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Observational constraints on the generation process of the Earth's magnetic field

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Summary. The most likely origin of the Earth's magnetic field is a dynamo process acting in the liquid core, with motions of the fluid driven by thermal convection. This hypothesis is examined in relation to the constraints imposed by a knowledge of the surface magnetic field, the surface heat flux, and also to some extent the properties of the core. Assuming the Weidemann–Franz relation between the electrical and thermal conductivities for the core, it is possible to obtain lower bounds on the heat flux through the base of the mantle. The minimum heat flux from the core is found to be 2.10^{10} W, independent of either the electrical conductivity or the temperature gradient. By requiring that these bounds be less than the surface heat flux, the product of the electrical conductivity and the temperature gradient may be bounded both from above and from below.

The heat flux estimates are improved by considering ohmic heating from the toroidal magnetic field, which cannot be observed directly, and the convected heat flux. An expression is obtained for a bound of the ohmic heating of the toroidal field in terms of the radial fluid velocity. A comparison with existing dynamo models reveals that the true value of the ohmic heating is much larger than the ideal bound. This is evidence against the existence of a toroidal field as large as, say, 10 mT.

The heat fluxes are larger for heating by uniformly-distributed heat sources than for latent heat from growth of the inner core. Unless the adiabatic gradient is below about 0.1 K km^{-1}, it is not possible to generate the magnetic field by distributed heat sources. The latent heat case does not suffer the same difficulty, but then it is possible for the outer regions of the core to be density-stratified. Gravitational energy released by accretion of the inner core may also be an appreciable or even dominant factor.

Introduction

There is little doubt that the Earth's magnetic field arises from complex motions of the fluid in the liquid core. This is called the dynamo theory. However, the nature of the energy source required to drive the fluid and sustain the magnetic field against resistive dissipation throughout most of geological time is a matter for speculation. Of the several proposed energy sources, heat sources in the form either of radioactivity or latent heat released by the gradual freezing of the core seem the most plausible. These heat sources drive fluid motions by thermal convection. All the arguments for or against any particular energy source are limited by the appalling lack of information about the composition and state of the liquid core, and the absence of usable data. This paper sets out to determine how the various parameters needed to model a dynamo process are constrained by the observations made at the surface. The modelling problem is so complicated and the data so sparse that all the available observations can probably be explained simply by adjusting the many parameters of any particular model. Therefore bounding techniques such as those used in Backus (1957) are followed. This approach has been developed by R. L. Parker (1972) for dealing with 'grossly inadequate data'.

Suppose that any theory must account for the following observations. Firstly, it must match the observed magnetic field. By assuming the mantle to be a perfect insulator, the field may be continued downwards to provide boundary conditions at the core mantle interface. The reference field used is that of Cain *et al.* (1967), model GSFC67 for 1965. Secondly, the convective processes must not transmit more heat through the core mantle boundary than is seen to emerge from the surface of the Earth. This is an assumption, because it may be that the lower mantle is still heating up, and the heat is not transmitted rapidly enough for it to have reached the Earth's surface. However, a very substantial amount of heat coming from the core would cause the lower mantle to convect, or possibly to melt, and therefore it seems very likely that the core heat flux be rather small compared with the total surface flux. A reasonable upper limit on the heat flux flowing through the ocean floor, excluding the contribution from radioactive isotopes on the continents, is 10^{13} W. This allows for possible heat transport by hydrothermal circulation at ocean ridges (Williams *et al.* 1974). Lastly, the Wiedemann–Franz law is assumed to hold. In this law the electrical and thermal conductivities are related by assuming that heat is transmitted by conduction electrons alone, and that the conduction electrons form a Fermi gas. Of all the constitutive laws applied at extreme pressures, this seems the most reasonable. Some heat may be transmitted through kinetic energy of the atoms (Stacey 1969). If k and σ are the thermal and electrical conductivities respectively, and T is the temperature, the law states that,

$$k = 2.45 \times 10^{-8} \sigma T \, \mathrm{W\,m^{-2}\,s^{-1}}.$$

In view of the phonon contribution to k, the law may be applied as:

$$k \geqslant 2.45 \times 10^{-8} \sigma T_{\mathrm{min}},$$

a form that is useful when σ is very small and the phonon heat conduction is significant. Before trying to constrain the properties of the core with this meagre set of assumptions and data, some current ideas on the state of the core will be reviewed.

Energy sources other than thermal convection include precession, tidal forces, chemical differentiation of the core and earthquakes. Precessional torques can supply enough energy to sustain the magnetic field with only a modest rate of decrease of the obliquity, but there are serious objections to the passage of sufficiently large amounts of energy into the core by this process (Rochester *et al.* 1975; Loper 1975). These authors conclude that the available energy falls short by more than an order of magnitude. Earthquakes suffer from having insufficient energy, but a more convincing argument has been used by Gubbins (1975), showing that short-period oscillations which are of sufficient amplitude to generate the magnetic field would also be accompanied by large gravity anomalies observable from the surface of the Earth. Tidal forces have not been studied in detail but are likely to suffer from the same problems. The idea of chemical differentiation comes from Braginsky (1964) and has not been studied at all. It poses certain geochemical and thermal history problems but there is no heat flux problem. Thermal convection seems plausible, and it has also been well studied and is quite well understood, and will be the first theory to be thoroughly tested. Recent thought on thermal convection has been dominated by the paper by Higgins & Kennedy (1971), who suggest that the temperature gradient in the core must be subadiabatic in order to remain liquid. The arguments are not conclusive (Jacobs 1972; Verhoogen 1973), but plausible. Some ways out of the difficulty have been suggested. The inner core may not be ordinary solid iron but may be an electronic phase transition of iron (Elsasser & Isenberg 1949; Stacey 1975). This may also cause difficulties in estimating parameters like the electrical conductivity in the liquid core. However, recent calculations by Bukowinsky & Knopoff (1976) indicate that core pressures are too low for the transition to occur. An alternative possibility is that both solid and liquid phases are present and convection is occurring at the melting point (Elsasser 1972; Busse 1972). Malkus (1973) estimates that the whole core would freeze rather rapidly, and that the solid inner core is a relatively recent feature of the Earth's interior. This would be the appropriate regime of convection if Higgins & Kennedy (1971) are right, and the adiabatic gradient is steeper than the melting gradient, and also if the Earth's core is cooling everywhere along the melting curve. The issue is still an open question, and so in this paper the most straightforward hypothesis of ordinary thermal convection is adopted.

Most of the arguments about the core rely heavily on estimates of properties of iron at high pressure. Measurements and calculations of these properties are very difficult to make, and the presence of unknown impurities of questionable composition exacerbate the problem. These estimates provide invaluable guidelines in modelling the core, and some recent estimates are given in Gubbins (1974). However, there is a tendency to take estimates of quantities such as the electrical conductivity and viscosity to be more reliable than they really are.

Jain & Evans (1973) have calculated the electrical conductivity of pure iron at the appropriate pressure and reasonable temperatures using modern fluid state methods. This is the best estimate that can be made from theory at present, although in his review Ziman (1973) stresses the difficulty of performing calculations at very high pressure. The calculated value is in rough agreement with Mitchell & Keeler's (1971) shock wave data, and extrapolations of laboratory experiments by Gardiner & Stacey (1971). The latter is probably the least reliable method. Surprisingly little work has been done on the effects of impurities, Johnston & Strens (1973) and the discussions in Stacey (1969) being exceptions. The problem appears complicated for sulphur. This work has led to a value for σ of $5.10^5\,\mathrm{mho\,m^{-1}}$. The error in the value for pure iron may be as much as a factor of ten, possibly much more if electronic phase transitions are possible. The unknown factor of impurities may introduce further errors of comparable magnitude. Granted the possibility of such large errors, the dynamo theory may be able to contribute some useful constraints on the electrical conductivity.

The core viscosity ranks as one of the poorest-known geophysical quantities. For the purposes of thermal convection it may be taken as very small, rotational and magnetic effects being dominant. Two different varieties of estimates have been made, one set from seismology and motions of the whole Earth in space, and one from calculations for liquid iron. Sato & Espinosa (1967) give a seismological estimate of $10^6\,\mathrm{m^2\,s^{-1}}$, a very high value. Q values for multiply reflected *P*-waves are given in Qamar & Eisenberg (1974), Cormier & Richards (1976).

