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## OBSERVABILITY AND CONTROLLABILITY FOR SMOOTH NONLINEAR SYSTEMS\*

A. J. VAN DER SCHAFT†

**Abstract.** The definition of a smooth nonlinear system as proposed recently by Willems, is elaborated as a natural generalization of the more common definitions of a smooth nonlinear input–output system. Minimality for such systems can be defined in a very direct geometric way, and already implies a usual notion of observability, namely, local weak observability. As an application of this theory, it is shown that observable nonlinear Hamiltonian systems are necessarily controllable, and vice versa.

**1. Introduction.** In the last decade there has been important work on a differential geometric approach to nonlinear input state–output systems, which in local coordinates have the form

$$(1.1) \quad \dot{x} = g(x, u), \quad y = h(x),$$

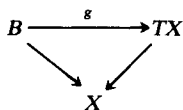
where  $x$  is the *state* of the system,  $u$  is the *input* and  $y$  the *output* (for a survey see Brockett [3]). Most of the attention has been directed to the formulation in this context of fundamental system theoretic concepts like controllability, observability, minimality and realization theory. Some basic papers are, for instance, Hermann–Krener [6], Sussmann [12], and recently Jakubczyk [9].

In spite of some very natural formulations and elegant results which have been achieved, there are certain disadvantages in the whole approach, from which we summarize the following points.

a) Normally the equations

$$(1.2) \quad \dot{x} = g(x, u)$$

are interpreted as a family of vector fields on a manifold parametrized by  $u$ ; i.e., for every fixed  $\bar{u}$ ,  $g(\cdot, \bar{u})$  is a globally defined vector field. As noted already by Brockett [4], Takens [15] and Willems [17] there are some serious objections to this setting. In fact, the last author proposes another framework by looking at (1.2) as a coordinatization of



where  $B$  is a *fiber bundle* above the state space manifold  $X$  and the fibers of  $B$  are the *state dependent* input spaces, while  $TX$  is as usual the tangent bundle of  $X$  (the possible velocities at every point of  $X$ ).

b) The usual definition of *observability* for this kind of system (cf. [6]) has some drawbacks. In fact, observability is defined as *distinguishability*; i.e., for every  $x_1$  and  $x_2$  (elements of  $X$ ) there exists a *certain* input function (in principle dependent on  $x_1$  and  $x_2$ ) such that the output function of the system starting from  $x_1$  under the influence of this input function is different from the output function of the system starting from  $x_2$  under the influence of this same input function. Of course, from a practical point of view this notion of observability is not very useful, and also is not in accord with the usual definition of observability or reconstructibility for general systems (cf. [10]).

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Hence, despite the work of Sussmann [13] on *universal* inputs, i.e., input functions which distinguish between every two states  $x_1$  and  $x_2$ , this approach remains unsatisfactory.

c) In the class of nonlinear systems (1.1) *memoryless* systems

$$(1.3) \quad y = h(u)$$

are not included! Of course, one could extend the system (1.1) to the form

$$(1.4) \quad \dot{x} = g(x, u), \quad y = h(x, u),$$

but this gives, if one wants to regard observability as distinguishability, the following rather complicated notion of observability. As can be seen from [2], distinguishability of (1.4) with  $y \in \mathbb{R}^p$ ,  $u \in \mathbb{R}^m$  and  $x \in \mathbb{R}^n$  is equivalent to distinguishability of

$$(1.5) \quad \dot{x} = g(x, u), \quad \bar{y} = \bar{h}(x)$$

where  $\bar{h} : \mathbb{R}^n \rightarrow (\mathbb{R}^p)^{\mathbb{R}^m}$  is defined by  $\bar{h}(x)(u) = h(x, u)$ .

Checking the Lie algebra conditions for distinguishability as described in [6] for the system (1.5) is not very easy!

d) As noted by Willems [17], in a description of a physical system ("physical" interpreted in a broad sense) it is often not clear how to distinguish a priori between inputs and outputs. Especially in the case of a nonlinear system, it could be possible that a separation of what we shall call *external variables* in input variables and output variables should be interpreted only *locally*. An example is the (nearly) ideal diode given by the  $I$ - $V$  characteristic in Fig. 1. For  $I < 0$  it is natural to regard  $I$  as the input and  $V$  as the output, while for  $V > 0$  it is natural to see  $V$  as the input and  $I$

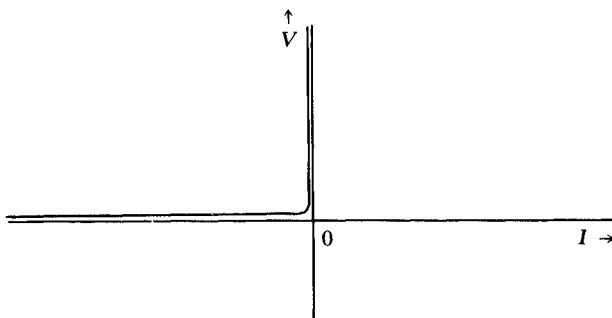


FIG. 1

as the output. Around 0 an input-output description should be given in the scattering variables  $(I - V, I + V)$ . Moreover, in the case of nonlinear systems it can happen that a *global* separation of the external variables in inputs and outputs is simply not possible! This results in a definition of a system which is a generalization of the usual input-output framework. It appears that various notions like the definitions of autonomous (i.e., without inputs), memoryless, time-reversible, Hamiltonian and gradient systems are very natural in this framework (see [16], [17]).

The organization of this paper is as follows. In § 2 we give the definition of a nonlinear system as proposed in [17], and give some connections with the more usual input-output settings. In § 3 we define *minimality* of such a system and derive *local* conditions from this global definition. It is very surprising that this results in the same

kind of conditions as given in recent papers on nonlinear disturbance decoupling; see [7], [8] and especially the setting proposed by Nijmeijer [11]. These local conditions imply local weak observability for systems which locally can be represented in an input–output form without a feedthrough term. Finally, in §4 the definition of minimality is tested in the case of Hamiltonian systems as defined in [16], and we can derive the theorem that an “observable” full Hamiltonian system is necessarily “controllable”, and vice versa. Surprisingly, it appears that this need not hold for gradient systems!

**2. Definition of a smooth nonlinear system.** As proposed in [17] and argued in [16], [17], smooth (say  $C^\infty$ ) systems can be represented in the commutative diagram

$$(2.1) \quad \begin{array}{ccc} B & \xrightarrow{f} & TX \times W \\ & \searrow \pi & \swarrow \pi_X \\ & & X \end{array}$$

where (all spaces are smooth manifolds)  $B$  is a fiber bundle above  $X$  with projection  $\pi$ ,  $TX$  is the tangent bundle of  $X$ ,  $\pi_X$  the natural projection of  $TX$  on  $X$  and  $f$  is a smooth map.  $W$  is the space of external variables (think of the inputs *and* the outputs).  $X$  is the state space and the fiber  $\pi^{-1}(x)$  in  $B$  above  $x \in X$  represents the space of inputs (to be seen initially as *dummy* variables), which is state dependent (think of forces acting at different points of a curved surface).

