

## Observational signatures and non-Gaussianities of general single-field inflation

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# Observational signatures and non-Gaussianities of general single-field inflation

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**Abstract.** We perform a general study of primordial scalar non-Gaussianities in single-field inflationary models in Einstein gravity. We consider models where the inflaton Lagrangian is an arbitrary function of the scalar field and its first derivative, and the sound speed is arbitrary. We find that under reasonable assumptions, the non-Gaussianity is completely determined by five parameters. In special limits of the parameter space, one finds distinctive ‘shapes’ of the non-Gaussianity. In models with a small sound speed, several of these shapes would become potentially observable in the near future. Different limits of our formulae recover various previously known results.

**Keywords:** CMBR theory, cosmological perturbation theory, string theory and cosmology, inflation

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**Contents**

<b>1. Introduction</b>	<b>2</b>
<b>2. Inflation models with a general Lagrangian</b>	<b>5</b>
<b>3. Several classes of models</b>	<b>6</b>
3.1. Slow-roll inflation . . . . .	6
3.2. DBI inflation . . . . .	7
3.3. Kinetically driven inflation . . . . .	10
3.3.1. The effective theory governing small fluctuations. . . . .	11
3.3.2. Basic phenomenology. . . . .	12
3.3.3. Exiting from K-inflation. . . . .	13
<b>4. Non-Gaussian perturbations</b>	<b>13</b>
4.1. The quadratic part . . . . .	16
4.2. The cubic part . . . . .	17
4.3. Correction terms . . . . .	20
4.4. Summary of final results . . . . .	21
<b>5. Size, shape and running of the non-Gaussianities</b>	<b>22</b>
5.1. Slow-roll inflation . . . . .	25
5.2. DBI inflation . . . . .	26
5.3. Kinetic inflation . . . . .	27
5.3.1. Crude estimate of non-Gaussianities. . . . .	27
5.3.2. Shape of non-Gaussianities in K-inflation. . . . .	28
<b>6. Non-Gaussianities as a probe of the inflationary vacuum</b>	<b>29</b>
<b>7. Conclusion</b>	<b>31</b>
<b>Acknowledgments</b>	<b>32</b>
<b>Appendix A. Corrections to <math>u_k</math></b>	<b>33</b>
<b>Appendix B. Details on the correction terms</b>	<b>35</b>
B.1. Final results . . . . .	41
B.2. The squeezed limit . . . . .	44
B.3. Some details . . . . .	44
<b>References</b>	<b>45</b>

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**1. Introduction**

Inflation has been a very successful paradigm for understanding otherwise puzzling aspects of big bang cosmology [1, 2]. It can naturally solve the flatness, homogeneity and monopole problems that otherwise seem to require a very high degree of fine tuning for the initial state of our universe. Furthermore, inflation generically predicts almost scale invariant Gaussian density perturbations [3], consistent with experimental observations of the

cosmic microwave background. Future experiments can constrain and distinguish between inflationary models in several ways.

In the next few years, we can expect to see increasingly precise determination of the scalar spectral index  $n_s$  and its running<sup>5</sup>. Planck will also lower bounds on the tensor-to-scalar ratio  $r$  well into the regime favoured by models of large-field inflation [6]. However,  $n_s$  and  $r$  are just two numbers. Although their precise determination will be a tremendous achievement, it will leave considerable ambiguity in reconstructing the correct inflationary model.

In contrast, the non-Gaussian component of the scalar fluctuations is characterized by a three-point function which is, *a priori*, a nontrivial function of three variables (momentum magnitude and ratios) on the sky. Furthermore, as demonstrated conclusively in [7, 8], slow-roll models where the density perturbations are produced by fluctuations of the inflaton itself, predict negligible non-Gaussianity. A detection of non-Gaussianity by the next generation of experiments would therefore strongly favour either an exotic inflationary model, or a model where density perturbations are generated by other dynamics (as in curvaton [9] and modulated reheating [10] scenarios). A crude measure of non-Gaussianity is the number  $f_{\text{NL}}$ . Values of  $|f_{\text{NL}}| \geq 5$  would almost certainly indicate some novelty in the dynamics of the inflaton itself.

In this paper, we determine the most general non-Gaussian perturbations possible in single-field inflationary theories. We assume that the inflaton itself generates the density perturbations, and that the Lagrangian is a function of the inflaton and its first derivative alone. Under these assumptions, we prove that the full non-Gaussianity (at the first order in various slow-variation parameters) is actually captured by five numbers. These numbers characterize the three-point function of fluctuations of the inflaton (or more precisely, its gauge-invariant analogue). For models with  $c_s \ll 1$ , which are known to produce the most significant non-Gaussianities, the result is stronger: the leading non-Gaussianity is characterized by two numbers, and two different possible qualitative shapes in momentum space; the subleading non-Gaussianity is characterized by three more numbers.

The elegant, gauge invariant calculation of the non-Gaussianities for slow-roll models appeared first in [7, 8]. The result is that slow-roll models produce a primordial  $f_{\text{NL}}$  of  $\mathcal{O}(10^{-2})$ , too small to measure. Creminelli stressed that models where the effects of higher derivatives are important may give larger  $f_{\text{NL}}$  (under the assumption that the non-Gaussianity is not diluted by the inflationary expansion itself, as happens in some models [11]). However, this effect can only reach  $f_{\text{NL}} \sim \mathcal{O}(1)$  in the regime where effective field theory applies [12]. A proof of principle that significantly larger  $f_{\text{NL}}$  can occur in sensible models was provided by the work of Alishahiha *et al* [13], who found that a fairly concrete string construction [14] could yield substantially larger  $f_{\text{NL}} \gg 1$ . Different models in this class were constructed [15, 16] in the context of warped compactification. The large non-Gaussianities in these models are compatible with the current observational bound, but are potentially observable in future experiments [13, 17]. (Another interesting model also providing large  $f_{\text{NL}}$ , but so far resisting embedding in a UV complete theory,

<sup>5</sup> Already, there are strong hints that models with a red spectrum are preferred [4]. Moreover, if the spectral index runs from blue to red, then there should be an approximate coincidence [5] between the length scale  $k$  at which  $n(k) - 1 = 0$  and the length scale at which the tensor to scalar ratio reaches a minimum. Such a coincidence of scales, if observed, can put constraints on inflationary model building.

appears in [18]. This model does not belong to the general class that we will study in Einstein gravity.) Our results build on these papers and further significant work by Seery and Lidsey [19], who found the fluctuation Lagrangian to cubic order for a general class of Lagrangians, and of Babich *et al* [20], who emphasized the importance of analysing the full shape of non-Gaussianities in  $k$ -space.

