from Phys. Rev. Lett. **63**, 348 (1989).

Observable Gauge Transformations in the Parton Picture

Y. S. Kim
Department of Physics and Astronomy, University of Maryland,
College Park, Maryland 20742

Abstract

The internal space-time symmetry of rapidly moving composite particles is studied in terms of the little groups of the Poincaré group. It is shown that Feynman's x parameter in his parton picture is a gauge transformation parameter.

The internal space-time symmetries of elementary particles are governed by the little groups of the Poincaré group [1]. The little groups for massive and massless particles are locally isomorphic to $O(3)$ and $E(2)$ (two-dimensional Euclidean group) respectively. It was shown recently that the $E(2)$-like little group for massless particles is an infinite-momentum/zero-mass limit of the $O(3)$-like little group [2, 3]. The role of the little groups is illustrated in the second row of Fig. 1.

The purpose of this Letter is to show that the internal space-time symmetry of composite particles can be formulated within the framework of Wigner's little groups, and that Feynman's x parameter in his parton picture [4] is a gauge transformation parameter. This will unify the second and third rows of Fig. 1. This figure is from the recent paper by Kim and Wigner which deals primarily with the third row [5].

Wigner's little group is the maximal subgroup of the Lorentz group whose transformations leave the four-momentum of a given particle invariant [1]. For a massive point particle, there is a Lorentz frame in which the particle is at rest. In this frame, the little group is the three-dimensional rotation group. This is the fundamental symmetry associated with the concept of spin.

The internal space-time symmetry of massless particles is governed by the cylindrical group which is locally isomorphic to $E(2)$ [3]. In this case, we can visualize a circular cylinder whose axis is parallel to the momentum. On the surface of this cylinder, we can rotate a point around the axis or translate along the direction of the axis. The rotational degree of freedom is associated with the helicity, while the translation corresponds to a gauge transformation in the case of photons [3].

This translational degree of freedom is shared by all massless particles, including neutrinos and gravitons [6]. Indeed, the requirement of invariance under this symmetry leads to the polarization of neutrinos [6, 7]. Since this translational degree of freedom is a gauge degree of freedom for photons, we can extend the concept of gauge transformations to all massless particles [8] and massive particles in the infinite-momentum limit.

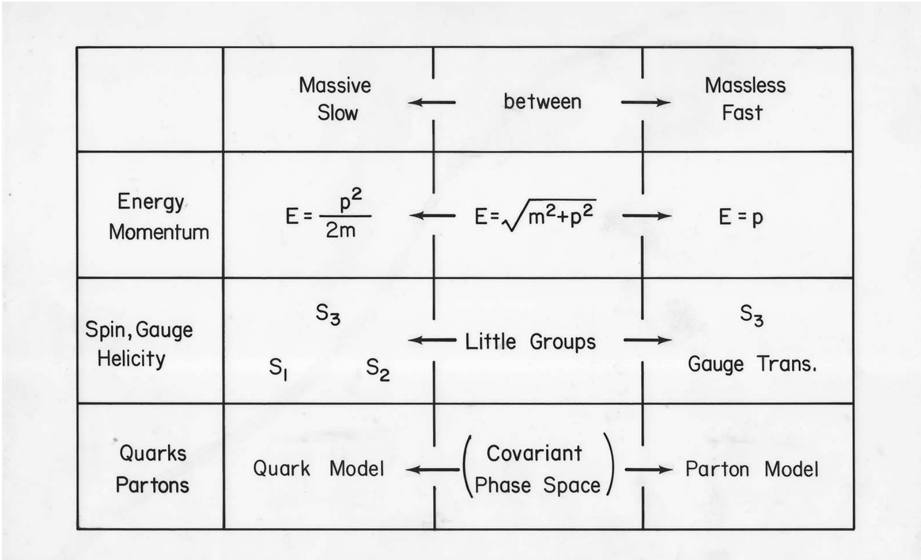

Figure 1: Slow and fast particles. Einstein's $E = [P^2 + m^2]^{1/2}$ unifies the energy-momentum relations for massive (nonrelativistic) particles and for massless particles. The second row indicates that the little group of the Poincaré group unifies the internal space-time symmetries of massive and massless particles, as is discussed in Ref. 2. The third row states that the covariant phase-space picture of quantum mechanics forms the physical basis for the covariant harmonic oscillator formalism which has been shown to give a unified picture of the quark model and the parton picture.