The viscosity values may be upper limits consistent with the observed attenuation of seismic waves through the whole Earth, but are not realistic values of the actual Newtonian viscosity relevant to the ponderous motions of dynamo theory. A large viscosity would present grave difficulties for longer term phenomena such as the Chandler and nearly diurnal wobbles. Verhoogen (1974) used the damping time of the Chandler wobble to estimate a reasonable core viscosity, but his argument appears to be in error (Rochester 1975). Toomre (1974) has given the tightest observational bound of $\nu \leqslant 10\,m^2 s^{-1}$ based on the absence of any observation of the nearly diurnal wobble, using a very idealized fluid flow in the core. This is an improvement over Hide's (1971) estimate of $10^2\,m^2 s^{-1}$, which was based on speculative arguments about the nature of core–mantle coupling and the geometry of the core–mantle boundary. Of the second variety of estimates for ν, the most frequently used is Gans' (1972): $\nu = 6 \times 10^{-6} m^2 s^{-1}$, using the Andrade (1934) hypothesis. Lepaluoto (1972) has used significant structure theory (Eyring & Jhon 1969) to arrive at values of 10^{-6} to 5×10^{-6} $m^2 s^{-1}$, close to Gans' value. Both the theories used involve *ad hoc* assumptions about the distribution of molecules in a liquid, for which there is no physical basis. Significant structure theory is criticised by Rowlinson (1970). Chapman's (1966) corresponding states theory is preferred because it is derivable from statistical mechanics. Somewhat surprisingly, this leads to values of about ten times lower than Gans and Lepaluoto. The preferred estimate for pure iron at 4000 K is $6 \times 10^{-7} m^2 s^{-1}$. Any solid particulate matter will alter the apparent viscosity.

The controversy over the adiabatic and melting point gradients has been mentioned. The two separate gradients are both very difficult to estimate, making resolution of the sign of the difference impossible. The difference may be detected in principle by seismology (e.g. Gilbert & Dziewonski 1975), but accuracy and resolution have not been estimated. The actual temperature can be estimated from the melting point of pure iron. Recent values range from 4000 to 7000 K (Higgins & Kennedy 1971; Verhoogen 1973; Boschi 1974). Although this is a poor state of affairs, temperature is not a crucial parameter.

Two different heat sources have been proposed. Verhoogen (1961) suggests that heat can be supplied by gradual cooling of the Earth and freezing of the inner core. This argument seems satisfactory provided the magnetic field can be maintained with the rather small quantity of heat available. The existence of radioactive isotopes in the core has always involved geochemical difficulties, although recent arguments favour sulphur and potassium (K^{40}) present in the core (Rama Murthy & Hall 1970; Lewis 1971). There are reasons to doubt the viability of such a mixture (Oversby & Ringwood 1972; Ganguly & Kennedy 1976).

Another property of the Earth's core is its toroidal magnetic field. The poloidal field is observed directly and in the core is about 0.50 mT, but the toroidal part of the magnetic field can only be inferred by indirect methods such as from observations of the secular

Table 1. Viscosities of pure iron at examples of core density and temperature

		η ($g\,m^{-1} s^{-1}$)	ν ($mm^2 s^{-1}$)
$T = 4000$ K, $\rho = 12\ Mg\,m^{-3}$	Pure iron	6.3	0.53
$T = 7000$ K, $\rho = 10\ Mg\,m^{-3}$	Pure iron	3.1	0.31

variation. Some authors (e.g. Elsasser 1946; Bullard & Gellman 1954; Braginsky 1964; Hide 1966) have required the toroidal field to be very large, from 10–100 mT. The idea receives some support from the presence of a strong field of this type on the Sun. However, there is no fundamental reason for its existence arising in the dynamo theory and it does imply a very large ohmic heat flux which may be too high for the Earth. Hide (1966) estimated the field as 10 mT from his theory of the westward drift, this strength being necessary to explain the rate of drift. The theory is only poorly understood and it may well be that the westward drift of the non-dipole field could have some other explanation, or be an inherent feature of the dynamo process. The existence of such large magnetic fields has yet to be demonstrated.

Because the properties of the core are so poorly known, we return to the problem stated earlier in this section and try to determine what can be deduced from fairly rigorous arguments and observable data.

Preliminary estimates of the heat flux

Backus (1975) and Hewitt, McKenzie & Weiss (1975) have shown that the 'efficiency' of a thermally-driven dynamo may be calculated from the entropy equation. In steady state, for a closed volume, we have

$$\int \frac{DS}{Dt}\, dV = \int \frac{\mathbf{J}^2}{\sigma T}\, dV + \int k\left(\frac{\nabla T}{T}\right)^2 + \int \left(\frac{q}{T} - \frac{q}{T_{\min}}\right) dV \qquad (1)$$

where S is specific entropy, q and Q the heat source per unit volume and total heat source respectively, and $\mathbf{J}$ the current density. $T_{\min}$ is the temperature at the core mantle boundary. Equation (1) leads to a formal bound on the 'efficiency' ratio Φ/Q, where Φ is the total ohmic heat generated

$$\Phi = \frac{\int \mathbf{J}^2 dV}{\sigma}.$$

Two different heat sources will be considered but the only important difference for the dynamics lies in their location.

Two extreme cases to be considered are those in which (a) all heat is released at the inner core boundary, and (b) heat sources are distributed uniformly throughout the liquid outer core, and are arbitrarily distributed in the inner core in such a way that the steady heat conduction equation is satisfied with the prescribed temperature at the inner core boundary. (a) Corresponds to latent heat released at the inner core boundary, but could equally well arise if, for some obscure geochemical reason, radioactive elements were concentrated in the inner core. For the former case, which will be referred to as 'freezing', heat may be released everywhere by gradual cooling of the whole core and in Verhoogen's (1961) calculations this appears as an important factor. (b) Is supposed to correspond to distributed radioactive heat sources and will be called 'internal heating'.

Returning to equation (1), and following the authors cited, a bound on Φ/Q may be obtained by setting:

$$\int \frac{\mathbf{J}^2}{T\sigma} \geqslant \frac{\Phi}{T_{\max}},$$

where T_{max} is the temperature at the inner core boundary. This device cannot be improved without a detailed model of the magnetic and temperature fields. The positive definite quantity

$$\int k\left(\frac{\nabla T}{T}\right)^2 dV$$

is omitted in deriving upper bounds on Φ/Q, but for any reasonable temperature gradient this entropy of conduction is at least an order of magnitude smaller than the contribution from the heat flux. For the freezing case heat is deposited at the highest temperature and therefore

$$\int \frac{q}{T}\, dV \geqslant \int \frac{q\, dV}{T_{max}} = \frac{Q}{T_{max}}.$$

This yields Backus' result

$$\frac{\Phi}{Q} \leqslant \frac{T_{max} - T_{min}}{T_{min}} = \frac{\Delta T}{T_{min}}. \quad (2)$$

For internal heating the 'efficiency' factor is smaller because the heat sources are not deployed in the most advantageous distribution. Hewitt *et al.* (1975) find an additional factor 2/5 in equation (2) for this case, using a Boussinesq argument. This factor is used here, although it can be calculated for a particular choice of temperature without Boussinesq assumptions. Some aspects of kinematic dynamo theory are inherently non-Boussinesq, such as the 'cyclonic' or 'helicity' arguments of E. N. Parker (1955) and Steenbeck, Krause & Rädler (1966), in which expansion of rising gas in a rotating system provides the necessary helical motion. The concepts may be important for the Sun where the fluid of the convection zone passes through many scale heights, but are probably not so for the Earth's core.

R. L. Parker (1972) found a correct lower bound on the ohmic heating, but concluded that a typical estimate would be too low to be geophysically interesting. However, coupled with the thermodynamic efficiency, the bound is more restrictive. The correct calculation for the ohmic heat flux is to minimize

$$\int (\nabla \times \mathbf{B})^2\, dV,$$

where the volume integral is over the whole core. Dissipation in the mantle is negligible by comparison (Rochester *et al.* 1975). The magnetic field is constrained by having to match the observed external field, which plays the part of a boundary condition. Values of the heating for harmonics up to degree 8 are given in Gubbins (1975). This assumes that permanent magnetization of the crust does not contribute to the lower harmonics in the geomagnetic field. The decreasing heat flux contribution with increasing order of the harmonics suggests that this is true. Parker (1972) had performed the calculation for the dipole field alone, and inclusion of higher harmonics increases the bound by a factor of 3. Thus, assuming uniform electrical conductivity, the ohmic heat flux may be bounded as

$$\Phi \geqslant \frac{7.558 \times 10^{13}}{\sigma}\ \mathrm{W}$$

so that the heat flux through the core–mantle boundary is at least

$$Q \geqslant \frac{7.558 \times 10^{13}\beta T_{\min}}{\sigma \Delta T} \tag{3}$$

where $\beta = 1$ or $5/2$ depending on the heating. Equation (3) is an estimate of the heat flux out of the core which may be bounded in terms of that at the Earth's surface. This leads to a lower bound on the product $\sigma\Delta T$.