There are several motivations for completing such a general analysis. Firstly, it provides a null hypothesis against which to compare any future measurement of the non-Gaussianity. Secondly, several string-inspired models, if realized in nature, will give rise to a very characteristic measurable non-Gaussianity. String theory is relevant here because models with significant non-Gaussianity tend to be governed by higher-derivative terms, and a UV completion is needed to make sense of such models (indeed, this is true even of slow-roll models with negligible non-Gaussianities [21], since the slow-roll conditions are sensitive to Planck-suppressed corrections to inflaton dynamics). Our analysis should make it straightforward to work out predictions for any such models. When we wish to provide specific examples, we use DBI inflation [14, 15] and K-inflation [22].

In addition, non-Gaussian fluctuations could contain a signature of any departure of the inflaton from its standard Bunch–Davies vacuum. This has been suggested as a possible signature of trans-Planckian physics, and there has been much debate of the plausibility of such modifications; some representative references are [11], [23]–[32]. It is a simple matter to translate any modification of the inflaton wavefunction into a modification of the three-point function, so the non-Gaussianities could serve as a test of any proposed modification.

Finally, the structure of the three-point function can in principle be determined by dS/CFT [33] or its generalization appropriate to models with  $c_s \ll 1$ . This could provide a useful laboratory for studying holographic descriptions of dS space. This perspective could be useful even if there is no exact relation between the dS gravity theory and a dual field theory, since the useful aspects of the duality for this purpose are purely kinematical. Recent work in this direction appears in [34, 35]. Our work on this connection will eventually appear in a companion paper [36].

The organization of this paper is as follows. In section 2, we present the general class of Lagrangians that we will analyse (those that are an arbitrary function of a single scalar field and its first derivative), and define the notation we will use in the rest of the paper. While our analysis applies much more broadly, in section 3 we describe three well-studied classes of inflationary theories whose non-Gaussianities we shall discuss in detail as special examples: slow-roll models, DBI models, and power-law K-inflation models. In section 4, we find the cubic fluctuation Lagrangian in appropriate gauge-invariant variables for the most general single-field Lagrangian, and compute the non-Gaussianities. Our main result is that there are only a few basic shapes, governed by five parameters in the most general model. We evaluate our results for the three special examples in section 5, and present the different qualitative shapes of the non-Gaussianities that may occur. In section 6, the effects of putting the inflaton in a vacuum other than the Bunch–Davies vacuum are described. We conclude in section 7. Readers who are interested only in the class of Lagrangians studied and the general structure of the non-Gaussianity for this class, can confine their attention to sections 2 and 4 (which are more or less self-contained).

## 2. Inflation models with a general Lagrangian

To set up our notation, let us first review the formalism in [37] where a general Lagrangian for the inflaton field is considered. The Lagrangian is of the general form

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [M_{\text{pl}}^2 R + 2P(X, \phi)], \quad (2.1)$$

where  $\phi$  is the inflaton field and  $X = -1/2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ . The reduced Planck mass is  $M_{\text{pl}} = (8\pi G)^{-1/2}$  and the signature of the metric is  $(-1, 1, 1, 1)$ . The energy of the inflaton field is

$$E = 2XP_{,X} - P, \quad (2.2)$$

where  $P_{,X}$  denotes the derivative with respect to  $X$ . Suppose the universe is homogeneous with a Friedmann–Robertson–Walker metric

$$ds^2 = -dt^2 + a^2(t) dx_3^2. \quad (2.3)$$

Here  $a(t)$  is the scale factor and  $H = \dot{a}/a$  is the Hubble parameter of the universe. The equations of motion of the gravitational dynamics are the Friedmann equation and the continuity equation

$$3M_{\text{pl}}^2 H^2 = E, \quad (2.4)$$

$$\dot{E} = -3H(E + P). \quad (2.5)$$

It is useful to define the ‘speed of sound’  $c_s$  as

$$c_s^2 = \frac{dP}{dE} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}} \quad (2.6)$$

and some ‘slow-variation parameters’ as in standard slow-roll inflation

$$\begin{aligned} \epsilon &= -\frac{\dot{H}}{H^2} = \frac{XP_{,X}}{M_{\text{pl}}^2 H^2}, \\ \eta &= \frac{\dot{\epsilon}}{\epsilon H}, \\ s &= \frac{\dot{c}_s}{c_s H}. \end{aligned} \quad (2.7)$$

These parameters are more general than the usual slow-roll parameters (which are defined through properties of a flat potential, assuming canonical kinetic terms), and in general depend on derivative terms as well as the potential. For example, in DBI inflation the potential can be steep, and kinetically driven inflation can occur even in the absence of a potential. We also note that the smallness of the parameters  $\epsilon$ ,  $\eta$ ,  $s$  does not imply that the rolling of inflaton is slow.

The primordial power spectrum is derived for this general Lagrangian in [37]

$$P_k^\zeta = \frac{1}{36\pi^2 M_{\text{pl}}^4 c_s(P + E)} \frac{E^2}{c_s \epsilon} = \frac{1}{8\pi^2 M_{\text{pl}}^2} \frac{H^2}{c_s \epsilon}, \quad (2.8)$$

where the expression is evaluated at the time of horizon exit at  $c_s k = aH$ . The spectral index is

$$n_s - 1 = \frac{d \ln P_k^\zeta}{d \ln k} = -2\epsilon - \eta - s. \quad (2.9)$$

In order to have an almost scale invariant power spectrum, we need to require the three parameters  $\epsilon$ ,  $\eta$ ,  $s$  to be very small, which we will denote simply as  $\mathcal{O}(\epsilon)$ . We note that in inflationary models with standard kinetic terms the speed of sound is  $c_s = 1$ , but here we do not require  $c_s$  to be close to 1. For example, in the case of DBI inflation, the speed of sound can be very small. In the case of arbitrary  $c_s$ , the formula (2.8), (2.9) for the power spectrum and its index at leading order is still valid as long as the variation of the sound speed is slow, namely  $s \ll 1$ . We will discuss this in more detail in section 4.1.