It is not difficult to associate the symmetry of a point particle with that of a composite particle if they are massive and at rest, because both of them are governed by the three-dimensional rotation group [9]. The story is quite different for rapidly moving composite particles or hadrons. Does a rapidly moving hadron have the same set of space-time degrees of freedom as that of photons? We can study this problem by constructing a cylindrical symmetry for a hadron with infinite momentum. Then, is this symmetry consistent with Feynman's parton picture? This is the question we would like to address in the present paper.

The group of Lorentz transformations is generated by three rotation and three boost generators [1, 6, 7]. If we use the four-vector convention $x^\mu = (x, y, z, t)$, the generators of rotations around and boosts along the z axis take the form

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{1}$$

respectively. The remaining four generators are readily available in the literature [7]. They are applicable also to the four-potential of the electromagnetic field or to a massive vector meson. These generators in differential form are also available in the literature [7]. J_3 and K_3 take the form

$$J_3 = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \qquad K_3 = -i\left(t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}\right), \tag{2}$$

applicable to the internal space-time variables of an extended object.

The $O(3)$-like little group for a particle at rest is generated by J_1, J_2, and J_3. If the particle is boosted along the z direction with the boost operator $B(\eta) = \exp(-i\eta K_3)$, the little group is generated by $J_i' = B(\eta) J_i B(-\eta)$.

Because J_3 commutes with K_3, J_3 remains invariant under this boost. J and J take the form

$$J_1' = (\cosh\eta) J_1 + (\sinh\eta) K_2, \qquad J_2' = (\cosh\eta) J_2 - (\sinh\eta) K_1. \tag{3}$$

For large values of η, we can consider N_1 and N_2 defined as $N_1 = -(\cosh\eta)^{-1} J$ and $N_2 = (\cosh\eta)^{-1} J$ respectively. Then, in the infinite-η limit [3]

$$N_1 = K_1 - J_2, \qquad N_2 = K_2 + J_1. \tag{4}$$

These operators satisfy the commutation relations

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0. \tag{5}$$

J_3, N_1, and N_2 are the generators of the $E(2)$-like little group for massless particles [1, 2, 3, 7]. In terms of the differential operators, the little group is generated by J_3 of Eq.(2), and by

$$N_1 = -i\left[x\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) - (z-t)\frac{\partial}{\partial x}\right], \quad N_2 = -i\left[y\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) - (z-t)\frac{\partial}{\partial y}\right]. \tag{6}$$

In the infinite-momentum limit, it is more convenient to use the light-cone coordinate system, in which z and t variables are replaced by

$$u = \frac{t+z}{\sqrt{2}}, \qquad v = \frac{z-t}{\sqrt{2}}. \tag{7}$$

When boosted along the z axis, u and v are multiplied by e^{η} and $e^{-\eta}$ respectively. The v variable can thus be ignored when η is very large.

The little group can now be generated by three-by-three matrices applicable to the three-dimensional space of (x, y, u) [3]:

$$J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}. \tag{8}$$

We know how to construct rotation matrices from J_3. As for N_1 and N_2,

$$\exp\left[-i\left(aN_1 + bN_2\right)\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}. \tag{9}$$

This means that the most general form of transformation is

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ a & b & 1 \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \\ u \end{pmatrix} = \begin{pmatrix} \cos(\phi+\alpha) \\ \sin(\phi+\alpha) \\ u + a\cos\phi + b\sin\phi \end{pmatrix}. \tag{10}$$

The rotation around the u axis by α is generated by J_3.N_1 and N_2 generate translations along the u axis. As we noted earlier in this paper, the translation along the u axis is a gauge transformation [3, 6, 7, 10].

Since $\partial/\partial u = (1/sqrt2)\,(\partial/\partial z + \partial/partialt)$, the differential operators N_1 and N_2 of Eq.(6) will become the generators of translations along the u axis:

$$N_1 = -ix\frac{\partial}{\partial u}, \qquad N_2 = -iy\frac{\partial}{\partial u}, \tag{11}$$

if the terms containing the $(z - t)$ can be dropped in Eq.(6). Then, the above operators will generate gauge transformations on the function to which they are applicable.

Let us see whether the covariant harmonic oscillator formalism which Kim and Wigner used in Ref. 5 to construct the third row of Fig. 1 satisfies the above-mentioned condition. The physical basis for this covariant formalism is the phase-space picture of quantum mechanics proposed by Wigner in 1932 [5, 11]. This harmonic oscillator model has been shown to be effective in describing many of the basic phenomenological properties of relativistic hadrons, including hadronic mass spectra, form factors and the parton phenomenon [7].

