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Oblique and Biorthogonal Multi-wavelet Bases with Fast-Filtering Algorithms

Akram Aldroubi

Biomedical Engineering and Instrumentation programs, National Institutes of Health,
13 south DR MSC 5766, Bethesda MD 20892-5766, USA
E-mail: aldroubi@helix.nih.gov

Jill McGowan

Dept. of Mathematics, Howard University
Washington DC 20059.

ABSTRACT

We construct oblique multi-wavelets bases which encompass the orthogonal multi-wavelets and the biorthogonal uni-wavelets of Cohen, Deaubechies and Feauveau. These oblique multi-wavelets preserve the advantages of orthogonal and biorthogonal wavelets and enhance the flexibility of wavelet theory to accommodate a wider variety of wavelet shapes and properties. Moreover, oblique multi-wavelets can be implemented with fast vector-filter-bank algorithms. We use the theory to derive a new construction of biorthogonal uni-wavelets.

Keywords: multi-wavelet, multi-scaling function, oblique wavelet bases, biorthogonal wavelet, semi-orthogonal wavelet, perfect reconstruction filter bank, perfect resolving filter bank, vector filter bank

1 INTRODUCTION

Multiresolution-type wavelet bases (MRA-type wavelets) have the important property that the wavelets' coefficients can be obtained with fast-filtering algorithms.[10,9] Mallat has constructed MRA-type wavelets that form orthogonal bases. These wavelets do not have compact support, and for certain applications, their shapes and properties are not appropriate. To circumvent the problem of infinite support, Daubechies has constructed compactly supported orthogonal MRA-type wavelets.[6] However, these wavelets are not symmetrical and are still restricted in their shape and properties. Cohen, Daubechies and Feauveau have overcome the problem of the lack of symmetry through the introduction of biorthogonal wavelets.[5] The construction of biorthogonal wavelets does not completely alleviate the problems with respect to shape and restrictive properties. For this reason, semiorthogonal wavelet bases (also called nonorthogonal wavelets) have been introduced to overcome this lack of flexibility.[11,3,14,4]

By relaxing some of the constraints on the multiresolution and on the wavelets themselves, we have been able to find other multi-wavelet bases with fast-filtering implementation for their coefficients. We have constructed

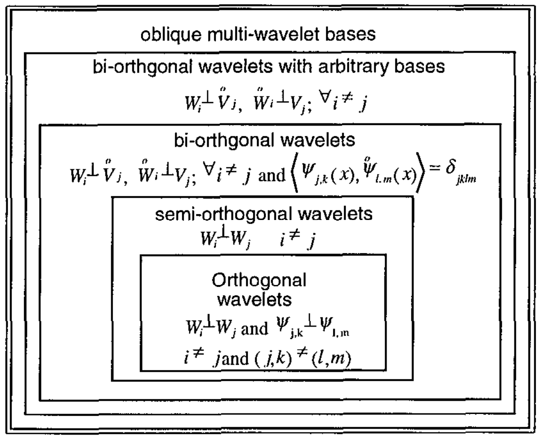

Figure 1: The different types of wavelet bases. The orthogonal wavelet bases are characterized by: $W_i \perp W_j$ for $i \neq j$, and $\langle \psi_{j,k}(x), \psi_{l,m}(x) \rangle = \delta_{jklm}$. semiorthogonal wavelet bases are characterized simply by $W_i \perp W_j$ for $i \neq j$. In the biorthogonal case, there are two pairs of multiresolution and wavelet spaces that must satisfy $W_i \perp \overset{o}{V}_j$, $\overset{o}{W}_i \perp V_j$ for all $i \neq j$, and $\left\langle \psi_{j,k}(x), \overset{o}{\psi}_{l.m}(x) \right\rangle = \delta_{jklm}$. The only difference between the biorthogonal case with arbitrary bases and the previous biorthogonal case is that we do not require the biorthogonality between the bases, but require only that $W_i \perp \overset{o}{V}_j$, $\overset{o}{W}_i \perp V_j$ for all $i \neq j$. In the oblique multi-wavelet case, there is only one underlying MRA with its wavelet spaces. However, the spaces W_i and W_j are not necessarily orthogonal. The only condition is that $V_{i+1} + W_{j+1} = V_j$ and that the wavelets, their shifts, and their dilates form an unconditional basis of $\mathcal{L}_2$. The oblique multi-wavelet case also contain the special cases of orthogonal and biorthogonal multi-wavelets.

oblique multi-wavelet bases. The theory underlying these bases encompasses the orthogonal theory of orthogonal multi-wavelets,[7,8,13,15] and, generalizes the concept of biorthogonal wavelets of Cohen Daubechies and Feauveau[5] (See Figure 1). In particular, the biorthogonal wavelet theory of Cohen, Daubechies and Feauveau (CDF) requires a pair of multiresolutions and a pair of wavelet spaces. Here, we only require one set of multiresolution spaces and one set of wavelet spaces, similar to the orthogonal case. Still, we omit the requirement that the wavelet spaces be orthogonal to the multiresolution spaces. In so doing, we are able to build a more general wavelet theory in which the wavelet spaces are merely complementary to the multiresolution spaces. Moreover, the computation of the oblique wavelet transform can be calculated using fast filter-bank algorithms with a complexity of the order of $\mathcal{O}(N^r)$ for a signal of length N^r.