The tensor perturbation spectrum  $P_k^h$  and the tensor spectral index  $n_T$  are given by

$$P_k^h \equiv \frac{2}{3\pi^2} \frac{E}{M_{\text{pl}}^4}, \quad (2.10)$$

$$n_T \equiv \frac{d \ln P_k^h}{d \ln k} = -2\epsilon \quad (2.11)$$

and they satisfy a generalized consistency relation  $P_k^h/P_k^\zeta = -8c_s n_T$ . This is phenomenologically different from standard inflation when the speed of sound is not one.

### 3. Several classes of models

In this section, we review three types of single-field inflationary models. We discuss the basic setups and results of the corresponding effective field theories. These models will be used as primary examples after we work out the general expression for non-Gaussianities.

#### 3.1. Slow-roll inflation

Slow-roll inflation models are the most popular models studied in the literature. The effective action takes the canonical non-relativistic form

$$P(X, \phi) = X - V(\phi). \quad (3.1)$$

One achieves inflation by starting the inflaton on top of a flat potential  $V(\phi)$ . The flatness of this potential is characterized by the slow-roll parameters

$$\begin{aligned} \epsilon_V &= \frac{M_{\text{pl}}^2}{2} \left( \frac{V'}{V} \right)^2, \\ \eta_V &= M_{\text{pl}}^2 \frac{V''}{V}, \end{aligned} \quad (3.2)$$

which are required to be much less than one. The energy

$$E = X + V \approx V \quad (3.3)$$

is dominated by the potential and the sound speed  $c_s = 1$ . During inflation, the inflaton speed is determined by the attractor solution

$$\dot{\phi} = -\frac{V'}{3H}. \quad (3.4)$$

This condition relates the slow-roll parameters in (3.2) to the slow-variation parameters in (2.7),

$$\epsilon = \epsilon_V, \quad \eta = -2\eta_V + 4\epsilon_V. \quad (3.5)$$

The primordial scalar and gravitational wave power spectrum are both determined by the potential

$$P_k^\zeta = \frac{1}{12\pi^2 M_{\text{pl}}^6} \frac{V^3}{V'^2}, \quad (3.6)$$

$$P_k^h = \frac{2V}{3\pi^2 M_{\text{pl}}^4}. \quad (3.7)$$

The spectral indices and the running can be computed using the relation

$$d \ln k = H dt = \frac{H}{\dot{\phi}} d\phi \quad (3.8)$$

and we get

$$\begin{aligned} n_s - 1 &= \frac{d \ln P_k^\zeta}{d \ln k} = M_{\text{pl}}^2 \left( -3 \frac{V'^2}{V^2} + 2 \frac{V''}{V} \right), \\ \frac{dn_s}{d \ln k} &= M_{\text{pl}}^4 \left( -6 \frac{V'^4}{V^4} + 8 \frac{V'^2 V''}{V^3} - 2 \frac{V' V'''}{V^2} \right), \\ n_T &= \frac{d \ln P_k^h}{d \ln k} = -M_{\text{pl}}^2 \frac{V'^2}{V^2}. \end{aligned} \quad (3.9)$$

There has been significant effort invested in developing slow-roll models of inflation in string theory. Some fairly recent reviews with further references are [38, 39].

### 3.2. DBI inflation

DBI inflation [13]–[16] is motivated by brane inflationary models [21], [40]–[43] in warped compactifications [44]–[49]. In particular, strongly warped regions or ‘warped throats’ with exponential warp factors, can arise when there are fluxes supported on cycles localized in small regions of the compactified space. A prototypical example of such a strongly warped throat is the warped deformed conifold [50, 51]. The effective field theory of compact models containing such throats [48, 49] has been explored in detail in [52]. In the slow-roll paradigm, inflation can happen when a brane is approaching anti-branes in a throat if the potential is flat enough. However, this is non-generic [21]. Both the degree of tuning involved, and various possible ways of engineering flat potentials, have been discussed in the literature [21], [53]–[59].

Perhaps the most interesting idea, which relies upon dynamics distinct from the usual slow-roll paradigm, arises in the DBI model. In this model, the warped space slows down the rolling of the inflaton on even a steep potential. (This ‘slowing down’ can also be understood as arising due to interactions between the inflaton and the strongly coupled large- $N$  dual field theory.) This scenario can naturally arise in warped string compactifications [15]. The inflaton  $\phi$  is the position of a D-brane moving in a warped throat. In the region where the back-reaction [14, 16, 60] and stringy physics [16, 17] can be ignored, the effective action has the following form

$$S = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} R - \int d^4x \sqrt{-g} \left[ f(\phi)^{-1} \sqrt{1 + f(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} - f(\phi)^{-1} + V(\phi) \right]. \quad (3.10)$$

The above expression applies for D3-branes in a warped background where  $f(\phi)$  is the warping factor. We will first express the results in terms of a general  $f(\phi)$ . For an AdS-like throat,  $f(\phi) \simeq \lambda/\phi^4$  (where  $\lambda$  in specific string constructions is a parameter which depends on the flux numbers)<sup>6</sup>. Two situations have been considered in the literature:

- In the UV model [13, 14], the inflaton moves from the UV side of the warped space to the IR side under the potential

$$V(\phi) \simeq \frac{1}{2}m^2\phi^2, \quad m \gg M_{\text{pl}}/\sqrt{\lambda}. \quad (3.11)$$

In this case the inflaton starts far away from the origin and rolls relativistically to the minimum of potential at the origin.