This definition formalizes the idea that at every point  $x \in X$  we have a set of possible velocities (elements of  $TX$ ) and possible values of the external variables (elements of  $W$ ), namely the space

$$f(\pi^{-1}(x)) \subset T_x X \times W.$$

We denote the system (2.1) by  $\Sigma(X, W, B, f)$ . It is easily seen that in local coordinates  $x$  for  $X$ ,  $v$  for the fibers of  $B$ ,  $w$  for  $W$ , and with  $f$  factored in  $f = (g, h)$ , the system is given by

$$(2.2) \quad \dot{x} = g(x, v), \quad w = h(x, v).$$

Of course one should ask oneself how this kind of system formulation is connected with the usual input–output setting. In fact, by adding more and more assumptions successively to the very general formulation (2.1) we shall distinguish among three important situations, of which the last is equivalent to the “usual” interpretation of system (1.1).

(i) Suppose the map  $h$  restricted to the fibers of  $B$  is an *immersive* map into  $W$  (this is equivalent to asking that the matrix  $\partial h/\partial v$  be injective). Then:

LEMMA 2.1. *Let  $h$  restricted to the fibers of  $B$  be an immersion into  $W$ . Let  $(\bar{x}, \bar{v})$  and  $\bar{w}$  be points in  $B$  and  $W$  respectively such that  $h(\bar{x}, \bar{v}) = \bar{w}$ . Then locally around  $(\bar{x}, \bar{v})$  and  $\bar{w}$  there are coordinates  $(x, v)$  for  $B$  (such that  $v$  are coordinates for the fibers of  $B$ ), coordinates  $(w_1, w_2)$  for  $W$  and a map  $\tilde{h}$  such that  $h$  has the form*

$$(2.3) \quad (x, v) \mapsto (w_1, w_2) = (\tilde{h}(x, v), v).$$

*Proof.* The lemma follows from the implicit function theorem.

Hence *locally* we can interpret a part of the external variables, i.e.,  $w_1$ , as the outputs, and a complementary part, i.e.,  $w_2$ , as the inputs! When we denote  $w_1$  by  $y$

and  $w_2$  by  $u$ , then system (2.2) has the form (of course only locally)

$$(2.4) \quad \dot{x} = y(x, u), \quad y = \tilde{h}(x, u).$$

(ii) Now we not only assume that  $\partial h/\partial v$  is injective, which results in a local input-output parametrization (2.4), but we also assume that the output set denoted by  $Y$  is globally defined. Moreover, we assume that  $W$  is a fiber bundle above  $Y$ , which we will call  $p: W \rightarrow Y$ , and that  $h$  is a bundle morphism (i.e., maps fibers of  $B$  into fibers of  $W$ ). Then:

LEMMA 2.2. *Let  $h: B \rightarrow W$  be a bundle morphism, which is a diffeomorphism restricted to the fibers. Let  $\bar{x} \in X$  and  $\bar{y} \in Y$  be such that  $h(\pi^{-1}(\bar{x})) = p^{-1}(\bar{y})$ . Take coordinates  $x$  around  $\bar{x}$  for  $X$  and coordinates  $y$  around  $\bar{y}$  for  $Y$ . Let  $(\bar{x}, \bar{v})$  be a point in the fiber above  $\bar{x}$  and let  $(\bar{y}, \bar{u})$  be a point in the fiber above  $\bar{y}$  such that  $h(\bar{x}, \bar{v}) = (\bar{y}, \bar{u})$ . Then there are local coordinates  $v$  around  $\bar{v}$  for the fibers of  $B$ , coordinates  $u$  around  $\bar{u}$  for the fibers of  $W$  and a map  $\tilde{h}: X \rightarrow Y$  such that  $h$  has the form*

$$(2.5) \quad (x, v) \xrightarrow{h} (y, u) = (\tilde{h}(x), v).$$

*Proof.* Choose a locally trivializing chart  $(0, \varphi)$  of  $W$  around  $\bar{y}$ . Then  $\varphi: p^{-1}(0) \rightarrow 0 \times U$ , with  $U$  the standard fiber of  $W$ . Take local coordinates  $u$  around  $\bar{u} \in U$ . Then  $(y, u)$  forms a coordinate system for  $W$  around  $(\bar{y}, \bar{u})$ . Because  $h$  is a bundle morphism,  $h$  has the form

$$(x, \tilde{v}) \xrightarrow{h} (y, u) = (\tilde{h}(x), h'(x, \tilde{v})).$$

where  $(x, \tilde{v})$  is a coordinate system for  $B$  around  $(\bar{x}, \bar{v})$ . Now adapt this last coordinate system by defining

$$v = (h')^{-1}(x, u) \quad \text{with } x \text{ fixed.}$$

Because  $h$  restricted to the fibers is a diffeomorphism,  $v$  is well defined and  $(x, v)$  forms a coordinate system for  $B$  in which  $h$  has the form

$$(x, v) \xrightarrow{h} (y, u) = (\tilde{h}(x), u). \quad \square$$

Hence under the conditions of Lemma 2.2 our system is locally (around  $\bar{x} \in X$  and  $\bar{y} \in Y$ ) described by

$$(2.6) \quad \dot{x} = g(x, u), \quad y = \tilde{h}(x).$$

This input-output formulation is essentially the same as the one proposed by Brockett [4] and Takens [15], who take the input spaces as the fibers of a bundle above a globally defined output space  $Y$ . In fact, this situation should be regarded as the normal setting for nonlinear control systems.

(iii) Take the same assumptions as in (ii) and assume moreover that  $W$  is a trivial bundle, i.e.,  $W = Y \times U$ , and that  $B$  is a trivial bundle, i.e.,  $B = X \times V$ . Because  $h$  is a diffeomorphism on the fibers, we can identify  $U$  and  $V$ . In this case the output set  $Y$  and the input set  $U$  are globally defined, and the system is described by

$$(2.7) \quad \dot{x} = g(x, u), \quad y = \tilde{h}(x),$$

where for each fixed  $\bar{u}$ ,  $g(\cdot, \bar{u})$  is a globally defined vector field on  $X$ . This is the "usual" interpretation of (1.1).

*Remarks on (i).*

1. When  $h$  restricted to the fibers of  $B$  is *not* an immersion we have a situation where we could speak of “hidden inputs”. In fact, in this case there are variables in the fibers of  $B$  which can affect the internal state behavior via the equation  $\dot{x} = g(x, v)$  but which cannot be directly identified with some of the external variables.

2. The splitting of the external variables into inputs and outputs as described in Lemma 2.1 is of course by no means unique! This fact has interesting implications, even in the linear case, which we shall not pursue further here.

*Remarks on (ii).*

1. From Lemma 2.2 it is clear that the coordinatization of the fibers of the bundle  $W$  uniquely determines, via  $h$ , the coordinatization of the fibers of  $B$ . It should be remarked that a coordinatization of the fibers of  $W$  is locally equivalent to the existence of an (integrable) *connection* on the bundle  $W$ , and that one coordinatization is linked with another by what is essentially an output feedback transformation, i.e., a bundle isomorphism from  $W$  into itself. Again we will not comment further on this point.

