, defined by (3.4) and (3.5) are (i) objective in the sense that they are equal to the strain measures $\underline{\boldsymbol{\gamma}}$ and $\underline{\boldsymbol{\kappa}}$ at the configuration $\underline{\mathcal{C}} = (\underline{\mathbf{r}}, \underline{\mathbf{A}})$, defined by (3.7) and (3.8) and (ii) independent of the history of deformation.*

The proof of this proposition is given in Appendix B a.

4. Simple finite-element implementation of the theory

In this section we attempt to illustrate that while the strain measures of the geometrically exact *theory* are objective, all of the available *finite-element implementations* of the theory are, to the best of our knowledge, non-objective. We analyse the interpolation-induced non-objectivity of the approximated strain measures and propose a basic solution to the problem. We will initially focus our attention on the analysis of the rotational strains (3.5) and choose an initially straight beam ($\mathbf{A}'_0 = \mathbf{0}$) of length L as the model.

(a) Traditional implementation (total form)

By parametrizing the rotation matrix \mathbf{A} in terms of its rotational vector $\boldsymbol{\psi}$ via (Argyris 1982)

$$\mathbf{A} = \exp \hat{\boldsymbol{\psi}} \mathbf{A}_0 \equiv \left(\mathbf{I} + \frac{\sin \psi}{\psi} \hat{\boldsymbol{\psi}} + \frac{1 - \cos \psi}{\psi^2} \hat{\boldsymbol{\psi}}^2 \right) \mathbf{A}_0; \quad \psi = \|\boldsymbol{\psi}\|,$$

the skew-symmetric matrix of rotational strains (3.5) reads (note that we have taken $\mathbf{A}'_0 = \mathbf{0}$)

$$\hat{\boldsymbol{\kappa}} = \left(\mathbf{I} + \frac{\sin \psi}{\psi} \hat{\boldsymbol{\psi}} + \frac{1 - \cos \psi}{\psi^2} \hat{\boldsymbol{\psi}}^2 \right)^t \left(\mathbf{I} + \frac{\sin \psi}{\psi} \hat{\boldsymbol{\psi}} + \frac{1 - \cos \psi}{\psi^2} \hat{\boldsymbol{\psi}}^2 \right)'. \quad (4.1)$$

After taking the indicated derivative on the second factor in (4.1) and noting the identities

$$\hat{\mathbf{a}} \hat{\mathbf{b}} \hat{\mathbf{a}} = -\mathbf{a}^t \hat{\mathbf{b}} \hat{\mathbf{a}} \quad \text{and} \quad \hat{\mathbf{a}} \hat{\mathbf{b}} \hat{\mathbf{a}}^2 = -\mathbf{a}^t \hat{\mathbf{b}} \hat{\mathbf{a}}^2$$

for any two 3D vectors \mathbf{a} and \mathbf{b} , a somewhat lengthy but otherwise straightforward algebra leads to the following skew-symmetric matrix of rotational strains

$$\hat{\boldsymbol{\kappa}} = \frac{1}{\psi^2} \left(1 - \frac{\sin \psi}{\psi} \right) \boldsymbol{\psi}^t \boldsymbol{\psi}' \hat{\boldsymbol{\psi}} + \frac{\sin \psi}{\psi} \hat{\boldsymbol{\psi}}' - \frac{1 - \cos \psi}{\psi^2} (\hat{\boldsymbol{\psi}} \hat{\boldsymbol{\psi}}' - \hat{\boldsymbol{\psi}}' \hat{\boldsymbol{\psi}}),$$

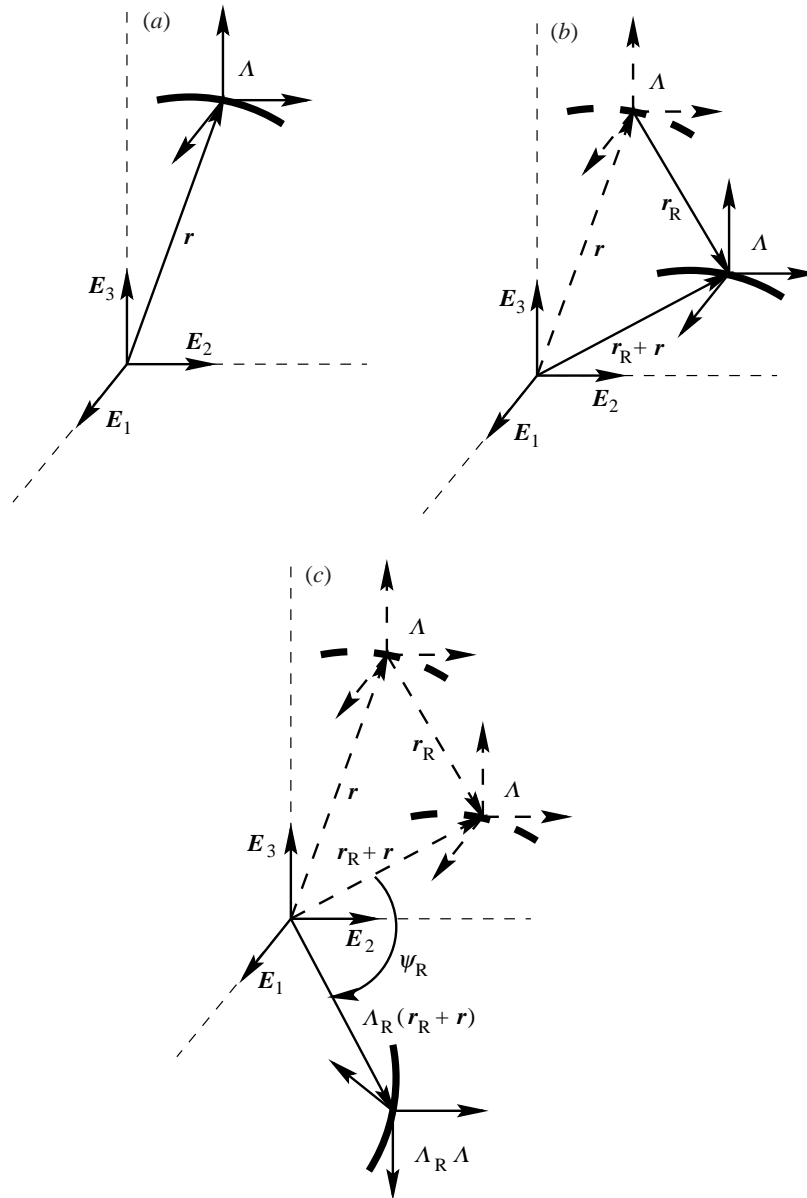


Figure 2. Rigid-body motion of the deformed configuration: (a) original deformed configuration, (b) deformed configuration after rigid-body translation, (c) deformed configuration after rigid-body translation and rotation.

with its axial vector being (note that $\hat{\psi}\hat{\psi}' - \hat{\psi}'\hat{\psi} = \widehat{\psi \times \psi'} = \widehat{\psi\psi'}$)

$$\kappa = \left[\frac{1}{\psi^2} \left(1 - \frac{\sin \psi}{\psi} \right) \psi \psi^t + \frac{\sin \psi}{\psi} \mathbf{I} - \frac{1 - \cos \psi}{\psi^2} \widehat{\psi} \right] \psi'. \quad (4.2)$$

This set-up (with the parametrization of the rotation matrix using a ‘total’ rotational vector between the initial and the current configuration) enables a straightforward

finite-element implementation by adopting a linear interpolation of the total rotational vector via

$$\boldsymbol{\psi}(s) \doteq \boldsymbol{\psi}^h(s) = \frac{L-s}{L}\boldsymbol{\psi}_1 + \frac{s}{L}\boldsymbol{\psi}_2, \quad (4.3)$$

where superscript ‘h’ here and throughout the text indicates a finite-element approximation of a particular field variable, and $\boldsymbol{\psi}_1$, $\boldsymbol{\psi}_2$ are the total rotational vectors at $s = 0$ and $s = L$, respectively. In a similar manner, we also interpolate the position of the centroid axis, so that we have

$$\mathbf{r}(s) \doteq \mathbf{r}^h(s) = \frac{L-s}{L}\mathbf{r}_1 + \frac{s}{L}\mathbf{r}_2. \quad (4.4)$$

The finite-element approximation $\boldsymbol{\kappa}^h$ of the rotational strain measure $\boldsymbol{\kappa}$ follows by substituting $\boldsymbol{\psi}$ in (4.2) with $\boldsymbol{\psi}^h$ defined by (4.3). In the standard finite-element approach, we would compute $\boldsymbol{\kappa}^h$ at the middle of the beam as

$$\boldsymbol{\kappa}^h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \frac{1}{2}L) = \frac{1}{L} \left[\frac{1}{\psi_a^2} \left(1 - \frac{\sin \psi_a}{\psi_a} \right) \boldsymbol{\psi}_a \boldsymbol{\psi}_a^t + \frac{\sin \psi_a}{\psi_a} \mathbf{I} - \frac{1 - \cos \psi_a}{\psi_a^2} \hat{\boldsymbol{\psi}}_a \right] \boldsymbol{\psi}_d, \quad (4.5)$$

where subscripts ‘a’ and ‘d’ are used to indicate ‘average’ and ‘difference’ quantities via $\boldsymbol{\psi}_a = \frac{1}{2}(\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2)$ and $\boldsymbol{\psi}_d = \boldsymbol{\psi}_2 - \boldsymbol{\psi}_1$.