This construction allows us more freedom in choosing the properties of the wavelet spaces. At the same time, we have enough control on the relationship between the multiresolution spaces and wavelet spaces to provide for fast-filtering algorithms and a reversible filter-bank. Hence, we have preserved some of the advantages of orthogonal and biorthogonal wavelet theory and yet enhanced the flexibility of wavelet theory to accommodate a wider variety of shapes and properties.

Organization of the paper. In Section 2, we introduce the notions of multi-scaling functions and multi-wavelets. We then show how to construct a multi-scaling function in 2.3. In Section 3, we define the concept of oblique multi-wavelets and show how to construct them. The implementation of the wavelet transform by vector-filter-banks is discussed in Section 4. The link with the new concept of perfect resolving vector-filter-banks is also discussed in this section. The special cases of biorthogonal and orthogonal multi-wavelets are then discussed in section 5. Finally in Section 6, as a special case of our theory, we derive a new construction of the biorthogonal wavelets of Cohen Deaubechies Feauveau, and an extension to those uni-wavelets.

Notation. In the following definitions, we will often use matrix-sequences, functions of matrices, and operators on matrices. However, these same definitions will be valid for vectors ($1 \times n$ matrices), and scalars (1×1 matrices). The signals that we consider belong to the finite energy space $\mathcal{L}_2(\mathcal{R})$, and l_2 will denote the discrete sequences of numbers with finite energy, while l_2^r will denote vector-sequences with finite energy. The symbol $*$ stands for three different types of convolutions:

If $\mathbf{F}(x)$ is an $m \times r$ matrix whose entries $\mathbf{F}_{i,j}(x)$ belong to $\mathcal{L}_2(\mathcal{R})$, and if $\mathbf{G}(x)$ is an $r \times n$ matrix whose entries $\mathbf{G}_{i,j}(x)$ belong to $\mathcal{L}_2(\mathcal{R})$, then the convolution $\mathbf{H}(x) = (\mathbf{F} * \mathbf{G})(x)$ between $\mathbf{F}(x)$ and $\mathbf{G}(x)$ is the $m \times n$ matrix-function $\mathbf{H}(x)$ whose entries $\mathbf{H}_{i,j}(x)$ are defined in terms of the convolution of the entries of $\mathbf{F}$, and $\mathbf{G}$ as

$$\mathbf{H}_{i,j}(x) := \sum_{l=1}^{l=r} \int_{\mathcal{R}} \mathbf{F}_{i,l}(\xi)\mathbf{G}_{l,j}(x-\xi)d\xi = \sum_{l=1}^{l=r} (\mathbf{F}_{i,l} * \mathbf{G}_{l,j})(x) \tag{1}$$

The convolution $\mathbf{C}(k) = (\mathbf{A} * \mathbf{B})(k)$ between the $m \times r$ matrix-sequence $\{\mathbf{A}(k)\}_{k \in \mathcal{Z}}$ and the $r \times n$ matrix-sequence $\{\mathbf{B}(k)\}_{k \in \mathcal{Z}}$ is the $m \times n$ matrix-sequence defined in terms of the convolution between the entries of $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{C}_{i,j}(k) := \sum_{l=1}^{l=r} \sum_{h \in \mathcal{Z}} \mathbf{A}_{i,l}(h)\mathbf{B}_{l,j}(k-h) = \sum_{l=1}^{l=r} (\mathbf{A}_{i,l} * \mathbf{B}_{l,j})(k) \tag{2}$$

The composite convolution $\mathbf{G}(x) = (\mathbf{A} * \mathbf{F})(x)$ between the $m \times r$ matrix-sequence $\mathbf{A}(k)$ and the $r \times n$ matrix-function $\mathbf{F}(x)$ is the $m \times n$ matrix-function whose entries $\mathbf{G}_{i,j}(x)$ are given by

$$\mathbf{G}_{i,j}(x) := \sum_{l=1}^{l=r} \sum_{k \in \mathcal{Z}} \mathbf{A}_{i,l}(k)\mathbf{F}_{l,j}(x-k) \tag{3}$$

Since matrix multiplication does not commute, we define $\mathbf{G}(x) = (\mathbf{F} * \mathbf{A})(x)$ as

$$\mathbf{G}_{i,j}(x) := \sum_{l=1}^{l=r} \sum_{k \in \mathcal{Z}} \mathbf{F}_{i,l}(x-k)\mathbf{A}_{l,j}(k) \tag{4}$$

The reflection of a matrix-function $\mathbf{F}(x)$ (resp., a sequence $\mathbf{B}(k)$) is the matrix-function $\mathbf{F}^{\vee}(x)$ (resp., the sequence $\mathbf{B}^{\vee}(k)$) given by

$$\mathbf{F}^{\vee}(x) := \mathbf{F}(-x), \quad x \in \mathcal{R} \tag{5}$$

$$\mathbf{B}^{\vee}(k) := \mathbf{B}(-k), \quad k \in \mathcal{Z} \tag{6}$$

The alternation (or modulation) $\tilde{\mathbf{B}}$ of a matrix sequence $\mathbf{B}(k)$ is defined to be

$$\tilde{\mathbf{B}}(k) := (-1)^k \mathbf{B}(k), \quad k \in \mathcal{Z} \tag{7}$$

The downsampling (or decimation) operator $\downarrow_2$ assigns to a matrix-sequence $\mathbf{B}(k)$, the sequence $\downarrow_2 [\mathbf{B}](k)$ that consists of the even samples of $\mathbf{B}$ only:

$$\downarrow_2 [\mathbf{B}](k) := \mathbf{B}(2k) \quad \forall k \in \mathcal{Z} \tag{8}$$