2. A beautiful example of this kind of system is the Lagrangian system (see Takens [15]). Here the output space is equal to the configuration space  $Q$  of a mechanical system. The state space  $X$  is the configuration space with the velocity space, so  $X = TQ$ . The space  $W$  is equal to  $T^*Q$  (the cotangent bundle of  $Q$ ), with the fibers of  $T^*Q$  representing the external forces. When we denote the natural projection of  $TQ$  on  $Q$  by  $\rho$ , then  $B$  is just  $\rho^*T^*Q$  (the pullback bundle via  $\rho$ ). Now given a function  $L : TQ \rightarrow \mathbb{R}$  (called the Lagrangian) we can construct a symplectic form  $d(\partial L / \partial \dot{q}) \wedge dq$  (with  $(q, \dot{q})$  coordinates for  $TQ$ ) on  $TQ$  which uniquely determines a map  $g : B \rightarrow TTQ$  (cf. [15]). Finally, in coordinates the system is given by

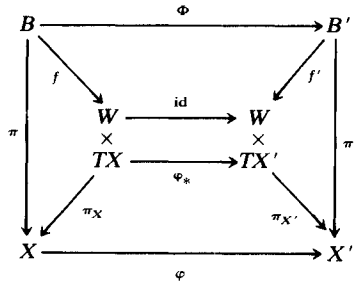
$$(2.8) \quad \ddot{q} = F(q, \dot{q}) + \sum_j u_j Z_j(q, \dot{q}), \quad y = q,$$

with the vector fields  $F(q, \dot{q})$  and  $Z_j(q, \dot{q})$  satisfying certain conditions. Moreover the vector fields  $Z_j$  commute, i.e.,  $[Z_i, Z_j] = 0$  for all  $i, j$ , a fact which has a very interesting interpretation (cf. [5], [15]).

*Remark on (iii).* Most cases where  $B$  can be taken as trivial are generated by a space  $X$  such that  $TX$  is a trivial bundle. For instance, when  $X$  is a Lie group  $TX$  is automatically trivial.

**3. Minimality and observability**

**3.1. Minimality.** We want to give a definition of minimality for a general (smooth) nonlinear system



**DEFINITION 3.1** (see [16]). Let  $\Sigma(X, W, B, f)$  and  $\Sigma'(X', W, B', f')$  be two smooth systems. Then we say  $\Sigma' \leq \Sigma$  if there exist surjective submersions  $\varphi : X \rightarrow X'$ ,  $\Phi : B \rightarrow B'$

such that the diagram

$$(3.1) \quad \begin{array}{ccc} & f & \\ B & \xrightarrow{\quad} & TX \times W \\ & \searrow & \swarrow \\ & X & \end{array}$$

commutes.

$\Sigma$  is called *equivalent* to  $\Sigma'$  (denoted  $\Sigma \sim \Sigma'$ ) if  $\varphi$  and  $\Phi$  are diffeomorphisms.

We call  $\Sigma$  *minimal* if  $\Sigma' \leq \Sigma \Rightarrow \Sigma' \sim \Sigma$ .

*Remark 1.* This definition formalizes the idea that we call  $\Sigma'$  *less complicated* than  $\Sigma$  ( $\Sigma' \leq \Sigma$ ) if  $\Sigma'$  consists of a set of trajectories in the state space, smaller than the set of trajectories of  $\Sigma$ , but which generates the same *external behavior*. (The external behavior  $\Sigma_e$  of  $\Sigma(X, W, B, f)$  consists of the possible functions  $w : \mathbb{R} \rightarrow W$  generated by  $\Sigma(X, W, B, f)$ . Hence, when we define

$$\Sigma := \{(x, w) : \mathbb{R} \rightarrow X \times W \mid x \text{ absolutely continuous and } (\dot{x}(t), w(t)) \in f(\pi^{-1}(x(t))) \text{ a.e.}\},$$

then  $\Sigma_e$  is just the projection of  $\Sigma$  on  $W^{\mathbb{R}}$ .)

*Remark 2.* Notice that we only formalize the *regular* case by asking that  $\Phi$  and  $\varphi$  be surjective as well as submersive. In fact we could, for instance, allow that at isolated points  $\varphi$  or  $\Phi$  are not submersive. However, we will at this time not go into this problem, and we will treat only the regular case as described in Definition 3.1.

*Remark 3.* Notice that  $\Sigma_1 \leq \Sigma_2$  and  $\Sigma_2 \leq \Sigma_1$  need not imply  $\Sigma_1 \sim \Sigma_2$ . This fact leads to very interesting problems which we will not pursue further at this time.

Of course, Definition 3.1 is an elegant but rather abstract definition of minimality. From a differential geometric point of view it is very natural to see what these conditions of commutativity mean *locally*. In fact, we will see in Theorem 3.7 that locally these conditions of commutativity do have a very direct interpretation. But first we have to state some preparatory lemmas and theorems.

Let us look at (3.1). Because  $\Phi$  is a submersion it induces an involutive distribution  $D$  on  $B$  given by

$$D := \{Z \in TB \mid \Phi_* Z = 0\}$$

(the foliation generated by  $D$  is of the form  $\Phi^{-1}(c)$  with  $c$  constant). In the same way  $\varphi$  induces an involutive distribution  $E$  on  $X$ . Now the information in the diagram (3.1) is contained in three subdiagrams (we assume  $f = (g, h)$  and  $f' = (g', h')$ ):

$$\begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\quad \Phi \quad} & B' \\ h \downarrow & & \downarrow h' \\ W & \xleftarrow{\quad \text{id} \quad} & W \\ \\ B & \xrightarrow{\quad \Phi \quad} & B' \\ \pi \downarrow & & \downarrow \pi' \\ X & \xrightarrow{\quad \varphi \quad} & X' \\ \\ B & \xrightarrow{\quad \Phi \quad} & B' \\ g \downarrow & & \downarrow g' \\ TX & \xrightarrow{\quad \varphi_* \quad} & TX' \end{array}$$

LEMMA 3.2. *Locally the diagrams I, II, III are equivalent, respectively, to*

$$(3.2) \quad \begin{aligned} \text{I':} & & D &\subset \ker dh, \\ \text{II':} & & \pi_* D &= E, \\ \text{III':} & & g_* D &\subset TE = T\pi_*(D). \end{aligned}$$

*Proof.* I' and II' are trivial. For III' observe that, when  $\varphi$  induces a distribution  $E$  on  $X$ , then  $\varphi_*$  induces the distribution  $TE$  on  $TX$ .  $\square$

Now we want to relate conditions I', II', III' with the theory of nonlinear disturbance decoupling, and especially with the formulation of it given in [11]. Consider in local coordinates the system

$$\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x) \quad \text{on a manifold } X.$$

We can interpret this as an affine distribution on  $X$  (for each  $x \in X$ , we give an affine subspace of  $T_x X$ ). We call this affine distribution  $\Delta$ . Now define

$$\Delta_0 := \Delta - \Delta := \{Y - Z \mid Y, Z \in \Delta\}.$$

It is easily seen that  $\Delta_0$  is a distribution on  $X$ , given in local coordinates by span  $\{g_1(x), \dots, g_m(x)\}$  (the directions in which we can steer). Define  $A(\Delta_0) := \{D \mid D \text{ an involutive distribution such that } D + \Delta_0 \text{ is involutive}\}$ . Then in [11] it is proved that

THEOREM 3.3. *Let  $D \in A(\Delta_0)$ . Then the condition*

$$(3.3) \quad [\Delta, D] \subseteq D + \Delta_0$$

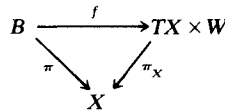
(we call such a  $D \in A(\Delta_0)$   $\Delta \pmod{\Delta_0}$  invariant) is equivalent to the two conditions

- a) there exists a vector field  $F \in \Delta$  such that  $[F, D] \subset D$ ;
- b) there exist vector fields  $B_i \in \Delta_0$  such that span  $\{B_i\} = \Delta_0$  and  $[B_i, D] \subset D$ .