This approach was presented and fully detailed by Ibrahimbegović *et al.* (1995). A somewhat different procedure was adopted by Cardona & Géradin (1988), who advocated an interpolation of the (material) *incremental* rotations (the rotations between the last converged configuration and the current deformed configuration) rather than the total rotations. The total approach is a special case of the incremental approach, where the last converged configuration coincides with the initial configuration. Indeed, it was shown in Cardona & Géradin (1988) that (4.5) is only a special case of a more general expression for rotational strains, which is also dependent on a particular reference configuration (in their case this is the last converged configuration) and on the rotational strains in the reference configuration. In a related fashion, in the first finite-element formulation of the theory, given by Simo & Vu-Quoc (1986), the ‘iterative’ rotations (i.e. those between the last known, but not necessarily converged configuration and the current configuration) were interpolated. There are a number of other differences between these formulations, including the choice of material or spatial quantities and the choice of additive or multiplicative update, but so far as the issue of interpolation is concerned, all of these formulations can be regarded as stemming from the same family, for which the following is valid:

The interpolation is applied to the rotations between a particular reference configuration and the current configuration.

With hindsight, the nature of this interpolation is bound to make all of these formulations non-objective. The rotations interpolated in this way in general include rigid-body rotations, so that the error, introduced by the interpolation, makes the resulting strain measures dependent on the rigid-body rotation. In other words, if we apply a constant rotation \mathbf{A}_R at nodes 1 and 2, so that the new, ‘rigidly rotated’, nodal triads $\underline{\mathbf{A}}_1 = \exp \hat{\underline{\boldsymbol{\psi}}}_1$ and $\underline{\mathbf{A}}_2 = \exp \hat{\underline{\boldsymbol{\psi}}}_2$ are given via (3.8), interpolate $\boldsymbol{\psi}(s)$ using (4.3) and compute the corresponding vector of rotational strains from (4.5), we realize that

$$\boldsymbol{\kappa}^h(\underline{\boldsymbol{\psi}}_1, \underline{\boldsymbol{\psi}}_2, \frac{1}{2}L) \neq \boldsymbol{\kappa}^h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \frac{1}{2}L)$$

for any genuine three-dimensional rotations. Evidence is given in Appendix C *a*. While the total formulation is not objective, it is path-independent in that the final solutions are defined by the final configuration only, and are thus independent of the precise nature of both the incremental and the iterative paths towards convergence. In contrast, the original iterative formulation of Simo & Vu-Quoc (1986) is not only non-objective but also dependent on the iterative path towards a converged solution, while the incremental formulation of Cardona & Géradin (1988) is non-objective and dependent on the history of incrementation. The path-dependence of these formulations is illustrated in Appendix C *b*. Other formulations based on the interpolation of incremental and iterative rotations, such as those due to Jelenić & Saje (1995) and Jelenić & Crisfield (1998), are also non-objective and path-dependent.

(*b*) *Proposed interpolation procedure*

The non-objectivity of the traditional approaches is eliminated if we first compute the rotation matrix that rotates the rotation matrix at node 1 onto the rotation matrix at node 2 so that $\exp \hat{\varphi}_{12} = \mathbf{A}_2 \mathbf{A}_1^t$ and then linearly interpolate the extracted rotational vector φ_{12} so that

$$\mathbf{A}(s) \doteq \mathbf{A}^h(s) = \exp\left(\frac{s}{L} \hat{\varphi}_{12}\right) \mathbf{A}_1.$$

In practice, it is more convenient to operate with a *local* rotational vector ϕ_{12} (with components given in the coordinate system defined by \mathbf{A}_1) so that

$$\exp \hat{\phi}_{12} = \mathbf{A}_1^t \exp \hat{\varphi}_{12} \mathbf{A}_1 \equiv \mathbf{A}_1^t \mathbf{A}_2, \quad (4.6)$$

while

$$\mathbf{A}(s) \doteq \mathbf{A}^h(s) = \mathbf{A}_1 \exp\left(\frac{s}{L} \hat{\phi}_{12}\right). \quad (4.7)$$

The extraction of ϕ_{12} from $\mathbf{A}_1^t \mathbf{A}_2$ can be performed using the Spurrier (1978) algorithm as given by Simo & Vu-Quoc (1986), Appendix B *a* and Crisfield (1997, § 16.10). The skew-symmetric matrix of approximated rotational strains is now obtained from (3.5) as

$$\begin{aligned} \kappa^h(s) &\equiv \mathbf{A}^{ht} \mathbf{A}^{h'} = \exp\left(\frac{s}{L} \hat{\phi}_{12}\right)^t \mathbf{A}_1^t \mathbf{A}_1 \left[\exp\left(\frac{s}{L} \hat{\phi}_{12}\right) \right]' \\ &\equiv \left(\mathbf{I} + \frac{\sin((s/L)\phi_{12})}{\phi_{12}} \hat{\phi}_{12} + \frac{1 - \cos((s/L)\phi_{12})}{\phi_{12}^2} \hat{\phi}_{12}^2 \right)^t \\ &\quad \times \left(\mathbf{I} + \frac{\sin((s/L)\phi_{12})}{\phi_{12}} \hat{\phi}_{12} + \frac{1 - \cos((s/L)\phi_{12})}{\phi_{12}^2} \hat{\phi}_{12}^2 \right)', \quad (4.8) \end{aligned}$$

where $\phi_{12} = \|\phi_{12}\|$. By observing the similarity between (4.1) and (4.8) and noting that the former is equivalent to (4.2), equation (4.8) is equivalent to

$$\begin{aligned} \kappa^h(s) &= \frac{1}{L} \left[\frac{1}{\phi_{12}^2} \left(1 - \frac{\sin((s/L)\phi_{12})}{(s/L)\phi_{12}} \right) \phi_{12} \phi_{12}^t + \frac{\sin((s/L)\phi_{12})}{(s/L)\phi_{12}} \mathbf{I} - \frac{1 - \cos((s/L)\phi_{12})}{(s/L)\phi_{12}^2} \hat{\phi}_{12} \right] \phi_{12}, \end{aligned}$$

which, after noting $\phi_{12}^t \phi_{12} = \phi_{12}^2$ and $\hat{\phi}_{12} \phi_{12} = \mathbf{0}$, provides a remarkably simple solution

$$\boldsymbol{\kappa}^h(s) = \frac{1}{L} \phi_{12}, \quad (4.9)$$

so that, for the proposed two-noded element, we have a constant vector of rotational strains. More importantly, in contrast to the traditional approaches, this method *does* provide objective strain measures in the sense that $\boldsymbol{\kappa}^h(\underline{\boldsymbol{\psi}}_1, \underline{\boldsymbol{\psi}}_2, s) = \boldsymbol{\kappa}^h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, s)$, where $\underline{\boldsymbol{A}}_1 = \exp \underline{\boldsymbol{\psi}}_1$ and $\underline{\boldsymbol{A}}_2 = \exp \underline{\boldsymbol{\psi}}_2$ are given by (3.8). The proof of this assertion is given in the following section. In Appendix C *c*, we use the numerical example used in Appendix C *a, b* to show that standard formulations are not objective, to demonstrate that the proposed formulation is objective.

(c) *Objectivity of the new formulation*

In order to check the objectivity of the strain measures, approximated by applying the technique from the previous section, we will superimpose an arbitrary constant rigid-body translation \mathbf{r}_R and a rotation \mathbf{A}_R onto the existing configuration at both nodes via (3.7) and (3.8), and examine how this affects the computed strain measures. Since $\underline{\mathbf{A}}_1^t \underline{\mathbf{A}}_2 = \mathbf{A}_1^t \mathbf{A}_R^t \mathbf{A}_R \mathbf{A}_2 = \mathbf{A}_1^t \mathbf{A}_2$, the relative rotation between the two nodes, given by (4.6), does not change as a result of the rigid-body rotation.† Objectivity of the rotational strain measures then follows immediately from (4.9).