A second estimate of the heat flux is obtained in terms of $\sigma\Delta T$ using the Wiedemann–Franz law. In a convecting system the total heat transport is equal to that conducted down a conduction gradient plus that convected by the fluid motion. The total heat transport must exceed the conducted heat flux, so that the conduction solution for the temperature in the core provides another lower bound on the heat flow. This estimate also depends on the nature of the heating. For distributed radioactive heat sources, the heat flowing across a spherical surface of radius r will increase as r^3, so the heat flowing across unit area and the temperature gradient must both increase linearly with r. On the other hand, by freezing the inner core the heat flowing across any spherical surface in the outer core will be constant, so that the temperature gradient will decrease as r^{-2}. In fact, for fixed ΔT, the radioactive heating case leads to a substantially larger temperature gradient at the core–mantle boundary. If r_o (=3485 km) and r_i (=1240 km) are the radii of the outer and inner cores respectively, the heat fluxes conducted down the temperature gradients for given ΔT and constant thermal conductivity are

$$Q_i \geqslant \frac{8\pi k\Delta T r_o^3}{r_o^2 - r_i^2} = 1.003 \times 10^8\, k\Delta T\,\text{W for internal heating}$$

and

$$Q_f \geqslant \frac{4\pi k\Delta T r_i r_o}{(r_o - r_i)} = 2.419 \times 10^7\, k\Delta T\,\text{W for freezing.}$$

The Wiedemann–Franz law may be used to relate k to σ giving

$$Q_i \geqslant 2.45 T_{\min}\,\sigma\Delta T\,\text{W}$$

and (4)

$$Q_f \geqslant 0.593 T_{\min}\sigma\Delta T\ \text{W}.$$

When equation (4) is bounded by the observed surface heat flux, an upper bound on the product $\sigma\Delta T$ is obtained. The result can be appreciated more easily when plotted on a diagram as in Fig. 1. The temperature $T_{\min}$ can be estimated with confidence, and in any case its exact value does not alter the appearance of Fig. 1 significantly. A value of 3000 K was taken. The figure shows the heat flux estimates plotted as functions of the parameter $\sigma\Delta T$. To be consistent with the data, the actual heat flux from the core and the value of $\sigma\Delta T$ must lie somewhere inside the triangle ABC for the freezing case, and the triangle A′B′C′ for heating from within. This area is very broad because Fig. 1 is plotted on a log–log scale, but this is to be expected because the derived bounds are so liberal. However, a number of interesting conclusions may be drawn from an examination of the figure.

Points A and A′ correspond to the smallest possible heat flux from the core in each case. Whatever the value of the electrical conductivity or the temperature gradient, the heat flux must exceed 5.62×10^{10} or 2.00×10^{10} W, depending on the type of heating. This heating is substantial and may be significant for lower mantle dynamics. Secondly, in the derivation of the ohmic heat flux on which Fig. 1 is based, the toroidal magnetic field was taken as

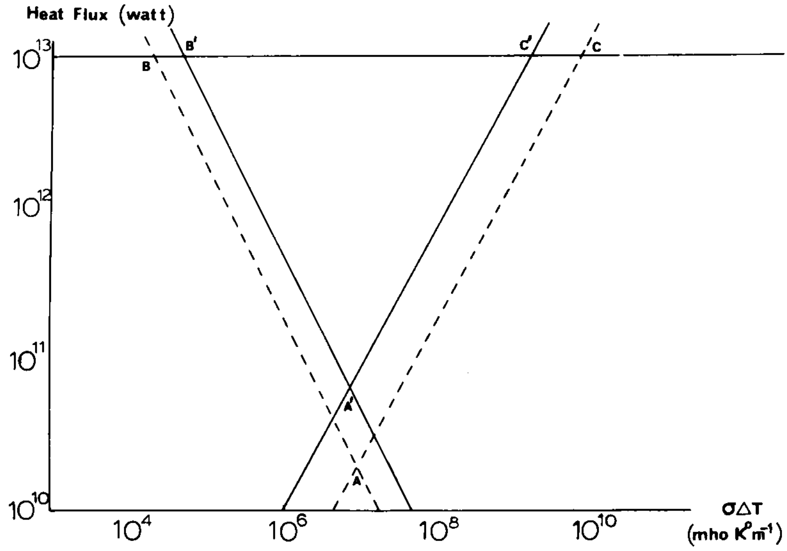

Figure 1. Lower bounds on the heat flux from the Earth's core as a function of $\sigma\Delta T$ from the ohmic heating and the conducted heat. The dotted lines in each case are for the 'freezing' mode of heating.

zero because it cannot be observed directly. However, the toroidal field is almost certainly present and it will contribute to the ohmic heating, with the tendency to move the lines AB to the right. The absolute limit will be to move the line to C or C′, a position which yields an upper bound on the toroidal magnetic field. Thus, using

$$\int (\nabla \times \mathbf{B})^2 \, dV \geqslant \left(\frac{4.493}{r_0}\right)^2 \int \mathbf{B}^2 \, dV$$

(Backus 1958), and defining $\bar{\mathbf{B}}$ the mean magnetic field by

$$\bar{\mathbf{B}}^2 = \int \mathbf{B}^2 \, dV \Big/ \int dV$$

leads to $\bar{B} \leqslant 99$ or 317 mT, the lower of which is below some estimates made by Braginsky (1964).

Convection will increase the heat flux above the conducted value, and this will tend to move the lines AC to the left, the limit being reached at points B or B′ which gives an upper limit on the Nusselt number of 2×10^4 or 6×10^4. These estimates are very high and not restrictive. Finally, the parameter $\sigma\Delta T$ is bounded by the extremes at B and C, giving

$2.27 \times 10^4 \leqslant \sigma\Delta T \leqslant 5.62 \times 10^9$ (freezing)

or

$5.67 \times 10^4 \leqslant \sigma\Delta T \leqslant 1.36 \times 10^9$ (internal heating).

Typical estimates of $\sigma\Delta T$ range from 2×10^8 to 8×10^8, using ΔT from Jacobs (1975), Higgins & Kennedy (1971) and Verhoogen (1973), and $\sigma = 5 \times 10^5 \, \text{mho m}^{-1}$. These estimates are to the right of the triangle in Fig. 1, so that the upper limit on ΔT is a significant bound. The lower limit is less restrictive but still proves important in eliminating whole classes of models of the core. The above estimates are quite far apart and could be made tighter by limiting the heat flux from the core to be some fraction, say 10 per cent, of the observed surface heat flux. This would not affect $\bar{B}$ or the Nusselt number by much but would close the gap in $\sigma\Delta T$ by two orders of magnitude. However, such arguments are speculative and a more satisfactory way to proceed is to search for additional physical effects that will further improve both the estimates of the ohmic heating and the convected heat flux, and this approach is pursued in the next two sections.

Improved estimates of the ohmic heat flux

The ohmic heat flux estimate may be improved by including the effect of the toroidal magnetic field. The author knows of no 'anti-dynamo' theorem prohibiting the maintenance, by a dynamo, of a magnetic field which is everywhere purely poloidal, or everywhere purely toroidal. The latter requires that the radial component of velocity be constant along magnetic field lines, which is a strong restriction but not necessarily strong enough to prohibit the existence of solutions to Maxwell's equations. This case is important because it would allow dynamo processes in the core which could not be detected by examining the surface magnetic field. This would be an 'annihilator' in the sense used by Parker & Huestis (1974). The case of zero toroidal field does not impose any simple restriction on the fluid flow. Childress (1968) claims to have proved, or found in the literature, proofs of both these theorems, but his references do not give any explicit derivation. Note that his reference to Backus (1957) is in error. The possibility of a zero toroidal magnetic field has not yet been eliminated.

The toroidal field may be partially accounted for using Busse's (1975a) relationship between the radial component of velocity for a dynamo and the ratio of energies in the poloidal and toroidal fields. The following argument suggests that both the ohmic and the conducted heat flux estimates may be increased, depending on the magnitude of the radial velocity. Busse (1975a) has shown that the toroidal field and radial velocity are inversely related. A small radial velocity implies a large toroidal field. Furthermore, the heat transport can be expected to be larger for a larger radial velocity, other factors being equal. In terms of Fig. 1, a small radial velocity would demand a high ohmic heat flux from the toroidal magnetic field, and the appropriate region of parameter space would be to the right of the triangle ABC. The convective contribution would have to be correspondingly small. Alternatively a small toroidal magnetic field, because of the smaller ohmic heating, would open up the left side of triangle ABC. The accompanying radial velocity would be large, implying a large convective contribution to the heat flux. In this case the left-hand part of ABC is appropriate. In general both estimates of the heat flux may be enlarged to a greater or lesser degree, depending on the radial velocity. It is possible to make this qualitative argument rigorous for the ohmic heating, but not for the convection. Probably both estimates could be improved in connection with a detailed dynamo model. The selection of such a model at this stage is premature, so the problem is first examined in as much generality as possible.