With the aid of this theorem the disturbance decoupling problem is readily solved.

The key to connecting our situation with this theory is given by the concept of the *extended system*, which is of interest in itself.

DEFINITION 3.4 (*extended system*). Let



Then we define the *extended system* of  $\Sigma(X, W, B, f)$  as follows: We define  $\Delta_0$  as the vertical tangent space of  $B$ , i.e.,

$$\Delta_0 := \{Z \in TB \mid \pi_* Z = 0\}.$$

Note that  $\Delta_0$  is automatically involutive.

Now take a point  $(\bar{x}, \bar{v}) \in B$ . Then  $g(\bar{x}, \bar{v})$  is an element of  $T_{\bar{x}} X$ . Now define

$$\Delta(\bar{x}, \bar{v}) := \{Z \in T_{(\bar{x}, \bar{v})} B \mid \pi_* Z = g(\bar{x}, \bar{v})\}.$$

So  $\Delta(\bar{x}, \bar{v})$  consists of the possible lifts of  $g(\bar{x}, \bar{v})$  in  $(\bar{x}, \bar{v})$ . Then it is easy to see that  $\Delta$  is an affine distribution on  $B$ , and that  $\Delta - \Delta = \Delta_0$ . We call the affine system  $(\Delta, \Delta_0)$  on  $B$  constructed in this way, together with the output function  $h : B \rightarrow W$ , the *extended system*  $\Sigma^e(X, W, B, f)$ .



We have the following:

LEMMA 3.5.

a) Let  $D$  be an involutive distribution on  $B$  such that  $D \cap \Delta_0$  has constant dimension. Then  $\pi_*D$  is a well-defined and involutive distribution on  $X$  if and only if  $D + \Delta_0$  is an involutive distribution.

b) Let  $D$  be an involutive distribution on  $B$  and let  $D \cap \Delta_0$  have constant dimension. Then the following two conditions are equivalent:

- i)  $\pi_*D$  is a well-defined and involutive distribution on  $X$ , and  $g_*D \subset T\pi_*D$ .
- ii)  $[\Delta, D] \subset D + \Delta_0$ .

*Proof.* a) Let  $D + \Delta_0$  be involutive. Because  $D$  and  $\Delta_0$  are involutive this is equivalent to  $[D, \Delta_0] \subset D + \Delta_0$ . Applying Theorem 3.3 to this case gives a basis  $\{Z_1, \dots, Z_k\}$  of  $D$  such that  $[Z_i, \Delta_0] \subset \Delta_0$ . In coordinates  $(x, u)$  for  $B$ , this last expression is equivalent to  $Z_i(x, u) = (Z_{ix}(x), Z_{iu}(x, u))$ , where  $Z_{ix}$  and  $Z_{iu}$  are the components of  $Z_i$  in the  $x$ - and  $u$ -directions, respectively. Hence  $\pi_*D = \text{span}\{Z_{1x}, \dots, Z_{kx}\}$  and is easily seen to be involutive. The converse statement is trivial.

b) Assume i); then there exist coordinates  $(x, u)$  for  $B$  such that  $D = \{\partial/\partial x_1, \dots, \partial/\partial x_k\}$  (the integral manifolds of  $D$  are contained in the sections  $u = \text{constant}$ ). Then  $g_*D \subset T\pi_*D$  is equivalent to

$$\left(\frac{\partial g}{\partial x_i}\right)_{j \in \text{comp}} = 0$$

with  $i = 1, \dots, k$  and  $j = k + 1, \dots, n$  ( $n$  is the dimension of  $X$ ). From these expressions  $[\Delta, D] \subset D + \Delta_0$  readily follows. The converse statement is based on the same argument. □

Now we are prepared to state the main theorem of this section. First we have to give another definition.

DEFINITION 3.6 (*local minimality*). Let  $\Sigma(X, W, B, f)$  be a smooth system. Let  $\bar{x} \in X$ . Then  $\Sigma(X, W, B, f)$  is called *locally minimal* (around  $\bar{x}$ ) if when  $D$  and  $E$  are distributions (around  $\bar{x}$ ) which satisfy conditions I', II', III' of Lemma 3.2, then  $D$  and  $E$  must be the zero distributions.

It is readily seen from Definition 3.1 that minimality of  $\Sigma(X, W, B, f)$  locally implies local minimality (locally every involutive distribution can be factored out).

Combining Lemma 3.2, Definition 3.4 and Lemma 3.5 we can state:

THEOREM 3.7.  $\Sigma(X, W, B, f = (g, h))$  is locally minimal if and only if the extended system  $\Sigma^e(X, W, B, f = (g, h))$  satisfies the condition that there exist no nonzero involutive distribution  $D$  on  $B$  such that

$$(3.4) \quad \begin{aligned} \text{(i)} \quad & [\Delta, D] \subset D + \Delta_0, \\ \text{(ii)} \quad & D \subset \ker dh. \end{aligned}$$

*Remark 1.* It is very surprising that the condition of minimality locally comes down to a condition on the extended system, which is in some sense an infinitesimal version of the original system.

*Remark 2.* Actually there is a conceptual *algorithm* to check local minimality (cf. [11]). Define

$$\Delta^{-1}(\Delta_0 + D) := \{\text{vector fields } Z \text{ on } B \mid [\Delta, Z] \subseteq \Delta_0 + D\}.$$

Then we can define the sequence  $\{D^\mu\}$ ,  $\mu = 0, 1, 2, \dots$  as follows:

$$\begin{aligned} D^0 &= \ker dh, \\ D^\mu &= D^{\mu-1} \cap \Delta^{-1}(\Delta_0 + D^{\mu-1}), \quad \mu = 1, 2, \dots \end{aligned}$$

Then  $\{D^\mu\}$ ,  $\mu = 0, 1, 2, \dots$ , is a decreasing sequence of involutive distributions, and for some  $k \leq \dim(\ker dh)$   $D^k = D^\mu$  for all  $\mu \geq k$ . Then  $D^k$  is the *maximal* involutive distribution which satisfies

- (i)  $[\Delta, D^k] \subset D^k + \Delta_0$ ,
- (ii)  $D^k \subset \ker dh$ .