The translational strain measures are equally unaffected by the rigid rotation. For the present linear interpolation, the approximated translational strain measures in the original configuration are computed from (3.4), (4.4) and (4.7) as

$$\boldsymbol{\gamma}^h(\mathbf{r}_1, \mathbf{r}_2, \mathbf{A}_1, \mathbf{A}_2, s) = \frac{1}{L} \exp\left(\frac{s}{L} \hat{\phi}_{12}\right)^t \mathbf{A}_1^t (\mathbf{r}_2 - \mathbf{r}_1) - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}.$$

By applying the rigid-body translation \mathbf{r}_R and rotation \mathbf{A}_R as mentioned above, substituting \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{A}_1 in the above equation with $\underline{\mathbf{r}}_1$, $\underline{\mathbf{r}}_2$ and $\underline{\mathbf{A}}_1$ as defined by (3.7) and (3.8), and bearing in mind that the relative rotation between the two nodes does not change as a result of the rigid-body rotation, we realize that both the rigid-body translation and the rigid-body rotation cancel out from the equation and we obtain

$$\boldsymbol{\gamma}^h(\underline{\mathbf{r}}_1, \underline{\mathbf{r}}_2, \underline{\mathbf{A}}_1, \underline{\mathbf{A}}_2, s) = \frac{1}{L} \exp\left(\frac{s}{L} \hat{\phi}_{12}\right)^t \mathbf{A}_1^t (\mathbf{r}_2 - \mathbf{r}_1) - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \boldsymbol{\gamma}^h(\mathbf{r}_1, \mathbf{r}_2, \mathbf{A}_1, \mathbf{A}_2, s),$$

so that the original translational strain measures are preserved.

The simplicity of this proof is a direct consequence of the adopted approximation for rotations via (4.7), which may be summarized in the following statement.

The interpolation is applied to the relative rotation between the nodes, which is free from any rigid-body motion.

† Strictly speaking, we have proved that the rotation matrix $\exp \hat{\phi}_{12}$ given by (4.6) does not change, rather than that the relative rotation ϕ_{12} itself does not change (the equivalence between the two is maintained only if $\phi_{12} \leq \pi$). However, the ϕ_{12} employed here is extracted from $\exp \hat{\phi}_{12}$, and the properly performed extraction always provides $\phi_{12} \leq \pi$ (see Jelenić & Crisfield 1998), so the above proof is not affected.

5. Objective formulations for general isoparametric finite elements

In the previous section, we proposed a simple objective formulation for two-noded beam finite element, which features linear interpolation of kinematic fields. In this section, we will generalize this result to account for isoparametric elements with an arbitrary number of nodes. This is needed because higher-order elements (in particular, three-noded elements, with the quadratic interpolation of fields) are often used due to their higher accuracy and ability to describe curved initial geometries.

(a) General condition for the objectivity of a finite-element formulation

If we want the objectivity properties of the strain measures to be inherited by the finite-element solution, we need to prove

$$\underline{\boldsymbol{\gamma}}^h = \boldsymbol{\gamma}^h \quad \text{and} \quad \underline{\boldsymbol{\kappa}}^h = \boldsymbol{\kappa}^h, \quad (5.1)$$

where the finite-element approximations of the strain measures in the original and a rigidly displaced configuration, $\boldsymbol{\gamma}^h$, $\boldsymbol{\kappa}^h$, $\underline{\boldsymbol{\gamma}}^h$ and $\underline{\boldsymbol{\kappa}}^h$, are obtained by introducing particular finite-element approximations of the field variables in both configurations, \mathbf{r}^h , \mathbf{A}^h , $\underline{\mathbf{r}}^h$ and $\underline{\mathbf{A}}^h$, into the strain-configuration relationships of the geometrically exact 3D beam theory (3.4) and (3.5) as

$$\boldsymbol{\gamma}^h = \mathbf{A}^{\text{ht}} \mathbf{r}^{h'} - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \underline{\boldsymbol{\gamma}}^h = \underline{\mathbf{A}}^{\text{ht}} \underline{\mathbf{r}}^{h'} - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad (5.2)$$

$$\hat{\boldsymbol{\kappa}}^h = \mathbf{A}^{\text{ht}} \mathbf{A}^{h'} - \mathbf{A}_0^{\text{ht}} \mathbf{A}_0^{h'} \quad \text{and} \quad \hat{\underline{\boldsymbol{\kappa}}}^h = \underline{\mathbf{A}}^{\text{ht}} \underline{\mathbf{A}}^{h'} - \mathbf{A}_0^{\text{ht}} \mathbf{A}_0^{h'}. \quad (5.3)$$

This is a generalization of the route we pursued in §4c and (numerically) in Appendix C. It is useful to note that in order to assess the objectivity properties of a formulation, a simple and general objectivity condition may be formulated, which makes it unnecessary to fully undertake this route. This assertion is the result of the following.

Proposition 5.1. *The objectivity of particular finite-element approximations of the strain measures $\boldsymbol{\gamma}^h$ and $\boldsymbol{\kappa}^h$ is equivalent to*

$$\underline{\mathbf{r}}^h = \mathbf{A}_R(\mathbf{r}_R + \mathbf{r}^h), \quad (5.4)$$

$$\underline{\mathbf{A}}^h = \mathbf{A}_R \mathbf{A}^h, \quad (5.5)$$

for any constant $(\mathbf{r}_R, \mathbf{A}_R) \in \mathcal{R}^3 \times SO(3)$. The proof is given in Appendix Bb.

(b) Formulation based on the interpolation of current local rotations

The simple invariant formulation for geometrically exact beams, proposed in §4b will here be generalized for higher-order elements with N nodes. The interpolation of the position vector of the centroid axis (4.4) is generalized as

$$\mathbf{r}(s) \doteq \mathbf{r}^h(s) = I_i(s) \mathbf{r}_i,$$

where the index i is here and throughout the text summational over the number of nodes N of the element and \mathbf{r}_i , $i = 1, \dots, N$, are the nodal values of the unknowns

of the problem. The shape functions $I_i(s)$, $i = 1, \dots, N$, are standard Lagrangian polynomials of degree $N - 1$, which satisfy the standard conditions,

$$I_i(s_j) = \delta_{ij}, \quad \sum_{i=1}^N I_i(s) = 1; \quad i, j = 1, \dots, N.$$

The finite-element approximation of the rotation matrix \mathbf{A} , given by (4.7), will be generalized as

$$\mathbf{A}(s) \doteq \mathbf{A}^h(s) = \mathbf{A}_r \exp \hat{\Psi}^{\text{lh}}(s), \quad (5.6)$$

with the actual interpolation being applied to the *local rotations in the body attached frame* $\Psi^{\text{lh}}(s)$ as

$$\Psi^l(s) \doteq \Psi^{\text{lh}}(s) = I_i(s) \Psi_i^l, \quad (5.7)$$

and the reference rotation matrix \mathbf{A}_r being fully dependent on \mathbf{A}_I and \mathbf{A}_J at two chosen nodes I and J and the fixed constant $c \in \mathcal{R}$ in the following way:

$$\left. \begin{aligned} \mathbf{A}_r &= \mathbf{A}_I \exp(c \hat{\phi}_{IJ}), \\ \exp \hat{\phi}_{IJ} &= \mathbf{A}_I^t \mathbf{A}_J \quad \longrightarrow \quad \phi_{IJ}, \\ \exp \hat{\Psi}_i^l &= \mathbf{A}_r^t \mathbf{A}_i \quad \longrightarrow \quad \Psi_i^l; \quad i = 1, \dots, N. \end{aligned} \right\} \quad (5.8)$$

It can be seen that for a two-noded element ($N = 2$), the choice of $I = 1$, $J = 2$ and $c = 0$ leads to the formulation given in § 4 *b* (closer inspection actually shows that for $N = 2$ any choice of $I, J \in \{1, 2\}$ and $c \in \mathcal{R}$ reduces the present solution to the earlier one). Assessment of this formulation against the objectivity condition, given in the previous section, is performed in Appendix D *a*. The objectivity proof is somewhat more involved than the one given in § 4 *c* for the two-noded element, but the path-independence of the formulation is immediately obvious from (5.6)–(5.8), where *no interpolation* is hidden in the history of the evolution of \mathbf{A}^h .

While both the objectivity and the path-independence of the formulation are satisfied for arbitrary choice of parameters $I, J \in \{1, \dots, N\} \subset \mathcal{N}$ and $c \in \mathcal{R}$ it is still possible to establish some criteria for the optimal choice of these parameters (i.e. for the optimal choice of the reference triad \mathbf{A}_r). Further comments on this subject will follow in § 6.

It is worth emphasizing a strong link between the proposed procedure and the ‘co-rotational method’ (Crisfield 1990). In the latter, one would start by defining an ‘element frame’ which is here the equivalent of \mathbf{A}_r . Then one would compute ‘local rotations’ with respect to that element frame (here the Ψ_i^l s). These would be effectively interpolated using a procedure similar to that in (5.7). However, in contrast to the current developments, with the co-rotational technique it is important to choose the element frame (here \mathbf{A}_r) to minimize the magnitudes of the local rotations (here the Ψ_i^l s). This is necessary because, with the co-rotational approach, the adopted strain measures are not ‘geometrically exact’ (as here) but rather are linearized approximations. The current approach is not limited by such concepts and we can choose \mathbf{A}_r for convenience. In particular, with a three-noded element, it makes sense to choose the central node (see § 6). Despite these differences, there are strong similarities between the co-rotational formulation and the present formulation and, in particular, by interpolating ‘local quantities’, both procedures produce strains that are invariant to a rigid-body rotation.