The upsampling operator $\uparrow_2$ assigns to a sequence of matrices $\mathbf{B}(k)$ a sequence of matrices $\uparrow_2 [\mathbf{B}](k)$ in which a zero has been inserted between two successive samples:

$$\uparrow_2 [\mathbf{B}](2k) := \mathbf{B}(k), \quad \forall k \in \mathcal{Z} \tag{9}$$

$$\uparrow_2 [\mathbf{B}](2k+1) := \mathbf{0}, \quad \forall k \in \mathcal{Z} \tag{10}$$

2 MULTI-SCALING FUNCTIONS AND MULTI-WAVELETS

2.1 Multiresolutions and multi-scaling functions

A multi-scaling function is a row vector of functions $\mathbf{\Phi} = (\varphi^1(x), \varphi^2(x), \varphi^3(x), ..., \varphi^r(x))$ that can be used to generate the multiresolution spaces

$$V_j := \left\{ \sum_{i=1}^{r} \sum_{k \in \mathcal{Z}} c^i(k) 2^{-j/2} \varphi^i \left(\frac{x-k}{2^j} \right)); c^i \in l_2 \right\} = \left\{ \sum_{k \in \mathcal{Z}} \mathbf{C}(k) \mathbf{\Phi}_j^T (x - 2^j k); \mathbf{C}(k) \in l_2^r \right\} \tag{11}$$

where $\mathbf{C}(k)$ is the vector $\mathbf{C} = (c^1(k), c^2(k), \ldots, c^r(k))$, T denotes transpose, and where the vector $\mathbf{\Phi}_j(x)$ is defined to be

$$\mathbf{\Phi}_j(x) = 2^{-j/2} \left(\varphi^1 \left(\frac{x}{2^j} \right), \varphi^2 \left(\frac{x}{2^j} \right), \ldots, \varphi^r \left(\frac{x}{2^j} \right) \right) \tag{12}$$

If we use the upsampling operator and the composite convolution (see definitions 3, 9), then the definition 11 can be rewritten into the simpler form

$$V_j = \left\{ \uparrow_{2^j} [\mathbf{C}] * \mathbf{\Phi}_j^T ; \mathbf{C} \in l_2^r \right\} \tag{13}$$

where the upsampling by a factor 2^j is denoted by $\uparrow_{2^j} = (\uparrow_2)^j$. The spaces V_j have to satisfy all the properties of multiresolutions. In particular, they must be closed and nested: $\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$, and they must also satisfy $\overline{\bigcup_{j \in \mathcal{Z}} V_j} = \mathcal{L}_2$, and $\bigcap_{j \in \mathcal{Z}} V_j = \{0\}$. From this nestedness property, it follows that the multi-scaling function must satisfy the vector two-scale equation

$$\frac{1}{\sqrt{2}} \mathbf{\Phi} \left(\frac{x}{2} \right) = \sum_{k \in \mathcal{Z}} \mathbf{\Phi}(x-k) \mathbf{H}_1(k) = \mathbf{\Phi} * \mathbf{H}_1 \tag{14}$$

where $\mathbf{H}_1(k)$ is an $r \times r$ matrix sequence called the generating sequence.

2.2 Wavelet spaces and multi-wavelets

The wavelet spaces $\{W_j\}_{j \in \mathcal{Z}}$ are complementary to the spaces V_j: $V_{j+1} + W_{j+1} = V_j$. We do not require orthogonality between the spaces V_j and W_j, so the spaces W_j are not necessarily orthogonal to each other, either. What is important to us is that the complementary wavelet spaces are generated by shifts and dilates of r wavelets $\mathbf{\Psi}(x) = (\psi^1(x), \psi^2(x), \cdots, \psi^r(x))$:

$$W_j = \left\{ \uparrow_{2^j} [\mathbf{D}] * \mathbf{\Psi}_j^T ; \mathbf{D} \in l_2^r \right\} \tag{15}$$

where

$$\mathbf{\Psi}_j(x) := 2^{-j/2} \mathbf{\Psi} \left(\frac{x}{2^j} \right) \tag{16}$$

The foregoing requirements on the wavelet spaces imply that the multi-wavelet must be a linear combination of shifts and dilates of the scaling function. Therefore, we are led to the relation

$$\mathbf{\Psi}_1 = \delta_1 * \mathbf{\Phi} * \mathbf{G} \tag{17}$$

where $\mathbf{G} = \mathbf{G}(k)$ is a matrix-sequence and where δ_1 serves to shift a sequence one term to the right, that is, $(\delta_1 * \mathbf{B})(k) = \mathbf{B}(k-1)$. Our requirements also imply that the set $\{2^{-j/2} \psi^i (2^{-j} x - k),\ i = 1, \ldots, r; (j,k) \in \mathcal{Z}^2\}$ form an unconditional Riesz basis of $\mathcal{L}_2$. Thus, any function $g \in \mathcal{L}_2(\mathcal{R})$ can be decomposed as

$$g = \sum_{j \in \mathcal{Z}} \uparrow_{2^j} [\mathbf{D}_j] * \mathbf{\Psi}_j^T (x) \tag{18}$$

A final requirement that we impose is that the coefficients $\mathbf{D}_j$ should be computable from g by a fast filter-bank algorithm, which consists of the repetitive application of a single procedure.