From Theorem 3.7 it follows that  $\Sigma(X, W, B, f)$  is locally minimal if and only if  $D^k = 0$ ! Notice that the maximum numbers of steps needed in this algorithm is equal to the dimension of  $\ker dh$ , and hence at least smaller than  $\dim B$ .

**3.2. Observability.** It is natural to suppose that our definition of minimality has something to do with controllability and observability. However, because the definition of a nonlinear system (2.1) also includes autonomous systems, (i.e., no inputs), minimality cannot be expected to imply, in general, some kind of controllability. In fact an autonomous linear system

$$\dot{x} = Ax, \quad y = Cx$$

is easily seen to be minimal if and only if  $(A, C)$  is observable (cf. [17]). Moreover, it seems natural to define a notion of *observability* only in the case that the system (2.1) has at least a local input–output representation; i.e., we make the standing assumption that  $(\partial h/\partial v)$  is injective (see Lemma 2.1). Therefore, *locally* we have as our system

$$(3.5) \quad (= (2.5)) \quad \dot{x} = g(x, u), \quad y = \tilde{h}(x, u)$$

for every possible input–output coordinatization  $(y, u)$  of  $W$  (see Remark (i) 2 in § 2). For such an input–output system local minimality implies the following notion of observability, which we will call *local distinguishability*.

**PROPOSITION 3.8.** *Choose a local input–output parametrization as in (3.5). Then local minimality implies that the only involutive distribution  $E$  on  $X$  which satisfies*

- i)  $[g(\cdot, u), E] \subset E$  for all  $u$  ( $E$  is invariant under  $g(\cdot, u)$ ),
- ii)  $E \subset \ker d_x \tilde{h}(\cdot, u)$  for all  $u$  ( $d_x \tilde{h}$  means differentiation with respect to  $x$ ) is the zero distribution.

*Proof.* Let  $E$  be a distribution on  $X$  which satisfies i) and ii). Then we can lift  $E$  in a trivial way to a distribution  $D$  on  $B$  by requiring that the integral manifolds of  $D$  be contained in the sections  $u = \text{constant}$ . Then one can see that  $D$  satisfies  $[\Delta, D] \subset D + \Delta_0$  and  $D \subset \ker dh$ . Hence  $D = 0$  and  $E = 0$ . □

*Remark.* It is easily seen that, under the condition  $(\partial h/\partial v)$  injective, local minimality is in fact equivalent to the condition in Proposition 3.8. This is because  $(\partial h/\partial v)$  injective implies that there cannot be a distribution  $D$  on  $B$  such that  $D \cap \Delta_0 \neq 0$  and  $D \subset \ker dh$ . So from Lemma 3.5 a) it follows that the only involutive distributions  $D$  with  $D \subset \ker dh$  and  $D + \Delta_0$  involutive are of the form  $E_{\text{lift}}$ , with  $E$  an involutive distribution on  $X$ .

Actually, for nonlinear systems which can be represented in the input–output form without a feedthrough term (2.6), we can state the following:

**COROLLARY.** *Suppose there exists an input–output coordinatization*

$$(3.6) \quad (= (2.6)) \quad \dot{x} = g(x, u), \quad y = \tilde{h}(x),$$

*Then local minimality implies local weak observability (cf. [6], [12]).*

*Proof.* As can be seen from Proposition 3.8, local minimality in this more restricted case implies that the only involutive distribution  $E$  on  $X$  which satisfies

- i)  $[g(\cdot, u), E] \subset E$  for all  $u$ ,
- ii)  $E \subset \ker d\tilde{h}$

is the zero distribution. It can be readily seen [cf. 8] that the biggest distribution which satisfies i) and ii) is given by the null space of the codistribution  $P$  generated by elements of the form

$$L_{g(\cdot, u^1)} L_{g(\cdot, u^2)} \cdots L_{g(\cdot, u^k)} d\tilde{h}, \quad \text{with } u^j \text{ arbitrary.}$$

Because this distribution has to be zero, the codistribution  $P$  equals  $T_x^*X$ , in every  $x \in X$ . This is, apart from singularities (which we don't want to consider), equivalent to local weak observability as defined in [6].

Moreover, let (3.6) be locally weakly observable. Then all feedback transformations  $u \mapsto v = \alpha(x, u)$  which leave the form (3.6) invariant (i.e.,  $y$  is only the function  $x$ ) are exactly the output feedback transformations  $u \mapsto v = \alpha(y, u)$ . It can be easily seen in local coordinates that after such output feedback is applied the modified system is still locally weakly observable.  $\square$

In Proposition 3.8 and its corollary we have shown that local minimality implies a notion of observability which generalizes the usual notion of local weak observability. Now we will define a much stronger notion. Let us denote the (defined only locally) vector field  $\dot{x} = g(x, \bar{u})$  for fixed  $\bar{u}$  by  $g^{\bar{u}}$  and the function  $\tilde{h}(x, \bar{u})$  by  $h^{\bar{u}}$  (with  $g$  and  $\tilde{h}$  as in (3.5)).

**DEFINITION 3.9.** Let  $\Sigma(X, W, B, f) = (g, h)$  be a smooth nonlinear system. It is called *strongly observable* if for every possible input-output coordinatization (3.5) the autonomous system

$$(3.7) \quad \dot{x} = g^{\bar{u}}(x), \quad y = h^{\bar{u}}(x)$$

with  $\bar{u}$  constant is *locally weakly observable* (for a definition see [6] or [12]), for all  $\bar{u}$ .

*Remark.* Let  $\Sigma(X, W, B, f) = (g, h)$  be strongly observable. Take one input-output coordinatization  $(y, u)$ . The system has the form (in these coordinates)

$$\dot{x} = g(x, u), \quad y = \tilde{h}(x, u).$$

Because the system is strongly observable, every *constant* input-function (constant in *this* coordinatization) distinguishes between two nearby states. However, in every other input-output coordinatization every constant (i.e., in *this* coordinatization) input function also distinguishes. This implies that in the first coordinatization every  $C^\infty$  input function distinguishes. Because the  $C^\infty$  input functions are dense in a reasonable set of input functions, every input function in this coordinatization distinguishes.

**PROPOSITION 3.10.** Consider the Pfaffian system constructed as follows:

$$P = dh^{\bar{u}} + L_{g^{\bar{u}}} dh^{\bar{u}} + L_{g^{\bar{u}}}(L_{g^{\bar{u}}} dh^{\bar{u}}) + \cdots + L_{g^{\bar{u}}}^{(n-1)} dh^{\bar{u}},$$

with  $n$  the dimension of  $X$  and  $L_{g^{\bar{u}}}$  the Lie derivative with respect to  $g^{\bar{u}}$ . As is well known from [6], the condition that the Pfaffian system  $P$  as defined above satisfies the condition  $P_x = T_x^*X$  for all  $x \in X$  (the so called observability rank condition) implies that the system

$$\dot{x} = g^{\bar{u}}(x), \quad y = h^{\bar{u}}(x)$$

is locally weakly observable. Hence, when the observability rank condition is satisfied for all  $\bar{u}$ , the system is strongly observable.

We will call the Pfaffian system  $P$  the *observability codistribution*.