The poloidal magnetic field equation is

$$\frac{\partial}{\partial t}(\mathbf{r}\,.\,\mathbf{B}) + \mathbf{v}\,.\,\nabla(\mathbf{r}\,.\,\mathbf{B}) = \eta\nabla^2(\mathbf{r}\,.\,\mathbf{B}) + \mathbf{B}\,.\,\nabla(\mathbf{r}\,.\,\mathbf{v})$$

where

$$\eta = (\mu_0\sigma)^{-1}$$

which for steady state gives

$$\int \mathbf{B}^2 \, dV \geqslant \frac{\eta^2}{(\upsilon . \mathbf{r})^2_{\max}} \int [\nabla(\mathbf{B}.\mathbf{r})]^2 \, dV$$

(Busse 1975a).

Defining the magnetic Reynolds number $R = (\mathbf{v} . \mathbf{r})_{\max}/\pi\eta$, and using the inequality

$$\int (\nabla \times \mathbf{B})^2 \, dV \geqslant \frac{\pi^2}{r_o^2} \int \mathbf{B}^2 \, dV = \Phi_1$$

gives

$$\int (\nabla \times \mathbf{B})^2 \, dV \geqslant \frac{1}{r_o^2 R^2} \int [\nabla(\mathbf{B}.\mathbf{r})]^2 \, dV. \tag{5}$$

Equation (5) forms the basis for a new estimate of the ohmic heat flux in terms of the magnetic Reynolds number R. The magnetic field is conveniently represented in terms of its poloidal and toroidal vector harmonics:

$$\mathbf{B} = \sum_{l,m} \nabla \times [T_l^m(r) \, Y_l^m(\theta, \phi) \, \hat{\mathbf{r}}] + \nabla \times \nabla \times [S_l^m(r) \, Y_l^m(\theta, \phi) \, \hat{\mathbf{r}}] \tag{6}$$

where $Y_l^m(\theta, \phi)$ are surface harmonics in common use in geomagnetism normalized so that

$$\int (Y_l^m)^2 \, d\Omega = \frac{4\pi}{(2l+1)}$$

and $\hat{\mathbf{r}}$ is the unit radial vector.

The integral on the right-hand side of (5) may be written as

$$\sum_{l,m} \frac{4\pi l^2(l+1)^2}{(2l+1)} \int_0^{r_o} \left\{ r^2 \left[\frac{S_l^m(r)}{r} \right]'^2 + l(l+1) \left[\frac{S_l^m(r)}{r^2} \right]^2 \right\} dr$$

where the prime denotes differentiation with respect to r. The functions $S_l^m(r)$ must behave at the origin like r^{l+1}, to ensure that the components of the magnetic field are continuous there. At the core–mantle boundary the components of $\mathbf{B}$ must match the external potential field in the insulator, which is known in terms of the scalar potential. Since both normal and tangential components of $\mathbf{B}$ must match, the boundary conditions are on both the value of the function S_l^m and its radial derivative. If g_l^m and h_l^m are the usual geomagnetic coefficients for the harmonics dependent on cosine and sine of longitude respectively, the magnetic field may be continued downwards from the surface of the Earth to the core–mantle boundary to give the following boundary conditions on S and its derivatives

$$S_l^{m(\text{c or s})}(r_o) = -\frac{(g_l^m \text{ or } h_l^m) \, a^{l+2}}{l r_o^l} = G_l^m$$

$$S_l^{m'(\text{c or s})}(r_o) = \frac{(g_l^m \text{ or } h_l^m) \, a^{l+2}}{r_o^{l+1}} = -\frac{l G_l^m}{r_o} \tag{7}$$

where a is the Earth's radius.

Suppose one tried to minimize the magnetic energy subject to these boundary conditions. The Euler equation is only of second order, making it impossible to satisfy both the boundary conditions and the continuity conditions at the origin. The difficulty is similar to one of finding the function $y(x)$, defined between $x = 0$ and $x = 1$, with, say, $y(0) = 0$ and $y(1) = 1$, that minimizes the area under the curve. The only solution is the pathological 'corner' path defined by the lines $x = 1$ and $y = 0$, which would not be available from the usual method of calculus of variations. It may be concluded that the observation of the magnetic field at the Earth's surface alone is not sufficient to infer the existence of any magnetic energy in the core at all. Exactly the same problem arises with the integral on the right-hand side of equation (5), and it cannot be bounded above zero by the available observations. This makes equation (5) useless in estimating heat fluxes. The straightforward miniminization of

$$\int (\nabla \times \mathbf{B})^2 \, dV,$$

done in Section 2, was elementary because this leads to a fourth-order Euler equation, for which the correct number of boundary conditions are available.

The obvious way out of this difficulty would be to find an alternative expression to equation (5) which relates the ohmic heating from toroidal fields to a quantity which can be bounded with observable data. This does not seem to be possible, and an inspection of the very complicated equation governing the toroidal scalar (Backus 1958, equation (78a)), suggests that such a result cannot be obtained, at least in a simple and usable form. This is resolved by combining two estimates of the heat flux. We have

$$\int (\nabla \times \mathbf{B})^2 \, dV \geqslant \int (\nabla \times \mathbf{B}_\mathrm{p})^2 \, dV = \Phi_2 \tag{8}$$

where $\mathbf{B}_\mathrm{p}$ is the poloidal magnetic field and Φ_2 is the ohmic heat flux derived in the previous section. The dissipation integral must exceed both Φ_1 and Φ_2 or any linear combination of the form

$$\int (\nabla \times \mathbf{B})^2 \, dV \geqslant \Phi(\alpha) = \alpha \Phi_1 + (1 - \alpha) \Phi_2$$

where $0 \leqslant \alpha \leqslant 1$, a parameter that may be chosen so as to maximize the estimate $\Phi(\alpha)$. Thus

$$\Phi(\alpha) = \sum_{l,m} \frac{4\pi l(l+1)}{(2l+1) r_o^3} \int_0^1 (1-\alpha) [D_l y]^2 + \frac{\alpha l(l+1)}{R^2} \left[x^2 \left(\frac{y}{x}\right)'^2 + l(l+1) \left(\frac{y}{x}\right)^2 \right] dx \tag{9}$$

where $x = r/r_o$, $y(x) = S_l^m(x)$ and $D_l y = y'' - l(l+1) y/x^2$ and prime here denotes differentiation with respect to x. Minimizing equation (9) leads to the Euler equation

$$y^{iv} - l(l+1) \left(\frac{y}{x^2}\right)'' - l(l+1) \frac{y''}{x^2} + l^2(l+1)^2 \frac{y}{x^4} + f \left\{ -y'' + l(l+1) \frac{y}{x^2} \right\} = 0 \tag{10}$$

where

$$f = \frac{\alpha l(l+1)}{R^2 (1-\alpha)}.$$

Equation (10) is solved by Frobenius' method to give a solution of the form

$$y(x) = b x^{l+1} + \sum_{k=0}^{\infty} a_k x^{k+l+3} \tag{11}$$

where the $a_{2k+1} = 0$ and the a_{2k} are proportional to f^k. Thus when $f = 0$, corresponding to no toroidal field, the solution reduces to $A x^{l+1} + B x^{l+3}$, which was obtained by Gubbins

(1975) and used in Section 2. Retaining only the first two terms in x^{l+1} and x^{l+3} and satisfying the boundary conditions gives a value for $\Phi(\alpha)$

$$\Phi(\alpha) = \frac{4\pi l(l+1)}{(2l+1)} \sum_{l,m} 4(2l+1)^2(2l+3)(1-\alpha) + \frac{\alpha}{R^2} l(l+1)\, p(l)\, (G_l^m)^2$$

where $p(l) = (2l+3)^2 - 2(2l+3)(2l+1) + (2l+1)^2(2l^2+5l+4)/(2l+5)$. The estimate for $\Phi(\alpha)$ can be improved beyond Φ_2 only if the coefficient of α is positive, or $R \leqslant R_c$. Using the GSFC67 magnetic field harmonics up to degree 8, one obtains $R_c = 0.454$. This value of R_c is quite small, and can be compared with Backus' (1958) necessary condition for dynamo action

$$R_m = \frac{v_{max}\, r_o}{\pi\eta} \geqslant 1.$$

Therefore the bound of $R < 0.454$ can only be useful for dynamo models in which the radial velocity, which appears in the magnetic Reynolds number R, is much less than the maximum component of velocity, which appears in R_m. Such models are of general interest and form a separate class, the Braginsky dynamo being an example.