- In the IR model [15, 16], the inflaton moves from the IR side of the warped space to the UV side under the potential

$$V(\phi) \simeq V_0 - \frac{1}{2}m^2\phi^2, \quad m \sim H. \quad (3.12)$$

The inflaton starts near the origin and rolls relativistically away from it.

The evolution of the inflaton in both cases was studied and the resulting power spectra were computed in [13]–[16]. Stages of DBI and slow-roll inflation can also be smoothly connected to each other [15, 62].

In the following we summarize the basic results of DBI inflation, following [13]–[16], and using the general formalism developed in [37]. For the zero-mode evolution, we assume the inflaton  $\phi$  is spatially homogeneous and denote  $X = \dot{\phi}^2/2$ . The pressure  $P$  and energy  $E$  are

$$\begin{aligned} P &= -f(\phi)^{-1}\sqrt{1 - 2Xf(\phi)} + f(\phi)^{-1} - V(\phi), \\ E &= 2XP_{,X} - P = \frac{f(\phi)^{-1}}{\sqrt{1 - 2Xf(\phi)}} - f(\phi)^{-1} + V(\phi) \end{aligned} \quad (3.13)$$

and the speed of sound  $c_s$

$$c_s = \sqrt{1 - \dot{\phi}^2 f(\phi)}. \quad (3.14)$$

In DBI inflation, the scalar rolls relativistically and a speed limit can be inferred by requiring positivity of the argument of the square root in the DBI action. So in this limit  $c_s \ll 1$ , we can approximate the inflaton speed during inflation by

$$\dot{\phi} \simeq \pm \frac{1}{\sqrt{f(\phi)}} = \pm \frac{\phi^2}{\sqrt{\lambda}}. \quad (3.15)$$

It is easy to see that the requirement  $\epsilon \ll 1$  (or equivalently  $|E + P|/E \ll 1$ ) implies that the potential energy  $V(\phi)$  dominates throughout inflation despite the fact that  $\phi$  is rolling relativistically. Hence, the Friedmann equation and the continuity equation reduce to

$$\begin{aligned} H^2 &= \frac{V(\phi)}{3M_{\text{pl}}^2}, \\ V'(\phi) &= -3H \frac{1}{c_s \sqrt{f(\phi)}}, \end{aligned} \quad (3.16)$$

<sup>6</sup> This is a good approximation if we assume that the last 60 e-foldings of inflation occur far from the tip of the throat. Otherwise, inflationary observables may depend on the details of the warp factor [61].

where we have used the universal speed limit relation (3.15) in the continuity equation. The number of e-foldings is computed as the following

$$N_e = \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} H \frac{d\phi}{\dot{\phi}} = \int_{\phi_i}^{\phi_f} \frac{d\phi}{M_{\text{pl}}} \sqrt{\frac{f(\phi)V(\phi)}{3}}. \quad (3.17)$$

The scalar power spectrum and gravitational power spectrum are computed in the general formalism to be

$$\begin{aligned} P_k^\zeta &= \frac{1}{36\pi^2 M_{\text{pl}}^4 c_s(P+E)} = \frac{f(\phi)V(\phi)^2}{36\pi^2 M_{\text{pl}}^4}, \\ P_k^h &= \frac{2E}{3\pi^2 M_{\text{pl}}^4} = \frac{2V(\phi)}{3\pi^2 M_{\text{pl}}^4}. \end{aligned} \quad (3.18)$$

The spectral indices and the running can also be computed using (3.8),

$$\begin{aligned} n_s - 1 &= \frac{d \ln P_k^\zeta}{d \ln k} = \frac{\sqrt{3} M_{\text{pl}} \phi^2}{\sqrt{\lambda V}} \left( -\frac{4}{\phi} + \frac{2V'}{V} \right), \\ \frac{dn_s}{d \ln k} &= \frac{3M_{\text{pl}}^2 \phi^2}{\lambda} \left( -\frac{4}{V} + \frac{8\phi V'}{V^2} + \frac{2\phi^2 V''}{V^2} - \frac{3\phi^2 V'^2}{V^3} \right), \\ n_T &= \frac{d \ln P_k^h}{d \ln k} = \sqrt{\frac{3M_{\text{pl}}^2 \phi^2 V'}{\lambda V^{3/2}}}, \end{aligned} \quad (3.19)$$

where we have evaluated  $f(\phi) = \lambda/\phi^4$ . Using the equations of motion it is easy to verify that the gravitational wave spectral index satisfies the generalized consistency constraint  $P_k^h/P_k^\zeta = -8c_s n_T$  in [37].

We make two remarks regarding the result (3.19). Firstly, as pointed out in [13], for the UV model, both the variation in the speed of sound and the small parameters  $\epsilon, \eta$  contribute to the scalar spectral index and their effects cancel each other in the case of a quadratic potential  $V(\phi) = \frac{1}{2}m^2\phi^2$ , so that the spectral index is a second-order quantity  $O(\epsilon^2)$  in this case. Indeed, we can directly see the cancellation from the first formula in (3.19) for a quadratic potential. Secondly, in the IR model the potential remains  $V(\phi) \simeq V_0$  during inflation, so the above expressions for the number of e-foldings, the scalar spectral index and its running (3.19) become simplified

$$\begin{aligned} N_e &= \int_{\phi_i}^{\phi_f} \frac{d\phi}{\phi^2} \frac{1}{M_{\text{pl}}} \sqrt{\frac{\lambda V_0}{3}} \simeq \frac{1}{M_{\text{pl}} \phi_i} \sqrt{\frac{\lambda V_0}{3}}, \\ n_s - 1 &= \frac{\sqrt{3} M_{\text{pl}} \phi_i^2}{\sqrt{\lambda V_0}} \left( -\frac{4}{\phi_i} \right) = -\frac{4}{N_e}, \\ \frac{dn_s}{d \ln k} &= \frac{3M_{\text{pl}}^2 \phi_i^2}{\lambda} \left( -\frac{4}{V_0} \right) = -\frac{4}{N_e^2}. \end{aligned} \quad (3.20)$$

In this IR model the gravitational wave production is very much suppressed compared to the UV model. This suppression is due to the consistency relation  $P_k^h/P_k^\zeta = -8c_s n_T$ , and to the fact that the gravitational wave spectral index  $n_T$  is much smaller than the scalar spectral index  $n_s - 1$  in this case since  $|V'/V| \ll |4/\phi|$ .