*Remark 1.* As is known from [6], local weak observability of the system

$$\dot{x} = g^{\bar{u}}(x), \quad y = h^{\bar{u}}(x)$$

implies that the observability rank condition (i.e.,  $\dim P_x = T_x^*X$ ) is satisfied almost everywhere (in fact, in the analytic case everywhere). Because we don't want to go into singularity problems, for us local weak observability and the observability rank condition are the same.

*Remark 2.* It is easily seen that when for one input-output coordinatization the observability rank condition for all  $u$  is satisfied, then for every input-output coordinatization the observability rank condition for all  $u$  is satisfied. This follows from the fact that the observability rank condition is an open condition.

**3.3. Controllability.** The aim of this section is to define a kind of controllability which is "dual" to the definition of local distinguishability (Proposition 3.8) and which we shall use in the following section. The notion of controllability we shall use is the so-called "strong accessibility", introduced in [14].

**DEFINITION 3.11.** Let  $\dot{x} = g(x, u)$  be a nonlinear system in local coordinates. Define  $R(T, x_0)$  as the set of points reachable from  $x_0$  in exactly time  $T$ ; in other words,

$$R(T, x_0) := \{x_1 \in X \mid \exists \text{ state trajectory } x(t) \text{ generated by } g \\ \text{such that } x(0) = x_0 \text{ and } x(T) = x_1\}.$$

We call the system *strongly accessible* if for all  $x_0 \in X$ , and for all  $T > 0$  the set  $R(T, x_0)$  has a nonempty interior.

For systems of the form (in local coordinates)

$$(3.8) \quad \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$$

(i.e., affine systems) we can define  $A$  as the smallest Lie algebra which contains  $\{g_1, \dots, g_m\}$  and which is invariant under  $f$  (i.e.,  $[f, A] \subset A$ ). It is known (cf. [14]) that  $A_x = T_x X$  for every  $x \in X$  implies that the system (3.8) is strongly accessible. In fact, when the system is analytic, strong accessibility and the rank condition  $A_x = T_x X$  for every  $x \in X$ , are equivalent. We will call  $A$  the *controllability distribution* and the rank condition the *controllability rank condition*. Now it is clear that for affine systems (3.8) this kind of controllability is an elegant "dual" of local weak observability.

We know that the extended system (see Definition 3.4) is an affine system. Hence for this system we can apply the rank condition described above. This makes sense because the strong accessibility of  $\Sigma(X, W, B, f)$  is very much related to the strong accessibility of  $\Sigma^e(X, W, B, f)$ , as can be seen from the following two propositions.

**PROPOSITION 3.12.** *If  $\Sigma^e(X, W, B, f = (g, h))$  is strongly accessible, then  $\Sigma(X, W, B, f = (g, h))$  is strongly accessible.*

*Proof.* In local coordinates the dynamics of  $\Sigma^e$  and  $\Sigma$  are given by

$$\begin{array}{ll} \text{I} & \dot{x} = g(x, u) \quad (\Sigma), \\ & \dot{x} = g(x, v) \quad (\Sigma^e), \\ \text{II} & \dot{v} = u. \end{array}$$

Now it is trivial that when for  $\Sigma^e$  we can steer to a point  $x_1$  then we also can for  $\Sigma$  (even with an input that is smoother).  $\square$

The converse is harder:

**PROPOSITION 3.13.** *Let  $\Sigma(X, W, B, f = (g, h))$  be strongly accessible. Assume moreover that the fibers of  $B$  are connected. Then also  $\Sigma^e(X, W, B, f = (g, h))$  is strongly accessible.*

*Proof* (sketch, see also [4]). Take the same representation of  $\Sigma$  and  $\Sigma^e$  as in the proof of Proposition 3.12. Let  $x_0 \in X$  and let  $x_1$  be in the (nonempty) interior of  $R_\Sigma(x_0, T)$  (the reachable set of system  $\Sigma$ ). Then it is possible that  $x_1$  is reachable from  $x_0$  by an input function  $v(t)$  which cannot be generated by the differential equation  $\dot{v} = u$  (with  $u$  for instance  $L^2$ ). However, we know that the set of the  $v$  generated in this way is dense in, for instance  $L^2$ . (For this we certainly need that the fibers of  $B$  are connected.) Because we only have to prove that the interior of a set is nonempty, this makes no difference. Now it is obvious from the equations

$$\dot{x} = g(x, v), \quad \dot{v} = u$$

that if we can reach an open set in the  $x$ -part of the (extended) state, then this is surely possible in the whole  $(x, v)$ -state.  $\square$

#### 4. Hamiltonian and gradient systems

##### 4.1. Hamiltonian systems. A linear input-output system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is called *Hamiltonian* if

$$\begin{aligned} A^T J + JA &= 0, \\ B^T J &= C, \quad \text{where } J \text{ is the symplectic form } \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \\ D &= D^T \quad (\text{see [16]}), \end{aligned}$$

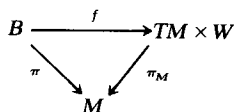
and is called a *gradient system* if

$$\begin{aligned} TA &= A^T T, \\ TB &= C^T, \quad \text{where } T \text{ is a nonsingular symmetric matrix,} \\ D &= D^T \quad (\text{see [18]}). \end{aligned}$$

It can be easily checked that for both kind of systems observability implies controllability, and vice versa.

We want to see whether we can derive a similar result for nonlinear Hamiltonian and gradient systems as defined in [16]. We start with the Hamiltonian case.

1. Let



be a smooth system.

Now take  $M$  a symplectic manifold with symplectic form  $\omega$  (see [1]). Because  $M$  is a symplectic manifold we can also define in a canonical way a symplectic form, denoted by  $\hat{\omega}$ , on  $TM$  (see [16]). Darboux's theorem tells us that we can find coordinates

$(q_i, p_i)$  for  $M$  and  $(q_i, p_i, \dot{q}_i, \dot{p}_i)$  for  $TM$  such that

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i \quad \text{and} \quad \dot{\omega} = \sum_{i=1}^n d\dot{q}_i \wedge dp_i + dq_i \wedge d\dot{p}_i.$$

Now also take  $W$  a symplectic manifold with symplectic form  $\omega^e$ .

Finally, we can make  $TM \times W$  into a symplectic manifold by defining the symplectic form

$$\Omega := \pi_1^* \dot{\omega} - \pi_2^* \omega^e \quad (\text{the minus sign is a matter of convention})$$

with  $\pi_1$  and  $\pi_2$  the natural projections of  $TM \times W$  on  $TM$  and  $W$  respectively.

DEFINITION 4.1 (see [16, Def. 4.1]).  $\Sigma(M, W, B, f)$  with  $M$  and  $W$  as above is called *full Hamiltonian* if  $f(B)$  is a Lagrangian submanifold of  $(TM \times W, \Omega)$

PROPOSITION 4.2 (see [16, Prop. 4.2]). *Let  $\Sigma(M, W, B, f)$  be full Hamiltonian. Then there exist coordinates for  $TM$  as above, coordinates  $\{y_1, \dots, y_m, u_1, \dots, u_m\}$  for  $W$  and a function  $H(q_1, \dots, q_n, p_1, \dots, p_n, u_1, \dots, u_m)$  such that the system is locally described by*

$$(4.1) \quad \begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, & i &= 1, \dots, n, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \\ y_j &= c_j \frac{\partial H}{\partial u_j}, & j &= 1, \dots, m, \quad \text{with } c_j = \pm 1, \\ \omega^e &= \sum_{j=1}^m c_j dy_j \wedge du_j. \end{aligned}$$

*Remark.* We see that in this case the freedom in the input–output parametrization is restricted to the so called *canonical* coordinates for  $\omega^e$ , i.e., only coordinates  $(y, u)$  such that  $\omega^e = \sum c_j dy_j \wedge du_j$ .