2.3 Construction of multi-scaling functions

If we take the Fourier transform of 14, we see that the Fourier transform $\hat{\mathbf{\Phi}}(f)$ of the multi-scaling function $\mathbf{\Phi}(x)$ and the Fourier transform $\hat{\mathbf{H}}_1(f)$ of the generating sequence $\mathbf{H}_1(k)$ are related by

$$\hat{\mathbf{\Phi}}(2f) = 2^{-1/2}\hat{\mathbf{\Phi}}(f)\hat{\mathbf{H}}_1(f) \tag{19}$$

From repeated applications of this relation, we obtain

$$\hat{\mathbf{\Phi}}(f) = \lim_{j\to\infty} \mathbf{1}_v 2^{-j/2}\hat{\mathbf{H}}_1\left(\frac{f}{2^j}\right)...\hat{\mathbf{H}}_1\left(\frac{f}{2^2}\right)\hat{\mathbf{H}}_1\left(\frac{f}{2}\right) \tag{20}$$

where $\mathbf{1}_v$ is the vector (1,1,...,1).

We can start from an appropriate sequence $\mathbf{H}_1(k)$ and construct a multi-scaling function $\hat{\mathbf{\Phi}}(f)$ from the infinite product above. However, for the infinite product to converge, the following condition is necessary:

$$\lim_{j\to\infty} 2^{-1/2}\hat{\mathbf{H}}_1\left(\frac{f}{2^j}\right) = 2^{-1/2}\hat{\mathbf{H}}_1(0) = \mathbf{I} \tag{21}$$

Thus, we will always require that $2^{-1/2}\hat{\mathbf{H}}_1(0) = \mathbf{I}$. For regularity purposes, we also require that

$$\hat{\mathbf{H}}_1(1/2) = \mathbf{0} \tag{22}$$

To ensure that the multi-scaling function and its shifts and dilates generate unconditional Riesz bases $\{2^{-j/2}\varphi^i(2^{-j}x-k),\ i=1,...r\}_{k\in\mathcal{Z}}$ for the spaces V_j, we need to impose additional conditions. For example, we require that $\sum_{k\in\mathcal{Z}} \|\mathbf{H}_1(k)\|\,|k| < \infty$, and that $\left\|\hat{\mathbf{H}}_1(f)\right\|^2 + \left\|\hat{\mathbf{H}}_1\left(f+\frac{1}{2}\right)\right\|^2 \leq 2$. These conditions, together with condition 21, also ensure that the multi-scaling function defined by the infinite product 20 is continuous and is an element of $\mathcal{L}_2^r(\mathcal{R})$.

3 Construction of oblique multi-wavelet bases

Whether we have arrived at a multiresolution by starting from a multi-scaling function or we have constructed it from a generating sequence, our next step in the construction of the multi-wavelet is to take the Fourier transform $\hat{\mathbf{H}}_1(f)$ of the generating sequence. We then find a matrix sequence $\mathbf{G}_1(k)$ whose Fourier transform $\hat{\mathbf{G}}_1(f)$ renders the matrix

$$\begin{bmatrix} \hat{\mathbf{H}}_1(f) & \hat{\mathbf{G}}_1(f) \\ \hat{\mathbf{H}}_1\left(f-\frac{1}{2}\right) & -\hat{\mathbf{G}}_1\left(f-\frac{1}{2}\right) \end{bmatrix} \tag{23}$$

invertible for almost all f. Having completed this construction, we are able to prove the following theorem:

THEOREM 3.1. *If $\mathbf{G}_1(k)$ is chosen so that 23 is invertible for almost all f, then the function $\mathbf{\Psi}_1(x)$ defined by*

$$\mathbf{\Psi}_1(x) = 2^{-1/2}\mathbf{\Psi}\left(\frac{x}{2}\right) = \delta_1 * \mathbf{\Phi} * \mathbf{G}_1 \tag{24}$$

is a multi-wavelet associated with the MRA generated by the multi-scaling function $\mathbf{\Phi}(x)$ whose two-scale sequence is $\mathbf{H}_1(k)$.

Our construction shows that there are infinitely many wavelet spaces associated with a given MRA, as depicted schematically in Figure 2. These multi-wavelets are not necessarily orthogonal, i.e., although the set

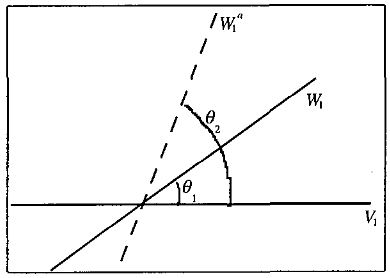

Figure 2: Schematic representation of two different wavelet spaces for a given MRA. if V_1 is represented by the horizontal line $\mathcal{R}$, and V_0 by the plane $\mathcal{R}^2$, then the slanted line W_1 complements V_1 to give V_0. Another complement is W_1^a, which is different from W_0, and is depicted by a dashed line.

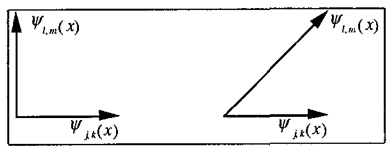

Figure 3: Schematic representation of two different bases for the same space. There are many bases that can generate a fixed wavelet space W_j depicted here schematically by the plane $\mathcal{R}^2$. Left panel represents an orthogonal basis. Right panel represents a nonorthogonal basis of the same space.