(c) Formulation based on the interpolation of incremental local rotations

For certain applications, it may be necessary to use a formulation based on the interpolation of *incremental* rotations (similar to those proposed by Cardona & G eradin 1988). In particular, interpolating incremental rotations appears to be an indispensable tool in designing energy-conserving dynamical integrators for systems with 3D rotations (e.g. Simo *et al.* 1995). As a first step towards introducing the present objective technique into conserving dynamical integrators for 3D beams, we here outline the theoretical framework for the objective formulation based on the interpolation of *incremental local rotations*.

By applying the rotational decomposition (5.6) at two consecutive increments n and $n + 1$, $\mathbf{A}_n(s) = \mathbf{A}_{r,n} \exp \hat{\Psi}_n^1(s)$ and $\mathbf{A}_{n+1}(s) = \mathbf{A}_{r,n+1} \exp \hat{\Psi}_{n+1}^1(s)$, and by noting that $\mathbf{A}_n(s)$ and $\mathbf{A}_{n+1}(s)$ are related through the (spatial) rotational vector $\vartheta(s)$ via (Argyris 1982) $\mathbf{A}_{n+1}(s) = \exp \hat{\vartheta}(s) \mathbf{A}_n(s)$, which can be rewritten as $\exp \hat{\vartheta}(s) = \mathbf{A}_{n+1}(s) \mathbf{A}_n^t(s)$, the rotation matrix of the incremental rotation $\vartheta(s)$ can be expressed as $\exp \hat{\vartheta}(s) = \mathbf{A}_{r,n+1} \exp \hat{\Psi}_{n+1}^1(s) \exp \hat{\Psi}_n^{1t}(s) \mathbf{A}_{r,n}^t$, which gives

$$\mathbf{A}_{n+1}(s) = \mathbf{A}_{r,n+1} \exp \hat{\Psi}_{n+1}^1(s) \exp \hat{\Psi}_n^{1t}(s) \mathbf{A}_{r,n}^t \mathbf{A}_n(s).$$

Obviously, by introducing interpolations $\Psi^{lh} = I_i \Psi_i^l$ for both the local rotations at increment n and at increment $n + 1$, the earlier invariant formulation of § 5*b* is restored. In the present incremental formulation, however, we introduce the notion of *incremental local rotations*, defined as

$$\exp \hat{\Theta}^l(s) = \exp \hat{\Psi}_{n+1}^1(s) \exp \hat{\Psi}_n^{1t}(s),$$

which we interpolate in the standard manner via

$$\Theta^l(s) \doteq \Theta^{lh}(s) = I_i(s) \Theta_i^l, \quad (5.9)$$

so that the finite-element approximation of the current rotation matrix \mathbf{A}_{n+1} is obtained as

$$\mathbf{A}_{n+1}^h(s) = \mathbf{A}_{r,n+1} \exp \Theta^{lh}(s) \mathbf{A}_{r,n}^t \mathbf{A}_n^h(s). \quad (5.10)$$

Note that this procedure requires the storage, either directly or indirectly, of both $\mathbf{A}_{r,n}$ and $\mathbf{A}_n^h(s)$, with the latter usually being at the numerical integration points. In order to apply interpolation (5.9), the nodal values of incremental local rotations are provided from (5.10) as

$$\exp \hat{\Theta}_i^l = \mathbf{A}_{r,n+1}^t \mathbf{A}_{i,n+1} \mathbf{A}_{i,n}^t \mathbf{A}_{r,n} \longrightarrow \Theta_i^l; \quad i = 1, \dots, N, \quad (5.11)$$

with $\mathbf{A}_{r,n}$ and $\mathbf{A}_{r,n+1}$ being previously provided by using (5.8)₁ and (5.8)₂ at increments n and $n + 1$.

The objectivity of the formulation is proved in Appendix D*b*. However, unlike the total formulation of § 5*b*, the present incremental formulation *does* depend on the history of deformation. Its use is therefore recommended only with techniques that are path-dependent in any case, such as, for instance, the time-stepping techniques in structural dynamics.

6. Notes on the choice of reference triad \mathbf{A}_r

The reference triad \mathbf{A}_r , which is dependent on parameters $I, J \in \{1, \dots, N\} \subset \mathcal{N}$ and $c \in \mathcal{R}$, should be chosen according to the following criteria:

1. The formulation must be unaffected by the actual orientation of the particular isoparametric beam element (with equidistant element nodes) within the finite-element mesh. In other words, the results of a particular problem in which the element nodes 1 and N correspond to some structural nodes A and B , respectively, must remain the same if the element topology is changed so that the element node 1 corresponds to the structural node B and the element node N corresponds to the structural node A . We may say that we want the formulation to be ‘invariant to node-numbering’.
2. The extraction of the relative rotation ϕ_{IJ} between the nodes I and J , and the total (or incremental) nodal local rotations Ψ_i^1 (or Θ_i^1) $i = 1, \dots, N$, should be given particular attention—no rotation larger than π can be uniquely extracted from a rotation matrix, so it makes sense to state from the outset that these rotations are within that range.

In order to meet these criteria, we set-up the parameters I , J and c in the following way:

- (i) Criterion 1 naturally leads to setting-up \mathbf{A}_r as a sort of ‘midway’ triad between two ‘symmetric’ nodes. For even-noded elements, this implies $I + J = N + 1$ and $c = \frac{1}{2}$. For odd-noded elements this implies either the same as for the even-noded elements or $I = J = \frac{1}{2}(N + 1)$ and $c = 0$. Since for $I = J$ parameter c plays no role (see (5.8)_{1,2}), it can be set-up as $c = \frac{1}{2}$, and for both types of elements criterion 1 reads

$$I + J = N + 1 \quad \text{and} \quad c = \frac{1}{2}. \quad (6.1)$$

- (ii) Criterion 2 requires that the rotations that undergo extraction from rotation matrices (marked as long arrows in (5.8)_{2,3} and (5.11)) be less than π in magnitude. Since extraction of Ψ_i^1 (or Θ_i^1) $i = 1, \dots, N$ depends on the extraction of ϕ_{IJ} , we firstly analyse the latter. For $I = J$, ϕ_{IJ} is guaranteed to be zero. Further to (i), this is applicable to odd-noded elements, hence for these elements $I = J$ and (6.1) together define the optimal choice of parameters I and J as $I = J = \frac{1}{2}(N + 1)$, which means fixing the reference triad \mathbf{A}_r at the central node. For even-noded elements, $I \neq J$, and ϕ_{IJ} will in general be smaller if I and J are adjacent nodes, so that the optimal choice of parameters is given by (6.1) and $(I - J)^2 = 1$. This leads to the reference triad \mathbf{A}_r , which is placed halfway between the two central nodes. For both types of elements, these requirements and (6.1) may be expressed in a unique form by introducing an integer function ‘int’, defined on \mathcal{R} , which returns the whole (integer) part of the argument, so that we obtain

$$I = \text{int}\left(\frac{1}{2}(N + 1)\right), \quad J = \text{int}\left(\frac{1}{2}(N + 2)\right) \quad \text{and} \quad c = \frac{1}{2}. \quad (6.2)$$

- (iii) Equation (6.2) gives the general solution for the optimal choice of parameters I , J and c . As a special case, it can be shown that for any choice of I , J and c , the formulation based on the linear interpolation ($N = 2$) of total local rotations, which was previously considered in § 4 *b*, provides a unique solution.

7. Conclusions

Many of today's finite-element formulations for 3D beams are based on a geometrically exact theory that is derived directly from the resultant forms of the differential equations of equilibrium. The paper has demonstrated that these formulations do not satisfy the property of objectivity of strain measures with respect to rigid-body rotations. The problem can be tracked down to the choice of rotational variables to be interpolated within the finite-element framework. In particular, all of the current procedures interpolate the rotations between a chosen reference configuration and the current configuration and differ in whether the actual interpolation is applied to iterative, incremental or total rotations. It has been shown that

- (i) the formulation based on the interpolation of the total rotations between the initial and the deformed configuration is non-objective, and
- (ii) the formulations based on the interpolation of either incremental or iterative rotations are non-objective and path-dependent.

In order to circumvent the problem, we have proposed a method in which interpolation is applied to a particular choice of local rotations. In this way, we have laid the foundations for

- (iii) an objective and path-independent formulation based on the interpolation of total local rotations with respect to a chosen reference frame, capable of accommodating total local rotations of magnitude not larger than π for a suitably based reference frame, and
- (iv) an objective but path-dependent formulation based on the interpolation of incremental local rotations with respect to a chosen reference frame, capable of handling incremental local rotations of magnitude not larger than π ; due to its path-dependence, this formulation should only be used for providing the finite-element framework for path-dependent problems.