The size of the improvement in Φ can be determined by solving the Euler equation with more terms in the power series. Calculations retaining two more terms in (11) show that Φ cannot be increased by much more than a factor of two. A better approach is to find the asymptotic solution uniformly valid as $f \to \infty$. This singular perturbation problem may be solved by standard methods of matched asymptotic expansions (Cole 1968). Set $\epsilon^2 = f^{-1}$, and let $\epsilon \to 0$. The 'outer' solution, valid in the main body of the fluid and satisfying continuity conditions at the origin, is written

$$y_1(x;\epsilon) = h_0(x) + \epsilon h_1(x) + \dots$$

and there is an exact solution

$$y_1(x;\epsilon) = A(\epsilon)\, x^{l+1}.$$

This solution cannot satisfy the conditions at $x = 1$, the core–mantle boundary, because of the difficulty discussed in relation to bounding the magnetic energy. The 'inner' expansion, valid near $x = 1$, is written as

$$y_2(x;\epsilon) = g_0(\xi) + g_1(\xi) + \dots$$

where $\xi = (1-x)/\epsilon$ is the 'boundary layer' coordinate. This solution must satisfy both boundary conditions at $x = 1$, so that

$$g_0(0) = G_l^m \qquad g_0'(0) = 0$$

$$g_1(0) = 0 \qquad g_1'(0) = -\epsilon l G_l^m.$$

$$g_n(0) = g_n'(0) = 0 \quad \text{for } n \geqslant 1.$$

The equations for g_0 and g_1 are

$$g_0^{iv} - g_0'' = 0 \qquad g_0(\xi) = A_1 \exp(\xi) + A_1 \exp(-\xi) + A_3\xi + A_4$$

$$g_1^{iv} - g_1'' = 0 \qquad g_1(\xi) = B_1 \exp(\xi) + B_2 \exp(-\xi) + B_3\xi + B_4.$$

Fitting boundary conditions and matching 'inner' and 'outer' solutions as $x_\delta = (1-x)/\delta(\epsilon)$ remains finite, $\delta(\epsilon) \to 0$, $\delta(\epsilon)/\epsilon \to \infty$ and $\epsilon \to 0$, determines all the unknown constants and $A(\epsilon)$ to order ϵ

Outer: $y_1(x;\epsilon) = G_l^m x^{l+1}$

Inner: $y_2(x;\epsilon) = G_l^m [1 + \epsilon\{(2l+1)[\exp(-\xi) - 1] + (l+1)\xi\}]$.

The composite series, uniformly valid in $0 < x \leqslant 1$, is

$$y(x) = G_l^m \{x^{l+1} + \epsilon(2l+1)[\exp(-(1-x)/\epsilon) - 1]\}.$$

With this solution, the leading term in the integral for $\Phi(\alpha)$ is

$$\Phi(\alpha) = \sum_{l,m} \frac{\alpha l^2(l+1)}{R^2} G_l^m \qquad (12)$$

which is clearly a maximum when $\alpha = 1$. This new estimate for the ohmic heat flux may be compared with the value in Section 2, and exceeds it for magnetic field model GSFC67 only if $R \leqslant R_a = 0.217$. For sufficiently small radial velocities, the toroidal field must contribute to the heat flux. The critical value, R_a, is lower than that derived from Frobenius' method, R_c, because the asymptotic method requires the radial velocity to be very small.

Equation (12) may be compared with some actual kinematic dynamo models. The only numerical calculations available are those of Gubbins (1973), Pekeris, Accad & Shkoller (1973) and Kumar & Roberts (1975). The first two have small toroidal magnetic fields and relatively large radial velocities, with R between 1 and 10. Therefore equation (12) cannot be used to estimate the toroidal field. Kumar & Roberts' calculations are for Braginsky-type dynamos. They show large toroidal fields, and when the dipole is normalized to the Earth's value the ohmic heat flux is comparable with the surface heat flux. However, the radial velocities are rather large, again with $R \approx 1$. There is a great discrepancy between the true estimate of heat and the bound in equation (12). This is caused by the long succession of inequalities that were used in order to derive the general result equation (9). It may be that some dynamo schemes do have small radial velocities but have not yet been discovered.

By examining the Braginsky type of model in some detail it becomes clear why the bound is so low for this case. The velocity and magnetic fields are scaled with the magnetic Reynolds number, which is taken as a large parameter in the theory

$$\mathbf{v} = \mathbf{v}_0 + R_{\mathrm{m}}^{-1/2}\mathbf{v}' + R_{\mathrm{m}}^{-1}\mathbf{v}_m$$

$$\mathbf{B} = \mathbf{B}_0 + R_{\mathrm{m}}^{-1/2}\mathbf{B}' + R_{\mathrm{m}}^{-1}\mathbf{B}_m$$

where m denotes meridional component and the prime the non-axisymmetric part of the function. R_{m} is based on the azimuthal velocity. The other magnetic Reynolds number used here, R, will be $0(R_{\mathrm{m}}^{1/2})$ in the Braginsky formalism, and therefore large. It is not surprising that an assumption of small radial velocity failed. The Braginsky dynamo does require a very large radial velocity, but it is not clear whether or not this is true in general. A better estimate of the magnitude of the toroidal field comes from the azimuthal velocity rather than from the radial velocity, but with loss of generality. Kumar & Roberts (1975) give ohmic heating values of $10^{17}/\sigma$ W. With the thermodynamic efficiency factor, the heating curve would lie very close to the point C′ in Fig. 1, with an absolute minimum heat flux for any σ or ΔT of 3×10^{12} W, and $10^8 \leqslant \sigma\Delta T \leqslant 10^9$. Such a dynamo is possible in principle but would involve a large amount of heat emerging from the core and almost no convected heat.

We conclude that Braginsky dynamos cannot be driven by thermal convection, with the other assumptions in this paper, a conclusion arrived at by Braginsky (1964) himself. The general analysis indicates that large toroidal fields can only be satisfactorily generated by processes with small radial motions, and such a process has not yet been discovered.

The convected heat flux

If the convective contribution to the heat flux could be estimated from the radial velocity, this would quantify the argument given in the previous section. Unfortunately this cannot be done without some detailed knowledge of the convective pattern. At best one can use a rough order of magnitude argument of the sort employed by Frazer (1973). This is not adequate for the present purposes.

In order for a fluid to convect, the temperature gradient must somewhere exceed the adiabatic gradient. This gradient, which depends on the properties of core material at the appropriate temperature and pressure, is very poorly known. Calculations of its value are usually based on Gruneissen's law

$$\left(\frac{\partial T}{\partial r}\right)_s = -\frac{\Gamma g T}{v_p^2}$$

where $\Gamma = \mu k/c_p\rho$ is the Gruneissen constant, μ the isothermal compressibility, k the bulk modulus, c_p the specific heat at constant pressure, g the gravitational acceleration and v_p the velocity of seismic waves in the liquid. v_p and g are well determined seismic parameters, but T and Γ are uncertain. The temperature must be fixed at one point, usually the melting point of iron at the inner core boundary. Γ has a theoretical value of 1.67 but some authors prefer slightly different values. The net result is a great uncertainty in the actual value of the gradient, but all the estimates show the gradient decreasing with depth, becoming very small near the inner core. A typical estimate is shown in Fig. 2, using temperatures $T_{max} = 4115$ K, $T_{min} = 3000$ K and $\Gamma = 1.75$. It decreases roughly linearly with depth.

Suppose that the extreme temperatures are fixed. The conduction gradient must exceed the adiabat for convection to occur. The argument differs for the two different modes of heating. For freezing, the temperature gradient is largest near the inner core, at the place where the adiabatic gradient is smallest. It seems very likely that the conduction gradient exceeds the adiabat near the inner core, but falls below it in the outer regions.

Consider the usual estimate of heat flux from the core, which uses the adiabatic gradient at the core–mantle boundary and some estimate of the thermal conductivity. If the adiabatic gradient is replaced by the conduction gradient at the core–mantle boundary, a lower heat flux ensues, perhaps lower by as much as a factor of 20, but the fluid will still convect in the inner regions. This removes most of the difficulties raised by Kennedy & Higgins (1973), a point made by Verhoogen (1973). With internal heating there is always a very large heat flux associated with a superadiabatic conduction temperature. Although this discussion has been carried through with only a single example of an adiabatic curve, the results hold true provided the adiabat does not increase with depth. Recently Kennedy and co-workers (1976, private communication) have found that the adiabatic temperature gradient for iron drops drastically with pressure, and in the core it may well be below 0.1 K km^{-1}.