A concern in DBI inflation is how to get the large background charge  $\lambda$  ( $\sim 10^{14}$ ) which is needed to fit the field theory result to the observed density perturbations. Since this requirement just arises from requiring the compactification scale in the throat to be  $\sim M_{\text{GUT}}$  (combined with the standard AdS/CFT relation between  $g_s N$  and the compactification volume), it seems very likely that model building could significantly reduce the apparent tune. For discussions of this issue, see [13, 16, 17].

### 3.3. Kinetically driven inflation

One simple class of models which can give rise to large non-Gaussianities is the models of K-inflation, where the dynamics of inflation is governed by (non-standard) inflaton kinetic terms [22, 37]. The Lagrangians giving rise to K-inflation are not radiatively stable, and so this mechanism is UV sensitive. There are as yet no convincing limits of string theory which give rise to K-inflation, but because the models are so simple, we analyse them in detail nonetheless. It would be very interesting to find controlled limits of string theory which give rise to such models.

The simplest class of K-inflation models are the models of ‘power-law K-inflation’. The Lagrangian for power-law K-inflation is of the form

$$P(X, \phi) = \frac{4(4-3\gamma)}{9} \frac{1}{\gamma^2} \frac{1}{\phi^2} (-X + X^2), \quad (3.21)$$

where  $\gamma$  is a constant, not to be confused with the Lorentz factor ( $1/c_s$ ) that often appears in the literature of DBI inflation. (The form of the Lagrangian and our discussion can be straightforwardly generalized to a more arbitrary form where  $P \propto f(X)/\phi^2$  [22].) Before describing the physics which follows from (3.21), we should discuss some general concerns about K-inflation. The most obvious (also mentioned above) is that a Lagrangian of the form (3.21) is not radiatively stable, since it is not protected by any symmetry. (A shift symmetry of  $\phi$  could protect a Lagrangian of the form  $P(X)$  with generic coefficients.) A second concern is that a reasonable exit mechanism for inflation must be provided. A third, related concern is that the dominant energy condition

$$\frac{\partial P}{\partial X} \geq 0, \quad X \frac{\partial P}{\partial X} - P \geq 0 \quad (3.22)$$

is not satisfied by (3.21) for small values of  $X$ . Therefore, while in the inflating solution we will see that (3.22) is satisfied, one must provide an exit mechanism that changes the form of  $P$  drastically enough that the physics around flat space is sensible. We shall discuss these issues further after summarizing the key properties of the solution of interest.

One solution to the equations of motion [22] is to take

$$X = X_0 = \frac{2-\gamma}{4-3\gamma} \quad (3.23)$$

which gives rise to an FRW cosmology with

$$a(t) \sim t^{2/3\gamma} \quad (3.24)$$

for any  $0 < \gamma < 2/3$ . The speed of sound following from (3.21) is

$$c_s^2 = \frac{\gamma}{8-3\gamma}. \quad (3.25)$$

We are most interested in the regime with  $c_s \ll 1$ . Therefore, we will focus on models with small  $\gamma$ , and sometimes expand formulae around  $\gamma \rightarrow 0$ .

*3.3.1. The effective theory governing small fluctuations.* To get some intuition for these models, it is useful to construct an effective theory describing small fluctuations around the background inflating solution. The equations (3.23) imply that

$$\phi(t) \sim (1 + \gamma/8)t \quad (3.26)$$

where we have absorbed an overall constant into the definition of  $t$ , and have only written the solution to  $\mathcal{O}(\gamma)$ . Let us cast the Lagrangian into a more familiar form by performing the field redefinition

$$\Phi = \log(\phi) \quad (3.27)$$

valid for  $t > 0$ . Then the mini-superspace Lagrangian takes the form

$$\mathcal{L} = f(\gamma) \left( -\frac{1}{2}\dot{\Phi}^2 + \frac{1}{4}\dot{\Phi}^4 e^{2\Phi} \right) \quad (3.28)$$

where  $f(\gamma)$  is the complicated  $\gamma$ -dependent prefactor in (3.21); for small  $\gamma$

$$f(\gamma) \sim \frac{16}{9\gamma^2}. \quad (3.29)$$

Defining  $Y = -\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$ , the full Lagrangian is just

$$\mathcal{L} = f(\gamma) \left( -Y + Y^2 e^{2\Phi} \right). \quad (3.30)$$

Now, we introduce the effective field  $\pi$  which describes small fluctuations around the solution via

$$\Phi(x, t) = \pi(x, t) + \Phi_0(t) \quad (3.31)$$

where  $\Phi_0$  characterizes the inflationary solution, with

$$\Phi_0(t) = \log \left( \left( 1 + \frac{\gamma}{8} \right) t \right) \quad (3.32)$$

(and therefore  $Y_0 = (1/2)(1/t^2)$ ). We will also find it useful to define

$$Z = -\frac{1}{2}g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi. \quad (3.33)$$

Expanding  $\mathcal{L}$  around this solution, we find a Lagrangian for  $\pi$  of the form

$$\mathcal{L}(\pi, \dot{\pi}, \nabla\pi) = f(\gamma) \left( \tilde{\mathcal{L}}_0 + \tilde{\mathcal{L}}_2 + \tilde{\mathcal{L}}_3 \right). \quad (3.34)$$

Here, the subscript on the  $\tilde{\mathcal{L}}$  denotes its order in the fluctuation  $\pi$ . The first-order term is guaranteed to vanish in the background  $\Phi_0$ , since it solves the equations of motion. To compute the Lagrangian up to third order in  $\pi$ , we need the expansions of  $Y$  and  $e^{2\Phi}$ . Using the explicit solution (3.32) we see that

$$Y = \frac{1}{2t^2} + \frac{1}{t}\dot{\pi} + Z \quad (3.35)$$

and

$$e^{2\Phi} = t^2 \left( 1 + \frac{\gamma}{4} \right) \left( 1 + 2\pi + 2\pi^2 + \frac{4}{3}\pi^3 + \dots \right). \quad (3.36)$$