If the space-time position of two quarks bound together inside a hadron are specified by x_a and x_b respectively, the system can be described the variables [7]:

$$X = (x_a + x_b)/2, \qquad x = (x_a - x_b)/2. \tag{12}$$

The four-vector X specifies where the hadron is located in space and time, while the variable x measures the space-time separation between the quarks. As for the four-momenta of the quarks p_a and p_b, we can combine them into the total four-momentum and momentum-energy separation between the quarks [7]:

$$P = p_a + p_b, \qquad q = \sqrt{2}\,(p_a - p_b)\,, \tag{13}$$

where P is the hadronic four-momentum conjugate to X. The internal momentum-energy separation q is conjugate to x.

The covariant oscillator wave functions are Hermite polynomials multiplied by a Gaussian factor [7], which dictates the localization property of the wave function. The Gaussian factor takes the form [7]

$$\exp\left[-\frac{\Omega}{2}\left(x^2 + y^2 + z^2 + t^2\right)\right]. \tag{14}$$

This expression is localized in the four-dimensional space-time. Since the x and y components are invariant under Lorentz boosts along the z direction, and since the oscillator wave functions are separable in the Cartesian coordinate system, we can drop the x and y variables from the above expression, and restore them whenever necessary. The ground-state wave function can then be written as [5, 7]

$$\psi_\eta(z,t) = \left(\frac{\Omega}{\pi}\right)^{1/2} \exp\left[-\frac{\Omega}{2}\left(e^{-2\eta}u^2 + e^{2\eta}v^2\right)\right]. \tag{15}$$

The Lorentz-deformation property of this form is illustrated in Fig. 2.

As η becomes very large, the distribution in v becomes very narrow. Since $v = (z - t)$, the terms containing $(z - t)$ in Eq.(6) will produce a factor like $v\delta(v)$ when applied to the Lorentz-deformed function of Eq.(15), and can therefore be dropped. The operators N_1 and N_2 then become those given in Eq.(11). Since they satisfy the commutation relations of Eq.(5), N_1 and N_2 of Eq.(11) together with J_3

of Eq.(2) can be chosen for the generators of the little group for relativistic extended particles in the infinite-momentum limit.

The Lorentz deformation property of the Gaussian form is shared by all other functions localized in the zt plane [7]. The distribution becomes narrower in v, while it becomes wider along the u axis in the manner described for the oscillator case in Fig. 2. The generators of the little group do not depend on the shape of wave functions. Therefore, in the infinite-momentum limit, the above conclusion is valid for all distribution functions localized in space and time.

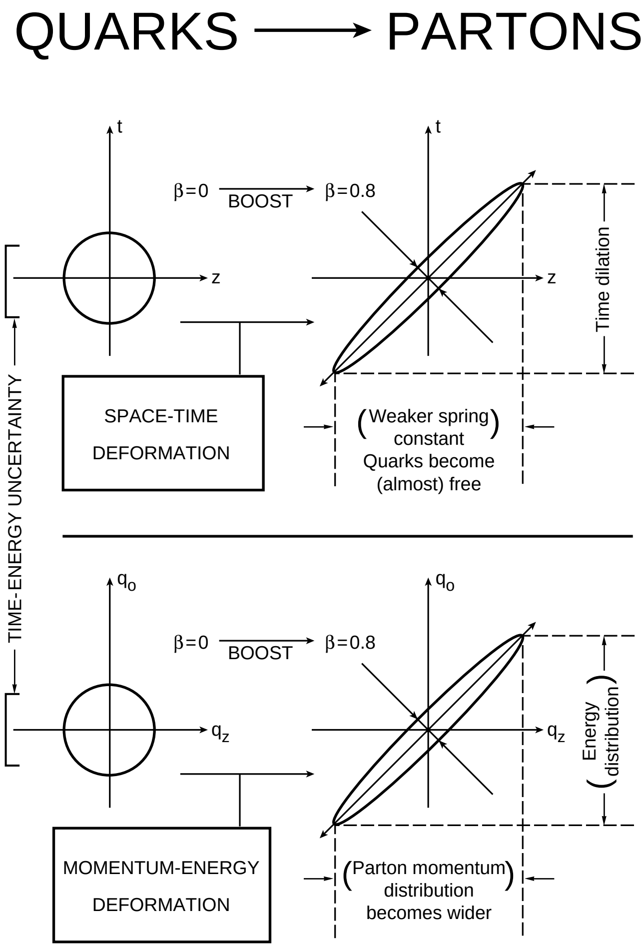

Figure 2: Lorentz deformations of space-time and momentum-energy wave functions. Both of them have the same Lorentz-deformation property. The major (minor) axis in the space-time ellipse is conjugate to the minor (major) axis of the momentum-energy ellipse. This figure explains why a hadron appears as a tightly bound state of quarks to an observer in the Lorentz frame where the hadron is at rest, while it appears as a collection of free partons with a wide-spread momentum distribution to an observer in the frame where the hadron moves very rapidly. This figure is reprinted from Ref. 7.