$\{2^{-j/2}\psi^i(2^{-j}x - k), i = 1, \dots, r\}_{(j,k)\in\mathcal{Z}^2}$ forms a basis of $\mathcal{L}_2(\mathcal{R})$, it is not necessarily an orthonormal one (see Figure 3). This is because we have not imposed any "Quadrature Mirror Filter"-type constraints. In fact, our multi-wavelets are not semiorthogonal either, in general (i.e., W_j is not orthogonal to V_j, see Fig. 1). Finally, unlike the Cohen-Daubechies-Feauveau biorthogonal wavelets, our construction is not necessarily associated with a pair of multiresolutions and wavelets. Thus, our construction produces all the well-known wavelet types as special cases, and it creates a new type of wavelet bases that we will call *oblique wavelets.* These wavelet bases are still associated with fast filter-banks algorithms as will be discussed in section 4.

Once a wavelet space has been found by the method described above, it is then possible to find other equivalent multi-wavelets by appropriate "linear combination". For example, if we choose a matrix-sequence $\mathbf{L}(k)$ such that both, $\mathbf{L}(k)$ and $\mathbf{L}^{-1}(k)$, satisfy the condition of Theorem 2.2 of,[2] then

$$\mathbf{\Psi}^{\approx} = \mathbf{\Psi}^b * \mathbf{L} \tag{25}$$

is an equivalent multi-wavelet generating the same wavelet spaces. This means that the both sets $\left\{(\psi^b)^i_{j,k}(x), i = 1, \dots r\right\}_{k\in\mathcal{R}}$ and $\left\{(\psi^{\approx})^i_{j,k}(x), i = 1, \dots r\right\}_{k\in\mathcal{R}}$ form unconditional bases for the same space W_j. The matrix-sequence $\mathbf{L}(k)$ can be chosen to obtain a desired multi-wavelet $\psi^{\approx}$ with some desired properties. This type of basis modification has been used to create semiorthogonal wavelet bases (also known as nonorthogonal wavelets[4,14]).

To prove Theorem 3.1, we need to show that for any vector $g = \mathbf{C}_0 * \mathbf{\Phi}^T$ in V_0, there are vectors v in V_1 and w in W_1 such that $v + w = g$ (here V_j and W_j are the spaces that are generated from $\mathbf{\Phi}(x)$ and $\mathbf{\Psi}(x)$ as in 13 and 15, respectively). For this purpose, we introduce the following two operators $\mathbf{P}_{V_1} : V_0 \rightarrow V_1$ and $\mathbf{P}_{W_1} : V_0 \rightarrow W_1$ that act on functions $g = \mathbf{C}_0 * \mathbf{\Phi}^T$ in V_0, and are defined by

$$\mathbf{P}_{V_1} g := \uparrow_2 \downarrow_2 [\mathbf{C}_0 * \mathbf{H}_2] * \mathbf{\Phi}_1^T \tag{26}$$

$$\mathbf{P}_{W_1} g := \uparrow_2 \downarrow_2 [\delta_{-1} * \mathbf{C}_0 * \mathbf{G}_2] * \mathbf{\Psi}_1^T \tag{27}$$

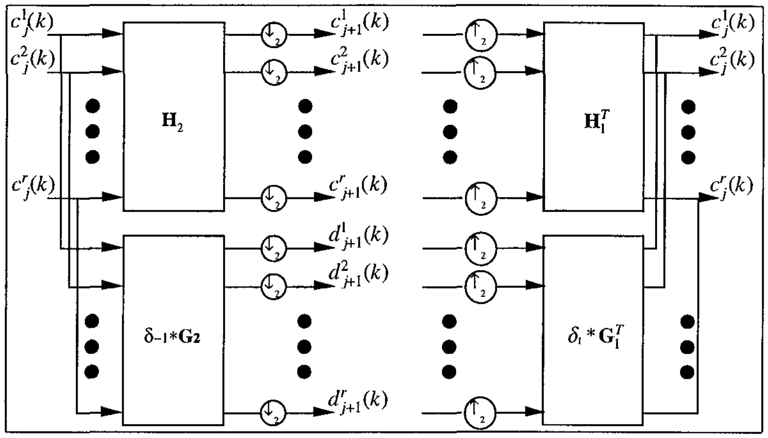

Figure 4: Perfect reconstruction filter-bank for computing the oblique multi-wavelet transform.

where $\mathbf{H}_2(k)$ and $\mathbf{G}_2(k)$ are defined in terms of $\mathbf{H}_1(k)$ and $\mathbf{G}_1(k)$ by the invertible system of equations

$$\begin{bmatrix} \hat{\mathbf{H}}_1(f) & \hat{\mathbf{G}}_1(f) \\ \hat{\mathbf{H}}_1\left(f-\frac{1}{2}\right) & -\hat{\mathbf{G}}_1\left(f-\frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{H}}_2^T(f) \\ \hat{\mathbf{G}}_2^T(f) \end{bmatrix} = \begin{bmatrix} 2\mathbf{I} \\ \mathbf{0} \end{bmatrix} \tag{28}$$

and where $\mathbf{I}$ is the $r \times r$ identity matrix, and $\mathbf{0}$ is the $r \times r$ zero matrix. We have the following theorem:

THEOREM 3.2. *The operators $\mathbf{P}_{V_1}$ and $\mathbf{P}_{W_1}$ are projectors (not necessarily orthogonal projectors); moreover, for any function $g \in V_0$, the following decomposition holds:*

$$g = \mathbf{P}_{V_1} g + \mathbf{P}_{W_1} g. \tag{29}$$

Theorems 3.1 and 3.2 provide the constructive approach for creating oblique multi-wavelets and for expanding any element $g \in \mathcal{L}_2(\mathcal{R})$ in terms of the oblique multi-wavelet basis.