The dynamo theory provides an additional constraint on the convective motion. Backus' (1958) necessary condition for dynamo action is a constraint on the fluid velocity **v**

$$v_{max} \geqslant \frac{\pi}{r_o \mu_0 \sigma}.$$

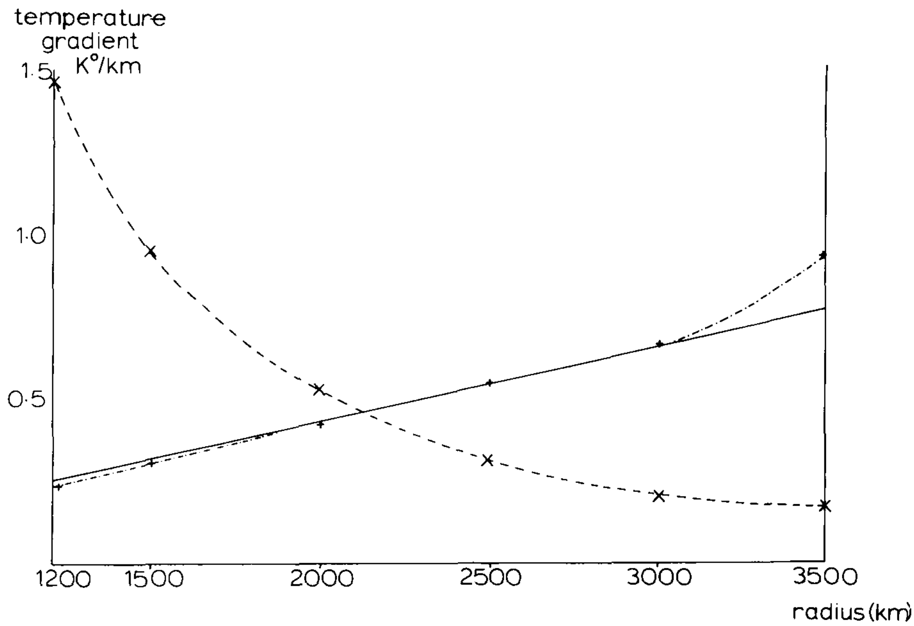

Figure 2. A comparison of the adiabatic temperature gradient (+ · · · +) and conduction temperature gradients for freezing (X · · · X) and heating from within for an inner core temperature of 4115 K and an outer core temperature of 3000 K.

This translates into a restriction on the Peclet number $\mathcal{T} = vr_0/K$, where $K = k/\rho c_p$, the thermal diffusivity. Using the Wiedemann–Franz law and rough values of $c_p = 0.5\ \mathrm{J\,g^{-1}}$ and $T = 4000$ K, then

$$\mathcal{T} \gtrsim \frac{4 \times 10^{16}}{\sigma^2} \tag{13}$$

which is very large for any reasonable value of the electrical conductivity. In the heat equation in non-dimensional form:

$$\frac{\partial T}{\partial t} = \mathcal{T}\,\mathbf{v}.\nabla T + \nabla^2 T$$

$\mathcal{T}$ represents the ratio of advection of heat to conduction. Because of equation (13), conduction must be negligible in any region where magnetic fields are generated. In steady state, the equation would be

$$\mathbf{v}.\nabla T \approx 0$$

which expresses the fact that a fluid parcel will simply carry its own temperature with it. In the core its temperature will follow an adiabatic curve.

There are several reasons for $\mathcal{T}$ not being large. If the temperature has a very small length scale, the appropriate distance to use in calculating $\mathcal{T}$ is not r_0 but something much less. This is true for Busse's (1975b) 'Geodynamo'. A related assumption is that the dynamics of the core are turbulent, and that the appropriate transport properties are eddy diffusivities, much larger than K or ν. A third and as yet unexplored possibility is for the radial velocity to be very small so that vertical conduction of heat becomes significant.

We consider only the case with $\mathscr{T} \gg 1$. In this case $\mathbf{v}.\nabla T = 0$. From the melting temperature of iron at the inner core boundary, draw an adiabatic curve to the core mantle boundary, giving an adiabatic temperature, T_a, there (Fig. 3). If the mantle is cooler than this value, conduction must be important somewhere, probably in a thermal boundary layer near the solid boundaries where heat cannot be transmitted by advection. The thickness of the layer, d, can be estimated by balancing terms $\mathbf{v}.\nabla T$ and $K\nabla^2 T$ in the heat diffusion equation. The heat flux would be $k[(T_a - T_{min})/d]$ or $(k\rho c_p v/r_0)^{1/2}(T_a - T_{min})$. With a typical velocity of $10^{-4}\,\mathrm{m\,s^{-1}}$ and $k = 50\,\mathrm{W\,m^{-2}\,(K)^{-1}}$. This gives $10^{13}(T_a - T_{min})$ W in all. The temperature difference across the boundary layer must be very small, of the order of a degree or so, for the heat flux to be reasonable. Therefore the mantle temperature must be at or above T_a, the adiabatic temperature, and at least part of the liquid core must be either stable or convecting with velocities much slower than the typical dynamo rate.

This returns the argument back to one of having vigorous convection in only the inner part of the core. The temperature gradients indicate that this might be the case, as shown in the upper part of Fig. 3. Convection will occur out to the point at which the adiabatic and conduction gradients cross ov.. Because the heat flux is enhanced by convection, circulation will occur past this point, perhaps to the point A, where the adiabatic gradient and temperature match to a conduction solution, or beyond. A stable region may exist in the outer layers. For the example chosen, these temperature gradients correspond to a very high stratification with Brunt–Vaisäla period of 1.5 hr, assuming the coefficient of thermal expansion to be $4 \times 10^{-6}\,(\mathrm{C}^\circ)^{-1}$. This stratification is actually higher than that calculated by Higgins & Kennedy (1971). Such a high stratification might be detectable seismologically by an inversion of the normal mode frequencies. This will be the subject of future work. The absence of convection or at least sufficiently vigorous convection in the outer regions might be viewed as inhibiting the generation of magnetic field. This is not the case, and some studies of kinematic dynamos (Gubbins 1976) show that circulation of global scale surrounded by a stable, electrically conducting region is more efficient for magnetic field generation than fluid motion throughout the entire core.

Very little is known about convection in compressible fluids in which an adiabatic gradient is defined and varies with depth. Convection may start at places where the conduction temperature exceeds the adiabat locally, but once started it is not confined to such regions.

Discussion

Most geochemists find it unlikely that the Earth's liquid core contains appreciable amounts of uranium or thorium. The third abundant radioactive element, potassium, may be present with sulphur, but there is no widespread agreement on the subject (Rama Murthy & Hall 1970; Oversby & Ringwood 1972; Ganguly & Kennedy 1976). Lewis (1971) estimates that 10^{13} W is available from decay of K^{40}, or all of the observed surface heat flux. The arguments in this paper indicate that such a high heat flux is necessary to generate the magnetic field, but it is difficult to imagine passage of such a large amount of heat through the mantle.

Verhoogen (1961) estimated that 5×10^{11} W is available by cooling the Earth, a calculation that seems satisfactory. This is better dynamically because of the distribution of apparent heat sources. This corresponds to a small gradient at the core mantle boundary of about $0.08\,\mathrm{K\,km^{-1}}$ for a thermal conductivity $k = 50\,\mathrm{W\,m^{-1}\,s^{-1}}$ and $T_{min} = 3000$ K. This might imply a stable region in the core, but certainly allows for a superadiabatic temperature gradient near the inner core. A uniformly-distributed source of 10^{13} W allows for a temperature gradient of $1.5\,\mathrm{K\,km^{-1}}$ at the core–mantle boundary, which may possibly be superadiabatic. This overall cooling of the core can only be accomplished if the base of the