The same reasoning can be carried out for the momentum-energy space, with $q_u = (q_z - q_o)$ and $q_v = (q_z + q_o)$. The momentum wave function is

$$\phi_\eta\left(q_z, q_o\right) = \left(\frac{1}{\pi\Omega}\right)^{1/2} \exp\left[-\frac{1}{2\Omega}\left(e^{2\eta}q_u^2 + e^{-2\eta}q_v^2\right)\right]. \qquad (16)$$

It has a narrow distribution in q_u or $(q_z - q_o)$, and the N_1 and N_2 operators take the form

$$N_1 = -iq_x \frac{\partial}{\partial q_v}, \qquad N_2 = -iq_y \frac{\partial}{\partial q_v}. \tag{17}$$

These are also the generators of gauge transformations. In the infinite-momentum limit with $t = z$ and $q_z = q_o, u$ and q_v become $\sqrt{2}$ and $\sqrt{2}q_z$ respectively. Both the z and q_z distributions become wide-spread. Furthermore, the momentum of each quark can be parameterized as

$$p_{az} = \xi P_z, \tag{18}$$

where the parameter ξ ranges approximately between zero and 1. This type of distribution was postulated by Feynman in his parton picture of hadrons in the infinite-momentum limit [4].

We can now integrate $|\phi_\eta(q_z, q_o)|^2$ over q_u to get the momentum distribution function. There are three quarks in the proton, and the generalization to the three-quark system is straight-forward. In the large-η limit, the momentum distribution function becomes [7, 12]

$$\rho(\xi) = \frac{3M}{\sqrt{2\pi\Omega}} \exp\left[-\frac{M^2}{2\Omega}(3\xi - 1)^2\right]. \tag{19}$$

This form of the parton distribution function can be compared with the experimental data [12].

The parameter ξ is essentially Feynman's x whose variation produces observable effects. It is linear in the q_v variable, whose variation is generated by N_1 and N_2 of Eq.(11). Feynman's x variable is therefore a gauge transformation parameter in the hadronic system with an infinite momentum.

The author is grateful to professor E. P. Wigner for encouraging him to find a common physical ground for his 1932 paper [11] on the phase-space picture of quantum mechanics and his 1939 paper on the Poincaré group [1].

References

[1] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[2] D. Han, Y. S. Kim, and D. Son, J. Math. Phys. **27**, 2228 (1986).

[3] Y. S. Kim and E. P. Wigner, J. Math. Phys. **28**, 1175 (1987).

[4] R. P. Feynman, in *High Energy Collisions*, Proceedings of the Third International Conference, Stony Brook, New York, edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969). See also J. D. Bjorken and E. A. Paschos, Phys. Rev. **185**, 1975 (1969).

[5] Y. S. Kim and E. P. Wigner, Phys. Rev. A **38**, 1159 (1988).

[6] S. Weinberg, Phys. Rev. **134**, B882 and **135**, B1049 (1964).

[7] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).

[8] In this context, the polarization of neutrinos is a consequence of the requirement of gauge invariance. See D. Han, Y. S. Kim, and D. Son, Phys. Rev. D **26**, 4717 (1982). See also Ref. 7.

[9] E. P. Wigner, *Group Theory and Its Applications to the Quantum Mechanics of Atomic Spectra*, translated from the German by J. J. Griffin (Academic Press, New York, 1959).

[10] J. Kuperzstych, Nuovo Cimento **31B**, 1 (1976).

[11] 11 E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[12] P. E. Hussar, Phys. Rev. D **23**, 2781 (1981). Hussar shows that the Gaussian form of Eq.(19) is numerically close to the input function of the valon model. For the valon model, see R. C. Hwa, Phys. Rev. D **22**, 759 (1980); R. C. Hwa and M. S. Zahir, Phys. Rev. D **23**, 2539 (1981).