We then find

$$\tilde{\mathcal{L}}_0 = -\frac{1}{4} \frac{1}{t^2} \left( 1 - \frac{\gamma}{4} \right), \quad (3.37)$$

$$\tilde{\mathcal{L}}_2 = \left( 1 + \frac{\gamma}{4} \right) \dot{\pi}^2 + \frac{\gamma}{4} Z + \frac{\pi^2}{2t^2} \left( 1 + \frac{\gamma}{4} \right) + 2 \frac{\pi\dot{\pi}}{t} \left( 1 + \frac{\gamma}{4} \right) \quad (3.38)$$

and

$$\tilde{\mathcal{L}}_3 = \left(1 + \frac{\gamma}{4}\right) \left(\frac{1}{3} \frac{\pi^3}{t^2} + 2 \frac{\pi^2 \dot{\pi}}{t} + 2\pi Z + 2\pi \dot{\pi}^2 + 2t\dot{\pi}Z\right). \quad (3.39)$$

Here, the field  $\pi$  has one dimension, and all terms should be rendered in four dimensions by appropriate powers of  $M_{\text{pl}}$ . One can use the relation  $H \sim 2/3\gamma t$  to replace explicit powers of  $t$  above with powers of  $H$  and  $\gamma$ , in estimating the size of various terms.

*3.3.2. Basic phenomenology.* Here, we describe the rough phenomenology of a ‘realistic’ model of power-law K-inflation. The exit from inflation will be discussed in a later subsection.

The power spectrum of these models was derived by Garriga and Mukhanov [37]. Working in the limit of small  $\gamma$ , their answer (equation (42) of [37]) becomes

$$P_k^\zeta = \frac{1}{c_s} \frac{2}{3\gamma} \frac{GH_1^2}{\pi} \left(\frac{k}{k_1}\right)^{-3\gamma} = \frac{1}{c_s} \frac{2}{3\gamma} \frac{H_1^2}{8\pi^2 M_{\text{pl}}^2} \left(\frac{k}{k_1}\right)^{-3\gamma} \quad (3.40)$$

where  $H_1$  is taken to be the Hubble scale at the time of horizon exit for the perturbations currently at our horizon, and  $k_1$  is the the associated comoving wavenumber.

It follows from (3.40) that the tilt

$$n_s - 1 = -3\gamma + \dots \quad (3.41)$$

which allows us to fix  $\gamma \sim 1/60$  using the central value of the spectral index in the WMAP results [4]. This justifies our use of perturbation theory in  $\gamma$  in earlier equations.

Using (3.25) and (3.40), as well as the fact that data determines  $P^\zeta \sim 10^{-9}$  at horizon crossing, we find

$$H^2 \sim \frac{3}{4\sqrt{2}} \gamma^{3/2} 8\pi^2 M_{\text{pl}}^2 \times 10^{-9} \quad (3.42)$$

at horizon crossing, where  $M_{\text{pl}}$  is the reduced Planck mass. Plugging in  $\gamma \sim \frac{1}{60}$  as determined from  $n_s$ , we see that the Hubble scale when the 60th from the last e-folding leaves our horizon is roughly  $H \sim 10^{-5} M_{\text{pl}}$ . So primordial gravitational waves will be unobservable in this model.

The reader may wonder about the following. In these models, with  $c_s \ll 1$ , the sound horizon where fluctuations freeze out at  $c_s/H$  can be much smaller than  $1/H$ . What happens if  $c_s/H < l_p$ , i.e.  $H > c_s M_{\text{pl}}$ ? This would seem to give rise to a ‘trans-Planckian problem’.

However, it is easy to see that this regime cannot be reached in any reliable fashion. The power spectrum (3.40) makes it clear that

$$\frac{\delta\rho}{\rho} \sim \frac{1}{\gamma^{3/4}} \left(\frac{H}{M_{\text{pl}}}\right). \quad (3.43)$$

Hence, for  $H > \gamma^{1/2} M_{\text{pl}}$ ,  $\delta\rho/\rho \geq \mathcal{O}(1)$  and there is no good semi-classical description of any resulting region which exits from inflation.

3.3.3. *Exiting from K-inflation.* Using  $a(t) \sim t^{2/3\gamma}$ , the requirement that one gets 60 e-foldings starting from some initial time  $t_i$  is simply

$$\log(t_f/t_i) = 90\gamma. \quad (3.44)$$

Then, we need to arrange for an appropriate exit mechanism to kick in at the  $t_f$  we determine in this way.

Here, we briefly describe a simple mechanism to exit from K-inflation, modelled on hybrid inflation [63]. Going back to the Lagrangian (3.30), we would like to arrange so that after 60 e-foldings, at some specific value of  $\phi$ , the inflationary stage ends and the exit to standard radiation domination occurs. An easy way to do this, while fixing the problem that (3.30) violates the DEC around  $X = 0$ , is to consider a more elaborate theory including also a second scalar field  $\psi$ . Then a Lagrangian of the schematic form

$$\mathcal{L} = -Y + Y^2 e^{2\Phi} + (\partial\psi)^2 + (\Phi_*^2 - \Phi^2)\psi^2 + \frac{1}{M_*^2}\psi^2 Y + \alpha(\psi^2 - \beta^2 M_*^2)^2 + \dots \quad (3.45)$$

can do the trick. Here  $M_*$  is some UV scale, perhaps the string scale or the Planck scale. For small  $\alpha\beta^2$ , assuming  $\Phi_* \sim M_*$ , then right around the time when  $\langle\Phi\rangle \sim \Phi_*$  the  $\psi$  field becomes tachyonic and condenses. It rolls to a vev  $\langle\psi\rangle \sim \beta M_*$ , and for  $\beta \geq \mathcal{O}(1)$ , can correct the sign of the  $\phi$  kinetic term.

The Lagrangian (3.45) can then support an early phase of K-inflation, and exit to a phase with normal kinetic terms for the various fields. The DEC is satisfied during both the inflationary phase and around the flat-space vacuum with  $\langle\psi\rangle = \beta M_*$ . It is an interesting question to check whether it is satisfied all along the trajectory from the inflationary phase to the final vacuum.