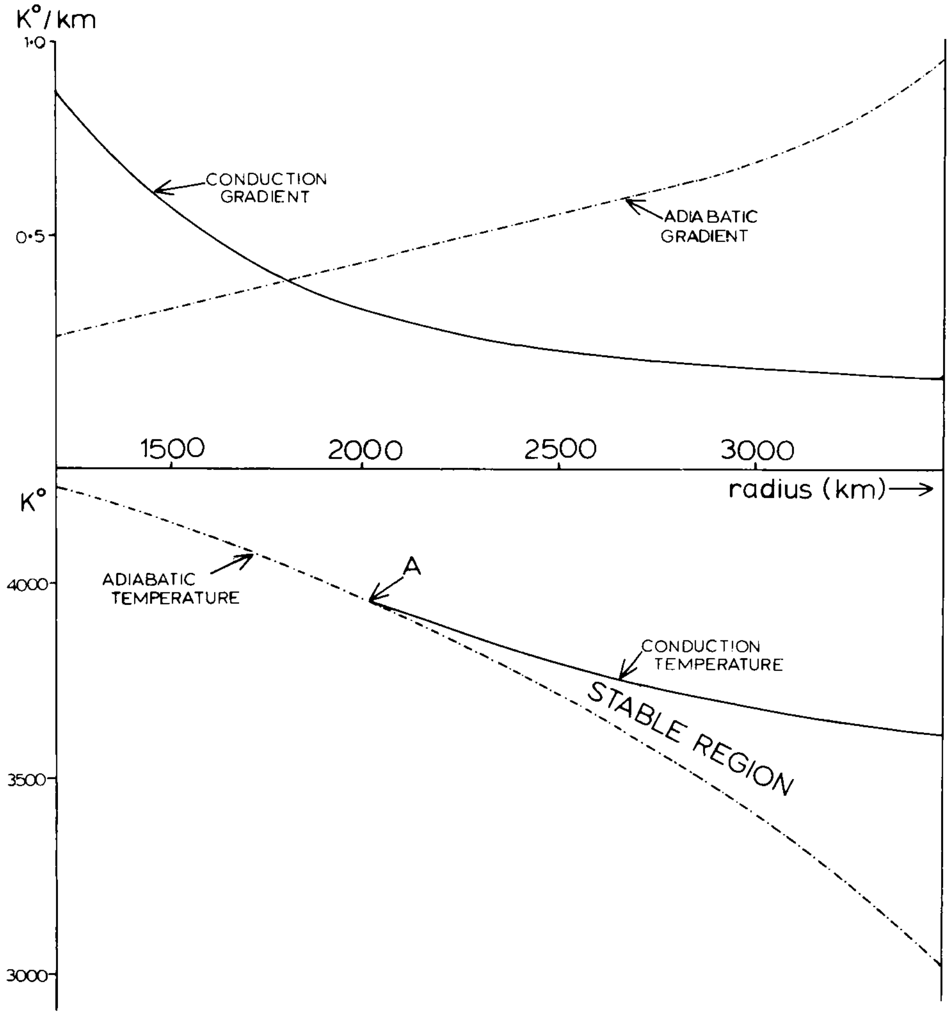

Figure 3. An example of convection near the inner core and a stable region outside. In the upper figure the conduction temperature gradient becomes subadiabatic near 1800 km. The actual temperature, shown in the lower figure, may be almost adiabatic for some distance to the point A after which it assumes a conduction solution.

mantle has cooled by several tens of degrees throughout the Earth's history. The effects of concomitant cooling of the whole of the mantle has not been considered, but this would release a quantity of heat much greater than that issuing from the core.

A feature of recent Earth models derived from free oscillation frequencies (Gilbert & Dziewonski 1975; Jordan & Anderson 1974) is the large density jump at the inner core boundary. The values are 0.87 and 0.56 $\mathrm{Mg\,m^{-3}}$ for models 1066A and 1066B respectively (Gilbert & Dziewonski 1975). It has not been shown that a density jump is needed in order to satisfy the data (Gilbert 1975, private communication). However, taking the results at face value one can calculate the gravitational energy released by freezing a small part of the liquid core to accrete onto the inner core, by assuming the solid–liquid transition to be accompanied by a volume contraction which accounts for the density change. The inner core is supposed to have grown at a steady rate for the past 4 Gy, the outer core contracting with the accretion process. The rate of release of gravitational energy is estimated crudely by adding a shell of high-density material to the inner core and reducing the volume of the whole core by a corresponding amount, using values of g and ρ from the seismological models 1066A and 1066B. The present rate of gravitational energy release is 2.6×10^{12} W for 1066A and 1.7×10^{12} W for 1066B. This exceeds the heat released by cooling, and it is therefore of great importance to tie down the density jump more closely. Melting transitions are rarely accompanied by large density changes and it is more likely that the jump at the inner core boundary is caused by compositional changes. If pure iron freezes out from the highly impure liquid core mixture, the gravitational energy released is lower than for pure melting but still appreciable. This idea is like Braginsky's (1964) settling, where he assumes that excess light components left in the bottom of the liquid core provide buoyancy to drive convection. The amount of gravitational energy available is comparable with the heat released and either source may be dominant.

If thermal convection is caused by gradual cooling of the core, the heat flux will ultimately be determined by the quantity of heat flowing through the lower mantle. This is a very complex problem because the mantle contains its own radiogenic heat sources, and because solid convection may be occurring. Ordinary thermal conduction seems inadequate. The calculations by Stacey (1969, p. 246) indicate that quite a large amount of heat could be transported by radiative transfer. Parts of the lower mantle may convect or event melt, giving rise to the lateral seismic inhomogeneities that seem to be present. If reliable calculations could be done for the mantle then this would become the controlling factor limiting the heat flux.

Conclusions

In this paper the generation of the Earth's magnetic field by thermal convection has been studied. It is not possible to rule out the thermal convection hypothesis with present observations. By considering both the effects of conduction and electric currents in the core, it has been shown that at least 2×10^{10} W and probably 5×10^{10} W of heat must emerge from the core, independent of temperature gradient or other parameters. The large toroidal field invoked by some authors probably cannot be generated by thermal convection, but it has not been possible to rule out this possibility rigorously. Very little can be said about the convected heat flux without a better knowledge of the adiabatic gradient, but it seems likely that the outer regions of the liquid core are either stably stratified or convecting at too slow a rate to help with generation of magnetic field. The dynamo can be driven with a much smaller total heat flux if the heat source is at depth, such as freezing of the inner core, whatever the value of the adiabatic gradient. Gravitational energy released by accretion of the inner core may be a significant factor. There are possibilities for better determination of the density jump at the inner core boundary and for estimates of the stratification, if any, both from seismology.

In constructing a thermally-driven dynamo model for the Earth, a large region of parameter space is open. The principal difficulties lie in the viscosity and thermal diffusivities being so very small. First attempts at solving the problem will have to invoke large eddy parameters, and there seems little hope of ever approaching realistic molecular values.

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References

Andrade, E. N. da C., 1934. A theory of the viscosity of liquids, *Phil. Mag.*, **17**, 497–511.

Backus, G. E., 1957. The axisymmetric self excited fluid dynamo, *Astrophys. J.*, **125**, 500–524.

Backus, G. E., 1958. A class of self-sustaining dissipative spherical dynamos, *Ann. Phys.*, **4**, 372–447.

Backus, G. E., 1975. Gross thermodynamics of heat engines in deep interior of Earth, *Proc. nat. Acad. Sci., Am.*, **72**, 1555–1558.

Batchelor, G. K., 1967. *An introduction to fluid dynamics*, Cambridge University Press.

Boschi, E., 1974. Melting of iron, *Geophys. J. R. astr. Soc.*, **38**, 327–334.

Braginsky, S. I., 1964. Magnetohydrodynamics of the Earth's Core. *Geomagn. Aeron.* (Engl. transl.), **4**, 698–712.

Bukowinski, M. S. T. & Knopoff, L., 1976. Electronic structure of iron and models of the Earth's core, *Geophys. Res. Lett.*, **3**, 45–48.

Bullard, E. C. & Gellman, H., 1954. Homogeneous dynamos and terrestrial magnetism. *Phil. Trans. R. Soc. Lond. A.*, **247**, 213–278.

Bullard, E. C., Freedman, C., Gellman, H. & Nixon, J., 1950. The westward drift of the Earth's magnetic field, *Phil. Trans. R. Soc. Lond. A.*, **243**, 67–92.

Busse, F. H., 1972. Comments on 'The adiabatic gradient and the melting point gradient in the core of the Earth' by G. Higgins and G. C. Kennedy, *J. geophys. Res.*, **77**, 1589–1590.

Busse, F. H., 1975a. A necessary condition for the dynamo, *J. geophys. Res.*, **80**, 278–280.

Busse, F. H., 1975b. A model of the geodynamo. *Geophys. J. R. astr. Soc.*, **42**, 437–460.

Cain, J. C., Daniels, W. E., Jensen, D. C. & Langel, A., 1967. A proposed model of the international geomagnetic reference field – 1965, *J. Geomag. Geoelect.*, **19**, 335–355.

Chapman, T. W., 1966. The viscosity of liquid metals, *A. I. Ch. E. Journal*, **12**, 395–400.

Childress, S., 1968. *Theorie magnetohydrodynamique de l'effet dynamo*, Lecture Notes, l'Institut Henri Poincaré.

Cole, J. D., 1968. *Perturbation methods in applied mathematics*, Blaisdell, Waltham, Massachusetts.

Cormier, V. F. & Richards, P. G., 1976. Comments on 'The damping of core waves' by Anthony Qamar and Alfredo Eisenberg, *J. geophys. Res.*, in press.

Elsasser, W. M., 1946. Induction effects in terrestrial magnetism. I. Theory, *Phys. Rev.*, **69**, 106–116.

Elsasser, W. M., 1972. Thermal stratification and core convection (abstract), *EOS Trans. Am. geophys. Un.*, **53**, 605.

Elsasser, W. M. & Isenberg, I., 1949. Electronic phase transition in iron at extreme pressures. *Phys. Rev.*, **76**, 469.

Eyring, H. & Jhon, M. S., 1969. *Significant liquid structures*, pp. 149, Wiley, New York.

Frazer, M. C., 1973. Temperature gradients and the convective velocity in the Earth's core. *Geophys. J. R. astr. Soc.*, **34**, 193–201.

Ganguly, J. & Kennedy, G. C., 1975. Solubility of K in FeS liquid, silicate–K–(FeS) liq equilibria lunar and terrestrial implications, *Inst. Geophys. Planet. Phys.*, Los Angeles, preprint no. 1437.