From Proposition 4.2 the following proposition easily follows:

PROPOSITION 4.3.

a) *Let  $\Sigma(M, W, B, f)$  be locally minimal. Then  $f$  must be an immersion.*

b) *Let  $\Sigma(M, W, B, f = (g, h))$  be full Hamiltonian and assume  $f$  is an immersion. Then  $h$  restricted to the fibers of  $B$  must be an immersion.*

*Proof.* a) From the definition of  $\Sigma(M, W, B, f)$  it follows that we only have to prove that  $f$  restricted to the fibers is an immersion. Now suppose that  $f$  restricted to the fibers is not an immersion. Then the distribution  $\ker d_v f \subset \Delta_0$  with  $d_v$  the derivative in the direction of the fiber is not equal to zero and satisfies (trivially) the conditions of Lemma 3.2., i.e., a contradiction.

b) Take a local input–output coordinatization  $(y, u)$  as in Proposition 4.2. Then the whole system is parametrized by the “input variables”  $u_1, \dots, u_m$ . Therefore the image of  $h$  restricted to the fibers has to be of dimension  $m$  and the image of  $g$  restricted to the fibers has to be of dimension *at most*  $m$ . Because  $f$  restricted to the fibers is an immersion, it is clear that the dimension of the fibers of  $B$  must be  $m$  and so  $h$  restricted to the fibers is an immersion.  $\square$

Now we can state the main theorem of this section.

THEOREM 4.4. *Let  $\Sigma(M, W, B, f = (g, h))$  be a full Hamiltonian system. Suppose  $f$  is an immersion. Then, for every input–output coordinatization of  $\Sigma(M, W, B, f)$  as in*

Proposition 4.2,

$$(4.2) \quad \begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \quad i = 1, \dots, n, \quad y_j = c_j \frac{\partial H}{\partial u_j}, \quad j = 1, \dots, m,$$

the following is true (see Proposition 3.8):

$$(4.2) \text{ is strongly accessible} \Leftrightarrow (4.2) \text{ is locally distinguishable.}$$

COROLLARY. If  $\Sigma(M, W, B, f)$  is locally minimal, then it follows from Proposition 4.3 that  $f$  is an immersion. Moreover, it follows from Proposition 3.8 that (4.2) is locally distinguishable. Therefore, by Theorem 4.4, the system (4.2) is also strongly accessible.

*Proof.* Let us denote by  $X_H$  the vector field

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

As is proved in Proposition 3.8, local distinguishability of (4.2), or equivalently local minimality, comes down to the following. Let  $\bar{O}$  be the vector space of functions spanned by  $\{u_1, \dots, u_m, \partial H/\partial u_1, \dots, \partial H/\partial u_m\}$  (for simplicity take  $c_j = 1$ ). Now add to  $\bar{O}$  all the functions generated by taking Lie derivatives of functions in  $\bar{O}$  with respect to the vector fields (on  $B$ )  $X_H$  and  $\partial/\partial u_1, \dots, \partial/\partial u_m$ . We denote the vector space spanned by all these functions by  $O$ . We shall give  $O$  the following notation:

$$O = \left\{ u_1, \dots, u_m, \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_m}, + \text{invariance under } X_H \text{ and } \frac{\partial}{\partial u_i} \right\}.$$

Then local distinguishability of (4.2) is equivalent to

$$dO(x, u) = T_{(x, u)}^* B \quad \text{for every } (x, u) \in B.$$

We can rewrite  $\partial H/\partial u_i$  as  $L_{\partial/\partial u_i} H$ . Also it is easy to prove that  $d_x(L_{\partial/\partial u_i} H) = L_{\partial/\partial u_i}(d_x H)$ ,  $i = 1, \dots, m$  ( $d_x$  denotes differentiation with respect to  $x$ ). Therefore:

$$\begin{aligned} dO &= \{du_1, \dots, du_m, d(L_{\partial/\partial u_1} H), \dots, d(L_{\partial/\partial u_m} H) + \text{invariance under } X_H \text{ and } \partial/\partial u_i\} \\ &= \{du_1, \dots, du_m, d_x(L_{\partial/\partial u_1} H), \dots, d_x(L_{\partial/\partial u_m} H) + \text{invariance under } X_H \text{ and } \partial/\partial u_i\} \\ &= \{du_1, \dots, du_m, L_{\partial/\partial u_1} d_x H, \dots, L_{\partial/\partial u_m} d_x H + \text{invariance under } X_H \text{ and } \partial/\partial u_i\} \end{aligned}$$

Now we turn to strong accessibility. As proved in Propositions 3.12 and 3.13, strong accessibility of (4.2) is equivalent to strong accessibility of the extended system of (4.2). Therefore when we define the vector space of vector fields  $A$  by the vector fields adding all the Lie derivatives of the vector fields  $\partial/\partial u_1, \dots, \partial/\partial u_m$  with respect to  $X_H$  and  $\partial/\partial u_i$ ,  $i = 1, \dots, m$ , i.e.,

$$A = \left\{ \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_m}, + \text{invariance under } X_H \text{ and } \frac{\partial}{\partial u_i} \right\},$$

then we have to show that

$$A(x, u) = T_{(x, u)} B \quad \text{for every } (x, u) \in B.$$

( $A$  is the controllability distribution of the extended system.) It immediately follows that

$$A = \left\{ \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_m}, L_{\partial/\partial u_1} X_H, \dots, L_{\partial/\partial u_m} X_H, + \text{invariance under } X_H \text{ and } \frac{\partial}{\partial u_i} \right\}$$

Now we will show that the map  $\alpha : TM \rightarrow T^*M$ , defined by  $\alpha(Y) = \omega(Y, -)$ , with  $Y \in TM$ , together with the map which sends  $\partial/\partial u_i$  to  $du_i$ ,  $i = 1, \dots, m$ , is an isomorphism between  $A$  and  $dO$ , and therefore  $A(x, u) = T_{(x, u)}B$  if and only if  $dO(x, u) = T_{(x, u)}^*B$ . The following observations are sufficient:

- i)  $\alpha(X_H) = d_x H$ .
- ii)  $\alpha(L_{\partial/\partial u_i} X_H) = L_{\partial/\partial u_i} \alpha(X_H) = L_{\partial/\partial u_i}(d_x H)$ .
- iii) Because  $L_{X_H} \omega = 0$  and also  $L_{\partial/\partial u_i} \omega = 0$ , and because Lie brackets of Hamiltonian vector fields (Hamiltonian with respect to the degenerate form  $\omega$  on  $B$ ) are again Hamiltonian,  $A$  is generated by Hamiltonian vector fields.
- iv) Take an arbitrary Hamiltonian vector field  $X_G$  in  $A$ . Then:

$$\alpha(L_{X_H} X_G) = L_{X_H} \alpha(X_G) \quad \text{because } L_{X_H} \omega = 0,$$

$$\alpha(L_{\partial/\partial u_i} X_G) = L_{\partial/\partial u_i} \alpha(X_G) \quad \text{because } L_{\partial/\partial u_i} \omega = 0.$$

This easily gives the induction argument that  $A$  is mapped onto  $dO$ . □

*Remark 1.* It is also possible to derive a duality result for strong observability (see Definition 3.9). The notion of dual controllability appears to be stronger than that of strong accessibility. However we will leave this for the moment.