4 Fast filter-bank algorithms

4.1 Perfect reconstruction filter-banks and perfect resolving filter-banks

From our construction of oblique wavelets, we know that any function $g \in \mathcal{L}_2(\mathcal{R})$ can be decomposed into a low resolution approximation $g_J \in V_J$ and the sum of the error terms in the spaces $\{W_j\}_{j \geq J}$

$$g = \uparrow_{2^J} [\mathbf{C}_J] * \mathbf{\Phi}_J^T + \sum_{j=J}^{-\infty} \uparrow_{2^j} [\mathbf{D}_j] * \mathbf{\Psi}_j^T \tag{30}$$

In practice, $g(x)$ belongs to a multiresolution space, e.g., $g = \mathbf{C}_0 * \mathbf{\Phi}^T$. In this case, the procedure for finding the coefficients $\mathbf{C}_J(k)$ and $\{\mathbf{D}_j(k)\}_{j=J,\ldots,1}$ from the coefficients $\mathbf{C}_0(k)$ can be obtained by a fast vector-filter-bank algorithm depicted in the left part of Figure 4. Similarly, the procedure for finding the coefficients $\mathbf{C}_0(k)$ from the knowledge of $\mathbf{C}_J(k)$ and $\{\mathbf{D}_j(k)\}_{j=J,\ldots,1}$ can be obtained by a vector-filter-bank algorithm depicted in the right part of Figure 4. The filters $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{G}_1$, and $\mathbf{G}_2$ of Figure 4 are the filters defined in the previous section and that are associated with the oblique wavelets. The difference between these vector-filter-banks and those used in the usual *uni-wavelet* algorithms is that the vector filter-banks have $r > 1$ inputs and $r > 1$ outputs instead of only one (compare Figures 4 and 7). The decomposition and reconstruction filter-banks together constitute the perfect reconstruction filter-bank structure.

4.2 Perfect resolving filter-bank

Another important filter-bank structure that we now introduce is the the dual filter-bank that we will call the *perfect resolving filter-bank* depicted in Figure 5. We can prove that a perfect reconstruction filter-bank can be built from a perfect resolving filter-bank, and vice versa. In particular, the filters in a perfect reconstruction filter-bank must satisfy the matrix equation

$$\begin{bmatrix} \hat{\mathbf{H}}_2(f) & \hat{\mathbf{G}}_2(f) \\ \hat{\mathbf{H}}_2\left(f-\frac{1}{2}\right) & -\hat{\mathbf{G}}_2\left(f-\frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{H}}_1^T(f) \\ \hat{\mathbf{G}}_1^T(f) \end{bmatrix} = \begin{bmatrix} 2\mathbf{I} \\ \mathbf{0} \end{bmatrix} \tag{31}$$

From 28, we can show that the system of equations above is satisfied by the filters used in the construction of the oblique multi-wavelets. This system of equations together with the duality between the reconstruction and resolving filter-banks, allows us to prove that the system 31 is equivalent to the following system of equations:

$$\hat{\mathbf{H}}_1^T(f)\hat{\mathbf{H}}_2(f) + \hat{\mathbf{H}}_1^T\left(f-\frac{1}{2}\right)\hat{\mathbf{H}}_2\left(f-\frac{1}{2}\right) = 2\mathbf{I} \tag{32}$$

$$\hat{\mathbf{G}}_1^T(f)\hat{\mathbf{G}}_2(f) + \hat{\mathbf{G}}_1^T\left(f-\frac{1}{2}\right)\hat{\mathbf{G}}_2\left(f-\frac{1}{2}\right) = 2\mathbf{I} \tag{33}$$

$$\hat{\mathbf{G}}_1^T(f)\hat{\mathbf{H}}_2(f) - \hat{\mathbf{G}}_1^T\left(f-\frac{1}{2}\right)\hat{\mathbf{H}}_2\left(f-\frac{1}{2}\right) = \mathbf{0} \tag{34}$$

$$\hat{\mathbf{H}}_1^T(f)\hat{\mathbf{G}}_2(f) - \hat{\mathbf{H}}_1^T\left(f-\frac{1}{2}\right)\hat{\mathbf{G}}_2\left(f-\frac{1}{2}\right) = \mathbf{0} \tag{35}$$

This equivalence between the two systems of equations is well-known in the case of uni-wavelets,[12,14] and has been obtained by algebraic manipulations. Because matrix multiplication does not commute in general, it is difficult to show this equivalence by algebraic manipulations in the multi-wavelet case. However, the filter-bank duality principle allows us to prove the equivalence between the two systems of equations in the multi-wavelet case. This equivalence is important in the construction of biorthogonal wavelets of Cohen, Deaubechies and Feauveau. Implicitly, it is the second system of equations that they used in their construction of biorthogonal wavelets. We will return to this issue in section 6.