Work is in progress on the numerical implementation of formulation (iii) in relation to both statics and dynamics. The potential of formulation (iv) in energy-momentum conserving dynamics is also being investigated.

This work has been financially supported by the Engineering and Physical Sciences Research Council.

Appendix A. Derivation of the strain-configuration relationships in the geometrically exact 3D beam theory

For the present purposes, it suffices to assume that the deformation from the initial to the deformed configuration is caused by distributed external forces and torques \mathbf{f} and \mathbf{t} . The differential equilibrium equations of the beam are given as (e.g. Reissner 1973, eqn 2; Simo 1985, eqn 3.3)

$$\mathbf{n}' + \mathbf{f} = \mathbf{0} \quad \text{and} \quad \mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \mathbf{t} = \mathbf{0},$$

where \mathbf{n} and \mathbf{m} are the vectors of (spatial) internal forces and moments acting over the cross-section at s . While the strain-configuration relationships can be derived

from the above equilibrium equations as they stand, they would be obtained in the *spatial* form (see Simo 1985, proposition 4.1; Ibrahimbegović 1994, proposition 3.4), which is less appropriate for studying the problems of invariance—an intrinsically material concept. By noting that the material and the spatial setting of a problem are related via pull-back/push-forward mappings with the orthogonal transformation $\mathbf{A} \in SO(3)$ (see Marsden & Hughes 1994), we introduce the material vectors of internal forces and moments (Simo 1985),

$$\mathbf{N} = \mathbf{A}^t \mathbf{n} \quad \text{and} \quad \mathbf{M} = \mathbf{A}^t \mathbf{m}, \quad (\text{A } 1)$$

which correspond to the classical notion of stress and stress-couple resultants over a cross-section, and recast the above differential equilibrium equations as

$$(\mathbf{A}\mathbf{N})' + \mathbf{f} = \mathbf{0}, \quad (\text{A } 2)$$

$$(\mathbf{A}\mathbf{M})' + \mathbf{r}' \times \mathbf{A}\mathbf{N} + \mathbf{t} = \mathbf{0}. \quad (\text{A } 3)$$

As indicated in §3*b*, in the geometrically exact beam theory we derive the strain–configuration relationships by *enforcing* the equivalence between the differential equations (A 2)–(A 3) and the equality of virtual work for the internal and external forces over a segment $[s_1, s_2] \subset \langle 0, L \rangle$. The virtual work equation is given as (see Reissner 1972, 1973, 1981)

$$\int_{s_1}^{s_2} (\delta\boldsymbol{\gamma}^t \mathbf{N} + \delta\boldsymbol{\kappa}^t \mathbf{M}) \, ds = \int_{s_1}^{s_2} (\delta\mathbf{r}^t \mathbf{f} + \delta\boldsymbol{\omega}^t \mathbf{t}) \, ds + [\delta\mathbf{r}^t \mathbf{n} + \delta\boldsymbol{\omega}^t \mathbf{m}]_{s_1}^{s_2}, \quad (\text{A } 4)$$

where $\delta\boldsymbol{\gamma}$ and $\delta\boldsymbol{\kappa}$ are (material) *virtual strain measures*, which are work-conjugate to the stress and stress-couple resultants \mathbf{N} and \mathbf{M} , while $\delta\mathbf{r}$ and $\delta\boldsymbol{\omega}$ are *virtual displacements and rotations*, i.e. the kinematically admissible and infinitesimally small perturbations of the configuration $\mathcal{C} = (\mathbf{r}, \mathbf{A})$. The definition of a kinematically admissible perturbation of the orthogonal triad attached to a cross-section by $\delta\boldsymbol{\omega}$ relies on recognizing the equivalence relationship $\mathbf{v} \times \mathbf{w} = \hat{\mathbf{v}}\mathbf{w} \, \forall \mathbf{w} \in \mathcal{R}^3$ between any element \mathbf{v} of a 3D vector space \mathcal{R}^3 endowed with a cross product operation and any element $\hat{\mathbf{v}}$ of a Lie algebra $so(3)$ (which is the skew-symmetric matrix defined earlier in (3.6)). In this way, the kinematically admissible perturbation of $\mathbf{A} \in SO(3)$ by $\delta\hat{\boldsymbol{\omega}} \in so(3)$, denoted by $\mathbf{A}_\epsilon \in SO(3)$, is defined through the standard exponential mapping between Lie algebras and Lie groups as $\mathbf{A}_\epsilon = \exp(\epsilon\delta\hat{\boldsymbol{\omega}})\mathbf{A}$, $\forall \epsilon \in \mathcal{R}$ and its infinitesimally small change is computed through the directional derivative along $\delta\hat{\boldsymbol{\omega}}$ as (Simo & Vu-Quoc 1986)

$$\delta\mathbf{A} = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathbf{A}_\epsilon = \delta\hat{\boldsymbol{\omega}}\mathbf{A}. \quad (\text{A } 5)$$

By following Reissner (1972, 1973), we express the distributed external forces and torques from differential equations (A 2) and (A 3) in terms of the stress resultants and the stress-couple resultants and insert them into the virtual work equation (A 4). After performing integration by parts on terms with $(\mathbf{A}\mathbf{N})'$ and $(\mathbf{A}\mathbf{M})'$ and making use of (A 1) and $\mathbf{r}' \times \mathbf{A}\mathbf{N} = \hat{\mathbf{r}}'\mathbf{A}\mathbf{N}$, the boundary terms cancel out and the remaining equation

$$\int_{s_1}^{s_2} (\delta\boldsymbol{\gamma}^t - \delta\mathbf{r}'^t \mathbf{A} + \delta\boldsymbol{\omega}^t \hat{\mathbf{r}}' \mathbf{A}) \mathbf{N} \, ds + \int_{s_1}^{s_2} (\delta\boldsymbol{\kappa}^t - \delta\boldsymbol{\omega}'^t \mathbf{A}) \mathbf{M} \, ds = 0$$

is non-trivially satisfied only for

$$\delta\boldsymbol{\gamma} = \mathbf{A}^t (\delta\mathbf{r}' + \hat{\mathbf{r}}' \delta\boldsymbol{\omega}) \quad \text{and} \quad \delta\boldsymbol{\kappa} = \mathbf{A}^t \delta\boldsymbol{\omega}'.$$

These virtual relations are easily integrated. By using (A 5) we obtain

$$\delta\boldsymbol{\gamma} = \mathbf{A}^t \delta \mathbf{r}' + \mathbf{A}^t \delta \hat{\boldsymbol{\omega}}^t \mathbf{r}' = \mathbf{A}^t \delta \mathbf{r}' + \delta \mathbf{A}^t \mathbf{r}' = \delta(\mathbf{A}^t \mathbf{r}') \quad (\text{A } 6)$$

and by using identities $\widehat{\mathbf{A}}\mathbf{v} = \mathbf{A}\hat{\mathbf{v}}\mathbf{A}^t \forall \mathbf{v} \in \mathcal{R}^3$, $\mathbf{A} \in SO(3)$ and $\hat{\mathbf{v}} = -\hat{\mathbf{v}}^t \forall \hat{\mathbf{v}} \in so(3)$ we have

$$\delta \hat{\boldsymbol{\kappa}} = \mathbf{A}^t \delta \hat{\boldsymbol{\omega}}' \mathbf{A} = \mathbf{A}^t (\delta \hat{\boldsymbol{\omega}} \mathbf{A})' - \mathbf{A}^t \delta \hat{\boldsymbol{\omega}} \mathbf{A}' = \mathbf{A}^t \delta \mathbf{A}' + \delta \mathbf{A}^t \mathbf{A}' = \delta(\mathbf{A}^t \mathbf{A}'). \quad (\text{A } 7)$$

Remembering that the initial configuration is undeformed (i.e. unstrained) and making use of (3.1), (3.2), and $\mathbf{G}_{0,1}(s) = \mathbf{A}_0(s) \mathbf{E}_1$, equations (A 6) and (A 7) are integrated to give the required strain-configuration relationships

$$\boldsymbol{\gamma} = \mathbf{A}^t \mathbf{r}' - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \hat{\boldsymbol{\kappa}} = \mathbf{A}^t \mathbf{A}' - \mathbf{A}_0^t \mathbf{A}_0'.$$

These equations and their spatial counterparts appear to have been first derived by Simo (1985, eqns 4.8b and 4.1b) (see also Simo & Vu-Quoc 1986, tables 1 and 2, and Simo *et al.* 1995, eqn (20)), who used the rate of change of energy over time rather than the virtual work. The approach used here corresponds to remark 4.5 in Simo (1985). It must be noted that neither the virtual rotation $\delta \boldsymbol{\omega}$ used here, nor the originally used rate of change of the rotation over time $\dot{\boldsymbol{\omega}}$, implies the existence of any ' $\boldsymbol{\omega}$ ' (see also Reissner 1981, p. 735).