Gans, R. F., 1972. Viscosity of the Earth's core, *J. geophys. Res.*, **77**, 360–366.

Gardiner, R. B. & Stacey, F. D., 1971. Electrical resistivity of the core, *Phys. Earth Planet. Int.*, **4**, 406–410.

Gilbert, F. & Dziewonski, A. M., 1975. An application of normal mode theory to the retrieval of structural parameters and source mechanisms from seismic spectra. *Phil. Trans. R. Soc. Lond. A.*, **278**, 187–269.

Gubbins, D., 1973. Numerical solutions of the kinematic dynamo problem. *Phil. Trans. R. Soc. Lond., A.*, **274**, 493–521.

Gubbins, D., 1974. Theories of the geomagnetic and solar dynamos, *Rev. Geophys. Space Phys.*, **12**, 137–154.

Gubbins, D., 1975. Can the Earth's magnetic field be sustained by core oscillations? *Geophys. Res. Lett.*, **2**, 409–412.

Gubbins, D., 1976. *Generation of magnetic fields by fluid motions of global scale*, in preparation.

Hewitt, J. M., McKenzie, D. P. & Weiss, N. O., 1975. Dissipative heating in convective flows, *J. Fluid Mech.*, **68**, 721–738.

Hide, R., 1966. Free hydromagnetic oscillations of the Earth's core and the theory of the geomagnetic secular variation, *Phil. Trans. R. Soc. Lond. A.*, **259**, 615–647.

Hide, R., 1971. Viscosity of the Earth's core, *Nature Phys. Sci.*, **233**, 100–101.

Higgins, G. & Kennedy, G. C., 1971. The adiabatic gradient and the melting point gradient of the core of the Earth, *J. geophys. Res.*, **76**, 1870–1878.

Jacobs, J. A., 1972. Physical state of the Earth's core, *Nature Phys. Sci.*, **243**, 113–114.

Jacobs, J. A., 1975. *The Earth's core*, pp. 253, Academic Press, London.

Jain, A. & Evans, R., 1973. Calculation of the electrical resistivity of iron in the Earth's core. *Nature Phys. Sci.*, **235**, 165–167.

Johnston, M. J. S. & Strens, R. G. J., 1973. Electrical conductivity of molten Fe–Ni–S–C core mix, *Phys. Earth Planet. Int.*, **7**, 217–218.

Jordan, T. H. & Anderson, D. L., 1974. Earth structure from free oscillations and travel times, *Geophys. J. R. astr. Soc.*, **36**, 411–459.

Kennedy, G. C. & Higgins, G. H., 1973. Temperature gradients at the core–mantle interface, *The Moon*, **7**, 14–21.

Kumar, S. & Roberts, P. H., 1975. A three-dimensional kinematic dynamo. *Proc. R. Soc. Lond. A.*, **344**, 235–258.

Lepaluoto, D., 1972. *PhD thesis*, University of California, Berkeley.

Lewis, J. S., 1971. Consequences of the presence of sulphur in the core of the Earth. *Earth Planet. Sci. Lett.*, **11**, 130–134.

Loper, D., 1975. Torque balance and energy budget for precessionally driven dynamos. *Phys. Earth Planet. Int.*, **11**, 43–60.

Malkus, W. V. R., 1973. Convection at the melting point: a thermal history of the Earth's core. *Geophys. Fluid Dyn.*, **4**, 267–278.

Mitchell, A. C. & Keeler, R. N., 1971. *Physics of high energy density XLVIII Corso*, p. 120, reported by R. N. Keeler and E. B. Royce, Academic Press.

Oversby, V. M. & Ringwood, A. E., 1972. Potassium distribution between metal and silicate in its bearing on the occurrence of potassium in the Earth's core, *Earth Planet. Sci. Lett.*, **14**, 345–347.

Parker, E. N., 1955. Hydromagnetic dynamo models, *Astrophys. J.*, **122**, 293–314.

Parker, R. L., 1972. Inverse theory with grossly inadequate data. *Geophys. J. R. astr. Soc.*, **29**, 123–138.

Parker, R. L. & Huestis, S. P., 1973. The inversion of magnetic anomalies in the presence of topography. *J. geophys. Res.*, **79**, 1587–1593.

Pekeris, C. L., Accad, Y. & Shkoller, B., 1973. Kinematic dynamos and the Earth's magnetic field. *Phil. Trans. R. Soc. Lond. A.*, **275**, 425–461.

Qamar, A. & Eisenberg, A., 1974. The damping of core waves, *J. geophys. Res.*, **79**, 758–765.

Rama Murthy, V. & Hall, H. T., 1970. On the possible presence of sulphur in the Earth's core. *Phys. Earth Planet. Int.*, **2**, 276–282.

Rochester, M. G., 1975. Chandler wobble and viscosity in the Earth's core. *Nature*, **255**, 655.

Rochester, M. G., Jacobs, J. A., Smylie, D. E. & Chong, K. F., 1975. Can precession power the geomagnetic dynamo? *Geophys. J. R. astr. Soc.*, **43**, 661–678.

Rowlinson, J. S., 1970. Book review, *Disc. Faraday Soc.*, **49**, 30–42.

Sato, R. & Espinosa, A. F., 1967. Dissipation in the Earth's mantle and viscosity in the Earth's core determined from waves multiply reflected from the core–mantle boundary, *Bull. seism. Soc. Am.*, **57**, 829–856.

Stacey, F. D., 1969. *Physics of the Earth*, pp. 324, Wiley & Sons Inc., New York.

Stacey, F. D., 1972. Physical properties of the earth's core, *Geophys. Surv.*, **1**, 99–119.

Stacey, F. D., 1975. Thermal regime of the Earth's interior, *Nature*, **255**, 44–45.

Steenbeck, M., Krause, F. & Rädler, K.-H., 1966. A calculation of the mean emf in an electrically conducting fluid in turbulent motion, under the influence of Coriolis forces. *Z. Naturforsch. A.*, **21**, 369–376.

Toomre, A., 1974. On the 'Nearly diurnal wobble' of the Earth, *Geophys. J. R. astr. Soc.*, **38**, 335–348.

Verhoogen, J., 1961. Heat balance of the Earth's core, *Geophys. J. R. astr. Soc.*, **4**, 276–281.

Verhoogen, J., 1973. Thermal regime of the Earth's core, *Phys. Earth Planet. Int.*, **7**, 47–58.

Verhoogen, J., 1974. Chandler wobble and viscosity in the Earth's core, *Nature*, **249**, 334–335.

Williams, D. L., Von Herzen, R. P., Sclater, J. G. & Anderson, R. N., 1974. The Galapagos spreading centre: lithospheric cooling and hydrothermal circulation, *Geophys. J. R. astr. Soc.*, **38**, 587–608.

Ziman, J. M., 1973. What do we not yet understand about liquid metals? *The properties of liquid metals*, p. xii–xvii, ed. S. Takeuchi, Taylor & Francis Ltd., London.

Appendix

Viscosity of the Earth's core

Chapman's (1966) corresponding states theory is used to obtain a new estimate of viscosity. The viscosity of a liquid, like all transport properties, may be written in terms of the intermolecular force and the appropriate distribution functions. Although such an expression cannot be evaluated explicitly, it is used to obtain a corresponding states formula to express the viscosity in terms of the density, temperature, and two parameters of the intermolecular potential function for the particular liquid, ϵ and δ. ϵ is the energy amplitude and δ the interatomic distance. Define reduced dynamic viscosity, η^*, temperature T^* and volume V^*

$$\eta^* = \frac{\eta\delta^2 N_0}{(MRT)^{1/2}}$$

$$T^* = \frac{kT}{\epsilon}$$

$$V^* = \frac{1}{n\delta^3}$$

where N_0 is Avogadro's number, R the Gas constant, M the molecular weight and n the molecule number density. Chapman (1966) arrives at:

$$\eta^*(V^*)^2 = G(T^*) \qquad \text{(A1)}$$

where $G(T^*)$ is a universal function which is determined experimentally for both sodium and potassium. (A1) assumes that the pair distribution function of molecules has the same form for each different metal and is independent of density. The latter assumption has been shown to be valid at the high pressures of the laboratory but may not hold at extreme core pressures. Using Chapman's values of ϵ and δ for iron, the dynamic viscosity has been calculated for two reasonable temperatures and densities. The results are shown in Table 1. The values are about an order of magnitude lower than previous estimates. The true value of the viscosity may differ significantly because of impurities. It will also be altered if any solid phase is present in a slurry. Batchelor (1967) gives an expression for a dilute suspension of solid spheres in viscous fluid. The modified viscosity is

$$(\rho\nu)' = (\rho\nu)\left(1 + \frac{5\alpha}{2}\right)$$

where α is the mass concentration of solid material. This does not alter the viscosity significantly. The viscosity may be drastically affected if the concentration of solid particles is very high, when Batchelor's formula no longer applies.