5 biorthogonal and orthogonal multi-wavelets bases

A special case of oblique multi-wavelets is that of *biorthogonal multi-wavelets.* Biorthogonal multi-wavelets are the generalization of the biorthogonal wavelets of Cohen, Daubechies and Feauveau[5] (see next section). In the biorthogonal multi-wavelet case, the oblique projections $\mathbf{P}_{V_1}$ and $\mathbf{P}_{W_1}$ are defined with respect to another pair of multiresolution and wavelet spaces $\{\mathring{V}_j\}_{j\in\mathcal{Z}}$ and $\{\mathring{W}_j\}_{j\in\mathcal{Z}}$ that are generated by multi-scaling and multi-wavelets $\mathring{\mathbf{\Phi}}(x)$ and $\mathring{\mathbf{\Psi}}(x)$, respectively. In particular, $\mathbf{P}_{V_1} = \mathbf{P}_{V_1 \perp \mathring{V}_1}$ is the projection on the space V_1 in a direction orthogonal to $\mathring{V}_1$, and $\mathbf{P}_{W_1} = \mathbf{P}_{W_1 \perp \mathring{W}_1}$ is the projection on the space W_1 in a direction orthogonal to $\mathring{W}_1$ (see Figure 6) . The orthogonal case is then simply the case in which $\mathring{V}_j = V_j$ and $\mathring{W}_j = W_j$. If $\mathring{\mathbf{\Phi}}(x)$ is a multi-scaling function for $\mathring{V}_0$, and if $\mathring{\mathbf{\Psi}}(x)$ is a multi-wavelet for $\mathring{W}_0$, then the oblique projections $P_{V_1 \perp \mathring{V}_1} g = \uparrow_2 [\mathbf{C}_1] * \mathbf{\Phi}_1^T$, and

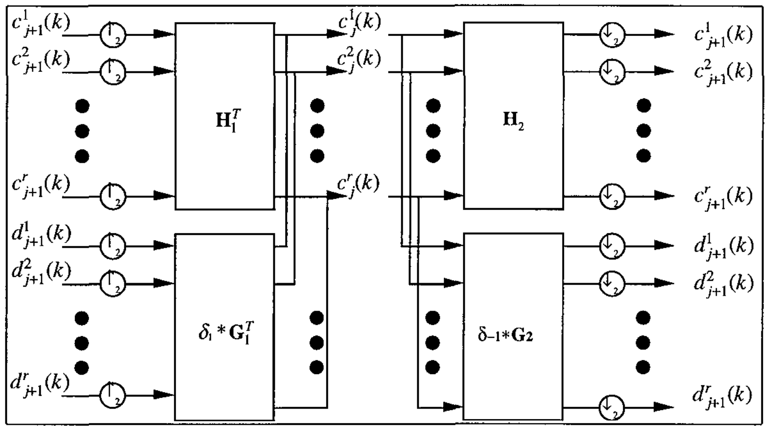

Figure 5: Perfect resolving filter-bank. Two transposed vector-sequences $\mathbf{C}_{j+1} = (c^1_{j+1}(k), \ldots, c^r_{j+1}(k))$ and $\mathbf{D}_{j+1} = (d^1_{j+1}(k), \ldots, d^r_{j+1}(k)$ are mixed together to form a single transposed vector-sequence ($\mathbf{C}_j = c^1_j(k), \ldots, c^r_j(k)$) (left filter-bank pairs). The two sequences $\mathbf{C}_{j+1}$ and $\mathbf{D}_{j+1}$ can then be resolved from $\mathbf{C}_j$ by the right pair of filter-banks. The whole structure is the perfect resolving filter-bank.

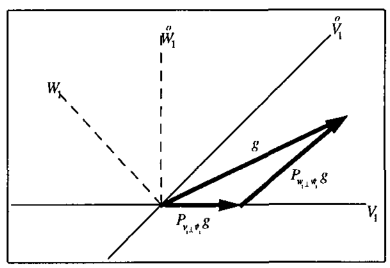

Figure 6: Schematic representation of oblique projections in the biorthogonal case.

$P_{W_1 \perp \mathring{W}_1} g = \uparrow_2 [\mathbf{D}_1] * \mathbf{\Phi}_1^T$ of a function $g = \mathbf{C}_0 * \mathbf{\Phi}^T(x)$ are given by[2]

$$\mathbf{C}_1 = \downarrow_2 [\mathbf{C}_0 * \mathbf{H}_2] \tag{36}$$

$$\mathbf{D}_1 = \downarrow_2 [\delta_1 * \mathbf{C}_0 * \mathbf{G}_2] \tag{37}$$

where the two matrix filters $\mathbf{H}_2(k)$ and $\mathbf{G}_2(k)$ are given in terms of the cross-correlation matrix sequence $\mathbf{X}_{i,j} = \left\langle \varphi^i(x), \varphi^j(x-k) \right\rangle$, and in terms of the generating sequences $\mathring{\mathbf{H}}_1(k)$, and $\mathring{\mathbf{G}}_1(k)$ of $\mathring{\mathbf{\Phi}}(x)$ and $\mathring{\mathbf{\Psi}}(x)$, respectively :

$$\mathbf{H}_2 = \mathbf{X} * \mathring{\mathbf{H}}_1 * \uparrow_2 \left[\mathbf{X}^{-1}\right] \tag{38}$$

$$\mathbf{G}_2 = \delta_{-1} * \mathbf{X} * \mathring{\mathbf{G}}_1^{\vee} * \uparrow_2 \left[\left(\downarrow_2 \left[\mathbf{H}_1^T * \mathbf{X} * \mathring{\mathbf{H}}_1^{\vee} \right] \right)^{-1} \right] \tag{39}$$

where we have used the notation $\mathbf{A}^{-1}$ to denote the convolution inverse of $\mathbf{A}(k)$. Specifically, $\mathbf{A}^{-1}(k)$ is the inverse Fourier transform of $\left(\hat{\mathbf{A}}(f)\right)^{-1}$.