#### 4. Non-Gaussian perturbations

Now in the general setup of section 2, we consider the non-Gaussian perturbations in the primordial power spectrum. There is a large amount of literature on this subject, see e.g. [7, 8, 13, 17, 19, 64, 69, 70]. However, most of the literature has been focused on the case where the speed of sound  $c_s$  is very close to one, where the primordial non-Gaussianities are generally too small to be detected in future experiments. In addition, in some of the literature, only the perturbations in the matter Lagrangian are considered, but not the gauge invariant perturbation that remains exactly constant after horizon exit<sup>7</sup>. Here we will consider a general Lagrangian of the form (2.1), and we will allow the speed of sound  $c_s$  to assume arbitrary values, only requiring the parameters in (2.7) (and one more parameter to be defined later) to be small and of order  $\mathcal{O}(\epsilon)$ . We will calculate the three-point correlation function for the gauge invariant scalar perturbation  $\zeta$  following the approach of Maldacena [7].

It is useful to define two parameters following [19]

$$\Sigma = X P_{,X} + 2X^2 P_{,XX} = \frac{H^2 \epsilon}{c_s^2}, \quad (4.1)$$

$$\lambda = X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX}. \quad (4.2)$$

<sup>7</sup> For models with significant non-Gaussianity, this may be a reasonable approximation, since the contributions from the gravitational sector yield an  $f_{\text{NL}}$  which is too small to measure in any case.

In the case of inflationary models with  $P_{,X\phi} = 0$ , the parameter  $\lambda$  can be written in terms of the speed of sound and small parameters  $\epsilon$ ,  $\eta$ ,  $s$  in (2.7). However, for inflation models where  $P_{,X\phi} \neq 0$ , such as DBI inflation or K-inflation, there is no simple formula for the parameter  $\lambda$  in terms of the slow-variation parameters, and we must treat each model individually.

To compute the Einstein action to the third order, it is useful to work in the ADM metric formalism

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (4.3)$$

This formalism is convenient because the equations of motion for the variables  $N$  and  $N^i$  are quite easy to solve. We will work in a comoving gauge where the three-dimensional metric  $h_{ij}$  takes the form

$$h_{ij} = a^2 e^{2\zeta} \delta_{ij}, \quad (4.4)$$

where we have neglected the tensor perturbations.  $a$  is the scale factor of the universe and  $\zeta$  is the scalar perturbation, and remains constant outside the horizon in this gauge. The index on  $N^i$  can be lowered by the three-dimensional metric  $h_{ij}$ . The inflaton fluctuation  $\delta\phi$  vanishes in this gauge, which makes the computations simpler. Using the ADM metric ansatz the action becomes

$$S = \frac{1}{2} \int dt d^3x \sqrt{h} N (R^{(3)} + 2P) + \frac{1}{2} \int dt d^3x \sqrt{h} N^{-1} (E_{ij} E^{ij} - E^2) \quad (4.5)$$

where we have set the reduced Planck mass  $M_{\text{pl}} = 1$  for convenience. The three-dimensional Ricci curvature  $R^{(3)}$  is computed from the metric  $h_{ij}$ . The symmetric tensor  $E_{ij}$  is defined as

$$E_{ij} = \frac{1}{2} (\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i). \quad (4.6)$$

The equations of motion for  $N$  and  $N^i$  are

$$\begin{aligned} R^{(3)} + 2P - 4X P_{,X} - N^{-2} (E_{ij} E^{ij} - E^2) &= 0, \\ \nabla_j (N^{-1} E_i^j) - \nabla_i (N^{-1} E) &= 0. \end{aligned} \quad (4.7)$$

We follow [7] and decompose  $N^i$  into two parts  $N_i = \tilde{N}_i + \partial_i \psi$  where  $\partial_i \tilde{N}^i = 0$ , and expand  $N$  and  $N^i$  in powers of  $\zeta$

$$\begin{aligned} N &= 1 + \alpha_1 + \alpha_2 + \dots, \\ \tilde{N}_i &= N_i^{(1)} + N_i^{(2)} + \dots, \\ \psi &= \psi_1 + \psi_2 + \dots, \end{aligned} \quad (4.8)$$

where  $\alpha_n, \tilde{N}_i^{(n)}, \psi_n \sim O(\zeta^n)$ . One can plug the power expansion into the equations of motion (4.7) for  $N$  and  $N^i$ . At first order in  $\zeta$ , the solutions [7, 19] are

$$\alpha_1 = \frac{\dot{\zeta}}{H}, \quad N_i^{(1)} = 0, \quad \psi_1 = -\frac{\zeta}{H} + \chi, \quad \partial^2 \chi = a^2 \frac{\epsilon}{c_s^2} \dot{\zeta}, \quad (4.9)$$

after choosing proper boundary conditions.

In order to compute the effective action to order  $\mathcal{O}(\zeta^3)$ , as pointed out in [7], in the ADM formalism one only needs to consider the perturbations of  $N$  and  $N^i$  to the

first-order  $\mathcal{O}(\zeta)$ . This is because their perturbations at order  $\mathcal{O}(\zeta^3)$  such as  $\alpha_3$  will multiply the constraint equation at the zeroth order  $\mathcal{O}(\zeta^0)$  which vanishes, and the second-order perturbations such as  $\alpha_2$  will multiply a factor which vanishes by the first-order solution (4.9). So the solution (4.9) is enough for our purpose. This general conclusion in the ADM formalism can be seen as follows.