Appendix B. Proofs of propositions 3.1 and 5.1

(a) Proof of proposition 3.1

By starting with the definition (3.4), the translational strains $\underline{\boldsymbol{\gamma}}$ at configuration $\underline{\mathcal{C}} = (\underline{\mathbf{r}}, \underline{\mathbf{A}})$ are computed as

$$\underline{\boldsymbol{\gamma}} = \underline{\mathbf{A}}^t \underline{\mathbf{r}}' - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}.$$

After inserting (3.7) and (3.8) we obtain

$$\underline{\boldsymbol{\gamma}} = \mathbf{A}^t \mathbf{A}_R^t [\mathbf{A}_R (\mathbf{r}_R + \mathbf{r})]' - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \mathbf{A}^t \mathbf{A}_R^t \mathbf{A}_R' (\mathbf{r}_R + \mathbf{r}) + \mathbf{A}^t \mathbf{A}_R^t \mathbf{A}_R (\mathbf{r}'_R + \mathbf{r}') - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}.$$

Since \mathbf{A}_R and \mathbf{r}_R are constant, it follows that $\mathbf{A}'_R = \mathbf{0}$ and $\mathbf{r}'_R = \mathbf{0}$, so by noting that $\mathbf{A}_R^t \mathbf{A}_R = \mathbf{I}$ the above equation reduces to

$$\underline{\boldsymbol{\gamma}} = \mathbf{A}^t \mathbf{r}' - \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \boldsymbol{\gamma}.$$

Similarly, for the rotational strain measures (3.5),

$$\hat{\underline{\boldsymbol{\kappa}}} = \underline{\mathbf{A}}^t \underline{\mathbf{A}}' - \mathbf{A}_0^t \mathbf{A}_0' = \mathbf{A}^t \mathbf{A}_R^t \mathbf{A}_R' \mathbf{A} + \mathbf{A}^t \mathbf{A}_R^t \mathbf{A}_R \mathbf{A}' - \mathbf{A}_0^t \mathbf{A}_0' = \mathbf{A}^t \mathbf{A}' - \mathbf{A}_0^t \mathbf{A}_0' = \hat{\boldsymbol{\kappa}},$$

which proves part (i) of the proposition. The strain measures $\boldsymbol{\gamma}$ and $\hat{\boldsymbol{\kappa}}$ depend only on the current configuration $\mathcal{C} = (\mathbf{r}, \mathbf{A})$. This makes them independent of the history of deformation and proves part (ii) of the proposition.

(b) *Proof of proposition 5.1*

We start by proving $\underline{\boldsymbol{\kappa}}^h = \boldsymbol{\kappa}^h \Rightarrow \underline{\boldsymbol{\Lambda}}^h = \boldsymbol{\Lambda}_R \boldsymbol{\Lambda}^h$. Inserting (5.3) into (5.1)₂ gives

$$\underline{\boldsymbol{\Lambda}}^{\text{ht}} \underline{\boldsymbol{\Lambda}}^{h'} = \boldsymbol{\Lambda}^{\text{ht}} \boldsymbol{\Lambda}^{h'}. \quad (\text{B } 1)$$

Since $\boldsymbol{\Lambda}^h, \underline{\boldsymbol{\Lambda}}^h \in SO(3)$, there exist $\boldsymbol{\Lambda}^*, \boldsymbol{\Lambda}^* \in SO(3)$ so that

$$\underline{\boldsymbol{\Lambda}}^h = \boldsymbol{\Lambda}^* \boldsymbol{\Lambda}^h \boldsymbol{\Lambda}^*. \quad (\text{B } 2)$$

Equations (B 1) and (B 2) imply

$$\boldsymbol{\Lambda}^{*t} \boldsymbol{\Lambda}^{\text{ht}} \boldsymbol{\Lambda}^{*t} (\boldsymbol{\Lambda}^{*'} \boldsymbol{\Lambda}^h \boldsymbol{\Lambda}^* + \boldsymbol{\Lambda}^* \boldsymbol{\Lambda}^{h'} \boldsymbol{\Lambda}^* + \boldsymbol{\Lambda}^* \boldsymbol{\Lambda}^h \boldsymbol{\Lambda}^{*'}) = \boldsymbol{\Lambda}^{\text{ht}} \boldsymbol{\Lambda}^{h'},$$

which is valid only for $\boldsymbol{\Lambda}^* = \boldsymbol{I}$ and $\boldsymbol{\Lambda}^* = \text{const}$. The latter requirement means that $\boldsymbol{\Lambda}^*$ is a rigid-body rotation, so the solution (B 2) is equivalent to (5.5), which proves that $\underline{\boldsymbol{\kappa}}^h = \boldsymbol{\kappa}^h \Rightarrow \underline{\boldsymbol{\Lambda}}^h = \boldsymbol{\Lambda}_R \boldsymbol{\Lambda}^h$. The reverse implication and hence the equivalence between (5.1)₂ and (5.5) is proved by inserting (5.5) into (5.3)₂ and showing that the latter becomes equal to (5.3)₁. Note that these implications are valid for (5.1) or (5.4) and (5.5) being satisfied along the whole domain $[0, L]$. In particular, satisfying (5.5) only at specific points does not imply the rotational strain invariance (5.1)₂ at these points.

The first part of the proposition is proved in a similar way. Inserting (5.2) into (5.1)₁ gives

$$\underline{\boldsymbol{\Lambda}}^{\text{ht}} \underline{\boldsymbol{r}}^{h'} = \boldsymbol{\Lambda}^{\text{ht}} \boldsymbol{r}^{h'},$$

which upon the introduction of (5.5) becomes

$$\boldsymbol{\Lambda}^{\text{ht}} \boldsymbol{\Lambda}_R^t \underline{\boldsymbol{r}}^{h'} = \boldsymbol{\Lambda}^{\text{ht}} \boldsymbol{r}^{h'}.$$

Since $\boldsymbol{\Lambda}_R$ is a constant rotation, the above reduces to the differential equation $\underline{\boldsymbol{r}}^{h'} = (\boldsymbol{\Lambda}_R \boldsymbol{r}^h)'$, with the solution

$$\underline{\boldsymbol{r}}^h = \boldsymbol{\Lambda}_R \boldsymbol{r}^h + \boldsymbol{c}; \quad \forall \boldsymbol{c} \in \mathcal{R}^3 \mid \boldsymbol{c} = \text{const}. \quad (\text{B } 3)$$

The constant vector \boldsymbol{c} in (B 3) can be expressed in terms of the arbitrary rigid motion $(\boldsymbol{r}_R, \boldsymbol{\Lambda}_R)$ as $\boldsymbol{c} = \boldsymbol{\Lambda}_R \boldsymbol{r}_R$, which leads to (5.4) and proves that $\underline{\boldsymbol{\gamma}}^h = \boldsymbol{\gamma}^h$ and

$$\underline{\boldsymbol{\kappa}}^h = \boldsymbol{\kappa}^h \Rightarrow \underline{\boldsymbol{r}}^h = \boldsymbol{\Lambda}_R (\boldsymbol{r}_R + \boldsymbol{r}^h).$$

Inserting (5.4) and (5.5) into (5.2)₂ proves the reverse implication.

Appendix C. Numerical results

(a) *Traditional implementation (total form): demonstration of non-objectivity*

Let us take particular values,

$$\boldsymbol{\psi}_1 = \begin{Bmatrix} 1 \\ -0.5 \\ 0.25 \end{Bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{Bmatrix} -0.4 \\ 0.7 \\ 0.1 \end{Bmatrix}, \quad \boldsymbol{\psi}_R = \begin{Bmatrix} 0.2 \\ 1.2 \\ -0.5 \end{Bmatrix},$$

where $\boldsymbol{\psi}_R$ is associated with the constant rotation $\boldsymbol{\Lambda}_R$ via $\boldsymbol{\Lambda}_R = \exp \hat{\boldsymbol{\Psi}}_R$, and assume $L = 1$. For the initial computation we omit the superimposed rigid-body rotation and compute

$$\boldsymbol{\psi}_a = \frac{1}{2}(\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2) = \begin{Bmatrix} 0.3 \\ 0.1 \\ 0.175 \end{Bmatrix} \quad \text{and} \quad \frac{1}{L} \boldsymbol{\psi}_d = \frac{1}{L}(\boldsymbol{\psi}_2 - \boldsymbol{\psi}_1) = \begin{Bmatrix} -1.4 \\ 1.2 \\ -0.15 \end{Bmatrix},$$

which, after insertion into (4.5), gives

$$\boldsymbol{\kappa}^h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \frac{1}{2}L) = \begin{Bmatrix} -1.2746 \\ 1.2676 \\ -0.4035 \end{Bmatrix}.$$