*Remark 2.* Of course duality between strong accessibility and local distinguishability is closely related to the existence of a Lie algebra morphism between a Lie algebra of Hamiltonian vector fields equipped with the Lie bracket and a Lie algebra of Hamilton functions provided with the Poisson bracket (cf. [1]). We will explore this relationship in a future paper [19].

*Remark 3.* Consider the expression  $\{\bar{u} \partial H / \partial u, H^{\bar{u}}\}$  with  $\{\cdot, \cdot\}$  the Poisson bracket on  $M$  and  $H^{\bar{u}}$  a function on  $M$  defined by  $H^{\bar{u}}(q, p) := H(q, p, \bar{u})$ . This expression equals

$$\sum_{j=1}^m \bar{u}_j \{h_j^{\bar{u}}, H^{\bar{u}}\} = \sum_{j=1}^m \bar{u}_j \cdot \frac{d}{dt} h_j^{\bar{u}},$$

with  $\bar{u} = (\bar{u}_1, \dots, \bar{u}_m)$  and  $h_j^{\bar{u}}(q, p) := j$ th component of  $(\partial H / \partial u)(q, p, \bar{u})$ , and has a direct interpretation in the sense that when we interpret  $u$  as the external force and  $y$  as the position (see [16]) the expressions equal the instantaneous external work.

**4.2. Gradient systems.** Following [16] a system  $\Sigma(M, W, B, f)$  is called a *full gradient system* if

- (i)  $M$  is a Riemannian manifold with (possibly indefinite) metric  $\langle \cdot, \cdot \rangle$ ;
- (ii)  $W$  is a symplectic manifold with symplectic form  $\omega^e$ ;
- (iii)  $\langle \cdot, \cdot \rangle$  induces a bundle isomorphism  $\alpha$  between  $TM$  and  $T^*M$  by setting  $\alpha(X) := \langle X, - \rangle$ , for  $X \in TM$ .

Because  $T^*M$  has a canonical 2-form  $\Omega$ ,  $TM$  has the symplectic form  $\alpha^* \Omega$ . Then  $TM \times W$  is also a symplectic manifold with symplectic form  $\pi_1^* \alpha^* \Omega - \pi_2^* \omega^e$  ( $\pi_1$  and  $\pi_2$  are the natural projections of  $TM \times W$  on  $TM$  and  $W$  respectively).

Now we ask that  $f(B)$  be a *Lagrangian submanifold* of  $(TM \times W, \pi_1^* \alpha^* \Omega - \pi_2^* \omega^e)$ .

**PROPOSITION 4.6.** *Parallel to Proposition 4.2 we can prove that locally this definition reduces to the existence of coordinates  $\{x_1, \dots, x_n\}$  for  $M$  and  $\{y_1, \dots, y_m$ ,*



$u_1, \dots, u_m\}$  for  $W$  and the existence of a potential function  $V(x, u)$  such that

$$(4.3) \quad \begin{aligned} G(x)\dot{x} &= \frac{\partial V}{\partial x}(x, u), \\ y_j &= c_j \frac{\partial V}{\partial u_j}(x, u) \quad \text{with } c_j = \pm 1 \end{aligned}$$

and  $\omega^e = \sum c_j dy_j \wedge du_j$  (so  $(y, u)$  are canonical coordinates).  $G(x)$  represents the Riemannian metric  $\langle \cdot, \cdot \rangle$ .

Now one could suppose, guided by the similarity in definition of Hamiltonian and gradient systems, and also by the linear situation as sketched before, that Theorem 4.3 should have an analogue in the gradient case. However it is easy to construct an example of a *nonlinear* gradient system which is strongly observable but not strongly accessible.

COUNTEREXAMPLE. Take  $V(x_1, x_2, x_3, u) = e^{x_3}x_1x_2 + x_3u + u^2$ , and as Riemannian metric the Euclidean metric on  $\mathbb{R}^3$ . This generates a gradient system

$$\begin{aligned} \dot{x}_1 &= e^{x_3}x_2 := g_1(x), & \dot{x}_2 &= e^{x_3}x_1 := g_2(x) & \dot{x}_3 &= e^{x_3}x_1x_2 + u := g_3(x) + u, \\ y &= x_3 + u := h(x) + u, \end{aligned}$$

which is locally weakly observable because

- i)  $dh = dx_3$ ,
- ii)  $L_g dh = d(e^{x_3}x_1x_2) = e^{x_3}x_2 dx_1 + e^{x_3}x_1 dx_2 + e^{x_3}x_1x_2 dx_3$ .

Because  $dh = dx_3$  and  $e^{x_3}$  is merely a factor  $> 0$  we only have to consider  $x_1 dx_2 + x_2 dx_1$ .

- iii)  $L_g(x_1 dx_2 + x_2 dx_1) = d(x_1 e^{x_3}x_1 + x_2 e^{x_3}x_2)$   
 $= e^{x_3}(x_1 dx_1 + x_2 dx_2) + (\text{factors in } dx_3)$ .

Now because  $x_1 dx_2 + x_2 dx_1$  and  $x_1 dx_1 + x_2 dx_2$  form a basis of  $T^*\mathbb{R}^2$  in almost every  $(x_1, x_2) \in \mathbb{R}^2$  the observability codistribution has full dimension, and so the system is locally weakly observable (even strongly observable, as can be readily seen). But the system is not strongly accessible, because

$$\left[ g, \frac{\partial}{\partial x_3} \right] = e^{x_3}x_2 \frac{\partial}{\partial x_1} + e^{x_3}x_1 \frac{\partial}{\partial x_2} + e^{x_3}x_1x_2 \frac{\partial}{\partial x_3} = g.$$

Therefore the controllability distribution has dimension at most two.

**5. Conclusion.** We have shown that the definition of a smooth nonlinear system in § 2 can be readily interpreted as a generalization of more usual input–output formulations. Further we can define a natural notion of minimality for such systems which implies the usual definition of observability for nonlinear systems. It would be interesting to look for a natural *realization theory* in this context. The definition of minimality suggests a more local theory than the realization theory of nonlinear input–output systems as developed in [9]. This aspect (see also [17]) is presently under investigation. We also expect to find a natural interpretation of the definition of strong observability in such a realization theory. The results of § 4 indicate that, contrary to the linear case, nonlinear gradient systems may be, at least from a system theoretic point of view, more complex than nonlinear Hamiltonian systems.

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