6 A new construction for the Cohen-Daubechies-Feauveau (CDF) biorthogonal wavelets

The theory of CDF is a viewed as a special case of of biorthogonal multi-wavelets. However, because we are dealing with scalar equations instead of equations in which the variables are matrices, we get some simplifications. For example, we can start from two arbitrarily chosen multiresolutions $\{V_j\}_{j\in\mathcal{Z}}$ and $\{\mathring{V}_j\}_{j\in\mathcal{Z}}$, and construct biorthogonal wavelets. This works because, unlike the multi-wavelet case, all products and convolution products commute in the scalar case. In particular, Let $\phi(x)$ and $\mathring{\phi}(x)$ be two scaling functions generating the spaces V_j and $\mathring{V}_j$, respectively. We do not require the scaling functions to be orthogonal, i.e., the sets $\{2^{-j/2}\phi(2^jx-k),\ k\in\mathcal{Z}\}$ and $\{2^{-j/2}\mathring{\phi}(2^jx-k),\ k\in\mathcal{Z}\}$ are bases of V_j and $\mathring{V}_j$ respectively, although they are not necessarily orthogonal. Moreover, we do not require the basis of V_j to be biorthogonal to the basis of $\mathring{V}_j$, i.e., we do not require $\left\langle \phi(x), \mathring{\phi}(x-k)\right\rangle \neq \delta(k)$. For this case, as long as the "angle" between the two spaces is not $\pi/2$, it is always possible to find wavelets that generate the associated wavelet spaces $\{W_j\}_{j\in\mathcal{Z}}$ and $\{\mathring{W}_j\}_{j\in\mathcal{Z}}$ such that any function $g\in V_0$ can be decomposed as

$$g = \mathbf{P}_{V_1 \perp \mathring{V}_1} g + \mathbf{P}_{W_1 \perp \mathring{W}_1} g \tag{40}$$

In particular, we can choose the two basic wavelets $\psi^b(x)$ and $\mathring{\psi}^b(x)$ defined by

$$\psi_1^b \;=\; g_1^b * \phi_0 \quad , \quad \mathring{\psi}_1^b \;=\; \mathring{g}_1^b * \mathring{\phi}_0 \tag{41}$$

with

$$\begin{aligned} g_1 &= \delta_1 * \tilde{\chi} * \tilde{\mathring{h}}_1^{\vee} \\ \mathring{g}_1 &= \delta_1 * \tilde{\chi}^{\vee} * \tilde{h}_1^{\vee} \end{aligned} \tag{42}$$

where $\chi = (\phi_0 * \mathring{\phi}_0^{\vee})(k)$ is the sampled cross-correlation function between ϕ and $\mathring{\phi}$, the symbol $\tilde{}$ denotes the alternation operator (see definition 7), and where h_1 and $\mathring{h}_1$ are the two-scale generating sequences for ϕ and $\mathring{\phi}$, respectively. The perfect reconstruction filter-bank that is associated with this system of biorthogonal wavelets is depicted in Figure 7.

Clearly, the basic wavelets obtained by our construction do not satisfy the biorthogonality condition between the bases, i.e., $\left\langle \psi_{j,k}^b(x), \mathring{\psi}_{j,k}^b(x)\right\rangle \neq \delta_{jklm}$. However, if we define the equivalent wavelet $\mathring{\psi}^{CDF}(x) := \chi^{-1} * \mathring{\psi}^b$, then we have a system of biorthogonal wavelets

$$\left\langle \psi_{j,k}^b(x), \mathring{\psi}_{j,k}^{CDF}(x)\right\rangle = \delta_{jklm} \tag{43}$$

and we recover the CDF biorthogonal wavelets.

This last modification, however, is not necessary for the theory of wavelets. Instead, any other equivalent wavelet $\psi^{\approx} = p * \psi^b$ can be constructed by an appropriate admissible sequence $p(k)^{4,1}$ that suits our needs. In all cases, the multiresolution and wavelet spaces are not modified. Therefore, the projections $\mathbf{P}_{V_1 \perp \mathring{V}_1} g$ and $\mathbf{P}_{W_1 \perp \mathring{W}_1} g$ are also unchanged. Only the coefficients $c_j(k)$ are affected by these changes, because they depend on the chosen basis. This is reflected in the filters of the filter-bank structure, which depends on the wavelet basis that is chosen.

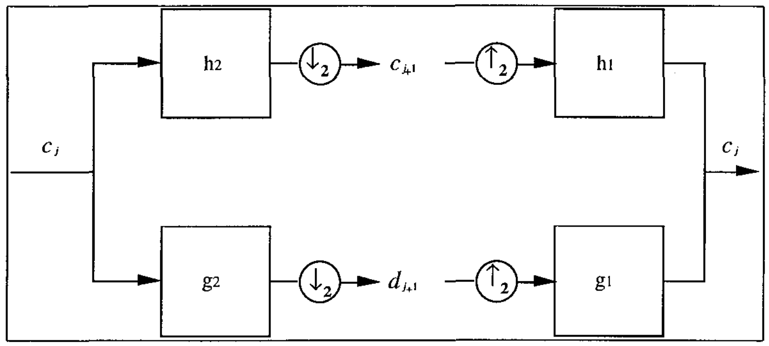

Figure 7: Perfect reconstruction filter-bank for computing the biorthogonal wavelet transform.

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