We now repeat the computations, having first superimposed the rigid-body rotation on the nodal rotation matrices so that, from $\underline{\mathbf{A}}_1 = \mathbf{A}_R \mathbf{A}_1$ and $\underline{\mathbf{A}}_2 = \mathbf{A}_R \mathbf{A}_2$, we extract the rotational vectors (see Spurrier (1978), Simo & Vu-Quoc (1986), and Crisfield (1997) for the actual algorithm to extract the rotational vector from a rotation matrix)

$$\underline{\boldsymbol{\psi}}_1 = \begin{Bmatrix} 1.0015 \\ 0.3468 \\ -0.8372 \end{Bmatrix} \quad \text{and} \quad \underline{\boldsymbol{\psi}}_2 = \begin{Bmatrix} 0.0885 \\ 1.9332 \\ -0.0819 \end{Bmatrix}.$$

Interpolation to the centre of the element ($s = \frac{1}{2}L$) then gives

$$\underline{\boldsymbol{\psi}}_a = \begin{Bmatrix} 0.545 \\ 1.14 \\ -0.45955 \end{Bmatrix} \quad \text{and} \quad \frac{1}{L} \underline{\boldsymbol{\psi}}_d = \begin{Bmatrix} -0.9130 \\ 1.5864 \\ 0.7553 \end{Bmatrix}$$

which, after inserting into (4.5), gives

$$\boldsymbol{\kappa}^h(\underline{\boldsymbol{\psi}}_1, \underline{\boldsymbol{\psi}}_2, \frac{1}{2}L) = \begin{Bmatrix} -1.2640 \\ 1.3137 \\ -0.3375 \end{Bmatrix}.$$

Clearly, $\boldsymbol{\kappa}^h(\underline{\boldsymbol{\psi}}_1, \underline{\boldsymbol{\psi}}_2, \frac{1}{2}L) \neq \boldsymbol{\kappa}^h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \frac{1}{2}L)$ and the underlying finite-element formulation is non-objective.

(b) *Traditional implementation (incremental form): a demonstration of path-dependence*

To illustrate the path-dependence of the incremental formulations (here this term includes both the formulation based on the interpolation of incremental rotations due to Cardona & Géradin (1988) and the formulation based on the interpolation of iterative rotations due to Simo & Vu-Quoc (1986)), we take the data from Appendix C a and apply the rotations $\boldsymbol{\psi}_1$, and $\boldsymbol{\psi}_2$ in two different steps via $\boldsymbol{\psi}_1 = \boldsymbol{\vartheta}_1 + \boldsymbol{\theta}_1$, and $\boldsymbol{\psi}_2 = \boldsymbol{\vartheta}_2 + \boldsymbol{\theta}_2$ with particular values,

$$\boldsymbol{\vartheta}_1 = \begin{Bmatrix} 0.775 \\ -0.3875 \\ 0.19375 \end{Bmatrix}, \quad \boldsymbol{\theta}_1 = \begin{Bmatrix} 0.225 \\ -0.1125 \\ 0.05625 \end{Bmatrix}, \quad \boldsymbol{\vartheta}_2 = \begin{Bmatrix} -0.16 \\ 0.28 \\ 0.04 \end{Bmatrix}, \quad \boldsymbol{\theta}_2 = \begin{Bmatrix} -0.24 \\ 0.42 \\ 0.06 \end{Bmatrix},$$

which in the first step gives the following interpolations to the centre of element

$$\boldsymbol{\vartheta}_a = \begin{Bmatrix} 0.3075 \\ -0.05375 \\ 0.116875 \end{Bmatrix} \quad \text{and} \quad \frac{1}{L} \boldsymbol{\vartheta}_d = \begin{Bmatrix} -0.935 \\ 0.6675 \\ -0.15375 \end{Bmatrix},$$

with

$$\mathbf{A}^h(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \frac{1}{2}L) = \exp \hat{\boldsymbol{\vartheta}}_a = \begin{bmatrix} 0.9830 & -0.1309 & 0.1288 \\ 0.1640 & 0.9413 & -0.2951 \\ -0.0826 & 0.3111 & 0.9468 \end{bmatrix}$$

and

$$\boldsymbol{\kappa}^h(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \frac{1}{2}L) = \begin{Bmatrix} -0.9006 \\ 0.6890 \\ -0.2343 \end{Bmatrix}.$$

In the second step, we have

$$\boldsymbol{\theta}_a = \begin{Bmatrix} -0.0075 \\ 0.15375 \\ 0.058125 \end{Bmatrix} \quad \text{and} \quad \frac{1}{L}\boldsymbol{\theta}_d = \begin{Bmatrix} -0.465 \\ 0.5325 \\ 0.00375 \end{Bmatrix}.$$

The vector of final rotational strains follows from the incremental form of (4.5), which reads (Jelenić & Saje 1995; Ibrahimbegović *et al.* 1995)

$$\boldsymbol{\kappa}^h(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \frac{1}{2}L) = \boldsymbol{\kappa}^h(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \frac{1}{2}L) + \boldsymbol{\Lambda}^{\text{ht}}(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \frac{1}{2}L) \frac{1}{L} \left[\frac{1}{\theta_a^2} \left(1 - \frac{\sin \theta_a}{\theta_a} \right) \boldsymbol{\theta}_a \boldsymbol{\theta}_a^t + \frac{\sin \theta_a}{\theta_a} \mathbf{I} + \frac{1 - \cos \theta_a}{\theta_a^2} \hat{\boldsymbol{\theta}}_a \right] \boldsymbol{\theta}_d$$

as

$$\boldsymbol{\kappa}^h(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \frac{1}{2}L) = \begin{Bmatrix} -1.2887 \\ 1.2518 \\ -0.4128 \end{Bmatrix},$$

which is different from $\boldsymbol{\kappa}^h(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \frac{1}{2}L)$, obtained in Appendix C *a* using a different deformation history.

(c) *Proposed implementation: a demonstration of objectivity*

We repeat the example from Appendix C *a* using the new formulation. Due to

$$\underline{\mathbf{A}}_1^t \underline{\mathbf{A}}_2 = \exp \hat{\boldsymbol{\psi}}_1^t \exp \hat{\boldsymbol{\psi}}_R^t \exp \hat{\boldsymbol{\psi}}_R \exp \hat{\boldsymbol{\psi}}_2 = \exp \hat{\boldsymbol{\psi}}_1^t \exp \hat{\boldsymbol{\psi}}_2 = \underline{\mathbf{A}}_1^t \underline{\mathbf{A}}_2,$$

equations (4.6) and (4.9) result in

$$\boldsymbol{\kappa}^h(\underline{\mathbf{A}}_1, \underline{\mathbf{A}}_2, s) = \boldsymbol{\kappa}^h(\mathbf{A}_1, \mathbf{A}_2, s) = \begin{Bmatrix} -1.263\ 825\ 266\ 688\ 60 \\ 1.271\ 015\ 367\ 499\ 35 \\ -0.422\ 941\ 202\ 531\ 96 \end{Bmatrix},$$

so that the rotational strain measures at any value of the arc-length parameter s are objective.

Appendix D. Objectivity of general isoparametric formulations

(a) *Objectivity of formulation based on interpolation of total local rotations*

By superposing a constant motion $(\mathbf{r}_R, \mathbf{A}_R)$ at $s = s_i$; $i = 1, \dots, N$ via

$$\mathbf{r}_i = \mathbf{A}_R(\mathbf{r}_R + \mathbf{r}_i)$$

the validity of (5.4) for all $s \in [0, L]$ is immediately seen through

$$\underline{\mathbf{r}}^h = I_i \mathbf{r}_i = I_i [\mathbf{A}_R(\mathbf{r}_R + \mathbf{r}_i)] = \mathbf{A}_R(\mathbf{r}_R + I_i \mathbf{r}_i) = \mathbf{A}_R(\mathbf{r}_R + \mathbf{r}^h).$$

By superposing a constant rotation \mathbf{A}_R at $s = s_i$; $i = 1, \dots, N$ via $\underline{\mathbf{A}}_i = \mathbf{A}_R \mathbf{A}_i$, $\underline{\mathbf{A}}^h$ on the left-hand side of (5.5) is expanded using (5.6) and (5.8)₁ as

$$\underline{\mathbf{A}}^h = \underline{\mathbf{A}}_r \exp \hat{\boldsymbol{\Psi}}^{\text{lh}} = \underline{\mathbf{A}}_l \exp(c \hat{\boldsymbol{\phi}}_{IJ}) \exp \hat{\boldsymbol{\Psi}}^{\text{lh}}. \quad (\text{D } 1)$$

Following the same arguments as in § 4 *c*, we conclude that $\underline{\phi}_{IJ} = \phi_{IJ}$, so that with the aid of (5.8)₁ equation (D 1) becomes

$$\underline{\mathbf{A}}^h = \underline{\mathbf{A}}_I \exp(c\hat{\phi}_{IJ}) \exp \hat{\underline{\Psi}}^{\text{lh}} = \underline{\mathbf{A}}_R \underline{\mathbf{A}}_I \exp(c\hat{\phi}_{IJ}) \exp \hat{\underline{\Psi}}^{\text{lh}} = \underline{\mathbf{A}}_R \underline{\mathbf{A}}_r \exp \hat{\underline{\Psi}}^{\text{lh}}. \quad (\text{D } 2)$$

The last factor in (D 2) will be equal to $\exp \hat{\underline{\Psi}}^{\text{lh}}$ if $\underline{\Psi}^{\text{lh}} = \hat{\underline{\Psi}}^{\text{lh}}$ (i.e. if $\underline{\Psi}_i^{\text{lh}} = \hat{\underline{\Psi}}_i^{\text{lh}}$; $i = 1, \dots, N$), so following earlier arguments on the limitations of the extraction of rotational vectors from rotation matrices (see footnote in § 4 *c*), the above is implied by the requirement $\exp \hat{\underline{\Psi}}_i^{\text{lh}} = \exp \hat{\underline{\Psi}}_i^{\text{lh}}$; $i = 1, \dots, N$. Hence, following (5.8)₃ and (5.8)₁ we obtain

$$\begin{aligned} \exp \hat{\underline{\Psi}}_i^{\text{lh}} &= \underline{\mathbf{A}}_r^t \underline{\mathbf{A}}_i = \exp(c\hat{\phi}_{IJ})^t \underline{\mathbf{A}}_I^t \underline{\mathbf{A}}_i \\ &= \exp(c\hat{\phi}_{IJ})^t \underline{\mathbf{A}}_I^t \underline{\mathbf{A}}_R^t \underline{\mathbf{A}}_R \underline{\mathbf{A}}_i = [\underline{\mathbf{A}}_I \exp(c\hat{\phi}_{IJ})]^t \underline{\mathbf{A}}_i = \underline{\mathbf{A}}_I^t \underline{\mathbf{A}}_i = \exp \hat{\underline{\Psi}}_i^{\text{lh}}. \end{aligned}$$

Consequently, $\underline{\Psi}_i^{\text{lh}} = \hat{\underline{\Psi}}_i^{\text{lh}}$; $i = 1, \dots, N$ and hence $\underline{\Psi}^{\text{lh}} = \hat{\underline{\Psi}}^{\text{lh}}$, so (D 2) becomes

$$\underline{\mathbf{A}}^h = \underline{\mathbf{A}}_R \underline{\mathbf{A}}_r \exp \hat{\underline{\Psi}}^{\text{lh}} = \underline{\mathbf{A}}_R \underline{\mathbf{A}}^h,$$

which shows that the new formulation maintains equation (5.5).

Since equations (5.4) and (5.5) are maintained, it follows from proposition 5.1 that the general isoparametric formulation based on the interpolation of total local rotations provides the required objective approximated strain measures.

(*b*) *Objectivity of formulation based on interpolation of incremental local rotations*

We prove the objectivity of the incremental formulation in the sense

$$\underline{\mathbf{A}}_{n+1}^h = \underline{\mathbf{A}}_{R,n+1} \underline{\mathbf{A}}_{n+1}^h, \quad (\text{D } 3)$$

where $\underline{\mathbf{A}}_{R,n+1}$ is the share of the total rigid rotation $\underline{\mathbf{A}}_R$ to be applied, that has been accumulated in $n + 1$ increments and *not* the incremental amount of the total rigid rotation applied between configurations n and $n + 1$. Note that the initial configuration is constant so $\underline{\mathbf{A}}_0^h = \underline{\mathbf{A}}_0^h$ and $\underline{\mathbf{A}}_{R,0} = \mathbf{I}$. As before, we assume that (D 3) is valid at nodes, i.e.

$$\underline{\mathbf{A}}_{n+1,i} = \underline{\mathbf{A}}_{R,n+1} \underline{\mathbf{A}}_{n+1,i}; \quad i = 1, \dots, N. \quad (\text{D } 4)$$

Expressing the left-hand side of (D 3) in terms of definition (5.10),

$$\underline{\mathbf{A}}_{n+1}^h = \underline{\mathbf{A}}_{r,n+1} \exp \hat{\underline{\Theta}}^{\text{lh}} \underline{\mathbf{A}}_{r,n}^t \underline{\mathbf{A}}_n^h,$$

and using (5.8)₁ at increments n and $n + 1$ gives

$$\underline{\mathbf{A}}_{n+1}^h = \underline{\mathbf{A}}_{I,n+1} \exp(c\hat{\phi}_{IJ,n+1}) \exp \hat{\underline{\Theta}}^{\text{lh}} \exp(c\hat{\phi}_{IJ,n})^t \underline{\mathbf{A}}_{I,n}^t \underline{\mathbf{A}}_n^h.$$

By following the same arguments as in § 4 *c* we prove $\underline{\phi}_{IJ} = \phi_{IJ}$ at both increments n and $n + 1$, so by using (D 4) and (5.8)₁, the previous equation turns into

$$\underline{\mathbf{A}}_{n+1}^h = \underline{\mathbf{A}}_{R,n+1} \underline{\mathbf{A}}_{r,n+1} \exp \hat{\underline{\Theta}}^{\text{lh}} \underline{\mathbf{A}}_{r,n}^t \underline{\mathbf{A}}_{R,n}^t \underline{\mathbf{A}}_n^h. \quad (\text{D } 5)$$

Next, we concentrate on the third factor in the right-hand side of (D 5) and note from (5.11) that $\underline{\phi}_{IJ} = \phi_{IJ}$ implies $\underline{\Theta}_i^{\text{lh}} = \hat{\underline{\Theta}}_i^{\text{lh}}$; $i = 1, \dots, N$, hence using the adopted interpolation (5.9) we have $\underline{\Theta}^{\text{lh}} = \hat{\underline{\Theta}}^{\text{lh}}$, which transforms (D 5) into

$$\underline{\mathbf{A}}_{n+1}^h = \underline{\mathbf{A}}_{R,n+1} \underline{\mathbf{A}}_{r,n+1} \exp \hat{\underline{\Theta}}^{\text{lh}} \underline{\mathbf{A}}_{r,n}^t \underline{\mathbf{A}}_{R,n}^t \underline{\mathbf{A}}_n^h. \quad (\text{D } 6)$$

The rest of the proof of objectivity of the incremental formulation is based on the mathematical induction:

- (i) If (D 3) is proved to be valid for $n = 0$ and
 - (ii) is assumed to be valid for $n = k - 1$ and
 - (iii) is proved to be valid for $n = k$ under the assumption (ii),
- then (D 3) is valid for any $n \in \mathcal{N}$.

- (i) Equation (D 6) for $n = 0$ reads (use has been made of the finite-element approximation for incremental formulation (5.10) and of $\mathbf{A}_{R,0} = \mathbf{I}$ and $\underline{\mathbf{A}}_0^h = \mathbf{A}_0^h$)

$$\underline{\mathbf{A}}_1^h = \mathbf{A}_{R,1} \mathbf{A}_{r,1} \exp \hat{\Theta}^{\text{lh}} \mathbf{A}_{r,0}^t \mathbf{A}_{R,0}^t \underline{\mathbf{A}}_0^h = \mathbf{A}_{R,1} \mathbf{A}_{r,1} \exp \hat{\Theta}^{\text{lh}} \mathbf{A}_{r,0}^t \mathbf{A}_0^h = \mathbf{A}_{R,1} \mathbf{A}_1^h,$$

which proves the objectivity requirement (D 3) for $n = 0$.

- (ii) Equation (D 3) is assumed to be valid for $n = k - 1$, i.e. $\underline{\mathbf{A}}_k^h = \mathbf{A}_{R,k} \mathbf{A}_k^h$.
- (iii) Equation (D 6) for $n = k$, and under assumption (ii) for the finite-element approximation (5.10), reads

$$\begin{aligned} \underline{\mathbf{A}}_{k+1}^h &= \mathbf{A}_{R,k+1} \mathbf{A}_{r,k+1} \exp \hat{\Theta}^{\text{lh}} \mathbf{A}_{r,k}^t \mathbf{A}_{R,k}^t \mathbf{A}_{R,k} \mathbf{A}_k^h \\ &= \mathbf{A}_{R,k+1} \mathbf{A}_{r,k+1} \exp \hat{\Theta}^{\text{lh}} \mathbf{A}_{r,k}^t \mathbf{A}_k^h = \mathbf{A}_{R,k+1} \mathbf{A}_{k+1}^h, \end{aligned}$$

which proves the objectivity requirement for increment $k+1$ if the same requirement is valid for increment k . Since it has been proved that the objectivity requirement is satisfied for the first increment, it follows that it is also satisfied for any increment $k \in [1, n] \subset \mathcal{N} \forall n \in \mathcal{N}$. Hence, further to proposition 5.1, the incremental formulation based on the finite-element approximations (5.9)–(5.11) and (5.8)_{1,2} is objective for any sequence of incremental constant rotations.

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