What we have done so far is solve the constraint equations for the Lagrange multipliers  $N$  and  $N_i$ , which result from the variation of the action with respect to them

$$\begin{aligned}\delta S &= \int d^4x \delta \mathcal{L}(\partial_i N, N) \\ &= \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial(\partial_i N)} \partial_i \delta N + \frac{\partial \mathcal{L}}{\partial N} \delta N \right],\end{aligned}\quad (4.10)$$

where for simplicity we schematically denote  $N$  as either  $N$  or  $N_i$ . We expand  $N = N^{(0)} + \Delta N = N^{(0)} + N^{(1)} + N^{(2)} + \dots$  for

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial(\partial_i N)} &= \frac{\partial \mathcal{L}}{\partial(\partial_i N)} \Big|_0 + \frac{\partial^2 \mathcal{L}}{\partial(\partial_i N) \partial(\partial_j N)} \Big|_0 \partial_j \Delta N + \dots, \\ \frac{\partial \mathcal{L}}{\partial N} &= \frac{\partial \mathcal{L}}{\partial N} \Big|_0 + \frac{\partial^2 \mathcal{L}}{\partial N^2} \Big|_0 \Delta N + \dots,\end{aligned}\quad (4.11)$$

where the subscript 0 means  $\Delta N = 0$ . In order to get  $N^{(1)}$  we can neglect the terms involving  $\partial^2 \mathcal{L} / \partial(\partial_i N) \partial N$ . This is because this term starts from  $\mathcal{O}(\zeta)$  as we can see from (4.5) and (4.6), so it does not contribute to  $\Delta N$  at the first-order  $\mathcal{O}(\zeta)$ .

The  $\mathcal{O}(\zeta^0)$  terms in (4.10) are

$$\int d^4x \left[ \frac{\partial \mathcal{L}}{\partial(\partial_i N)} \Big|_{0, \zeta^0} \partial_i \delta N + \frac{\partial \mathcal{L}}{\partial N} \Big|_{0, \zeta^0} \delta N \right] = 0. \quad (4.12)$$

This equation is consistent with the background equation of motion (2.4). The  $\mathcal{O}(\zeta)$  terms in (4.10) are

$$\begin{aligned}\int d^4x \left\{ \left[ \frac{\partial \mathcal{L}}{\partial(\partial_i N)} \Big|_{0, \zeta} + \frac{\partial^2 \mathcal{L}}{\partial(\partial_i N) \partial(\partial_j N)} \Big|_{0, \zeta^0} \partial_j N^{(1)} \right] \partial_i \delta N \right. \\ \left. + \left[ \frac{\partial \mathcal{L}}{\partial N} \Big|_{0, \zeta} + \frac{\partial^2 \mathcal{L}}{\partial N^2} \Big|_{0, \zeta^0} N^{(1)} \right] \delta N \right\} = 0,\end{aligned}\quad (4.13)$$

where the subscripts  $\zeta$  or  $\zeta^0$  denote the order of the perturbation  $\zeta$  that we take. After integration by parts this gives the constraint equations for  $\Delta N$  at order  $\mathcal{O}(\zeta)$ , namely  $N^{(1)}$ . A similar procedure can be used to solve for  $\Delta N$  to order  $\mathcal{O}(\zeta^n)$ , namely up to  $N^{(n)}$ .

We will next substitute these solutions for the Lagrange multipliers into the action and expand to order  $\mathcal{O}(\zeta^n)$ , where  $n \geq 3$ . We show that to do this the knowledge up to  $N^{(n-2)}$  is enough. Let us look at the terms that possibly contain  $N^{(n-1)}$  and  $N^{(n)}$ ,

$$\begin{aligned}\Delta S &= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_i N)} \Big|_0 \partial_i \Delta N + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial(\partial_i N) \partial(\partial_j N)} \Big|_0 (\partial_i \Delta N) (\partial_j \Delta N) \right. \\ &\quad \left. + \frac{\partial \mathcal{L}}{\partial N} \Big|_0 \Delta N + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{(\partial N)^2} \Big|_0 (\Delta N)^2 + \dots \right\}.\end{aligned}\quad (4.14)$$

The terms involving  $\partial^2 \mathcal{L} / \partial(\partial_i N) \partial N = \mathcal{O}(\zeta)$  are not written because such terms will not contain  $N^{(n-1)}$  or  $N^{(n)}$  at order  $\mathcal{O}(\zeta^n)$ .

The following are all terms containing  $N^{(n)} (n \geq 2)$  in (4.14),

$$\int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_i N)} \Big|_{0, \zeta^0} \partial_i N^{(n)} + \frac{\partial \mathcal{L}}{\partial N} \Big|_{0, \zeta^0} N^{(n)} \right\}. \quad (4.15)$$

Comparing with (4.12) we know that this term vanishes. This is because after integration by parts,  $N^{(n)}$  will multiply a term which is just the zeroth-order constraint equation coming from (4.12) after integration by parts. Next we look at all terms containing  $N^{(n-1)} (n \geq 3)$  in (4.14),

$$\begin{aligned} \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_i N)} \Big|_{0, \zeta} \partial_i N^{(n-1)} + \frac{\partial^2 \mathcal{L}}{\partial(\partial_i N) \partial(\partial_j N)} \Big|_{0, \zeta^0} \partial_i N^{(n-1)} \partial_j N^{(1)} \right. \\ \left. + \frac{\partial \mathcal{L}}{\partial N} \Big|_{0, \zeta} N^{(n-1)} + \frac{\partial^2 \mathcal{L}}{\partial N^2} \Big|_{0, \zeta^0} N^{(n-1)} N^{(1)} \right\}. \end{aligned} \quad (4.16)$$

This term also vanishes because, after integration by parts,  $N^{(n-1)}$  will multiply a term which is the first-order constraint equation coming from (4.13) after integration by parts.

Therefore our task is simplified. In order to expand the action (4.5) to quadratic and cubic order in the primordial scalar perturbation  $\zeta$ , we only need to plug in the solution for the first-order perturbation in  $N$  and  $N^i$  and do the expansion. The results can also be extracted<sup>8</sup> from [7, 19]

$$S_2 = \int dt d^3x \left[ a^3 \frac{\epsilon}{c_s^2} \dot{\zeta}^2 - a\epsilon (\partial\zeta)^2 \right], \quad (4.17)$$

$$\begin{aligned} S_3 = \int dt d^3x \left[ -\epsilon a \zeta (\partial\zeta)^2 - a^3 (\Sigma + 2\lambda) \frac{\dot{\zeta}^3}{H^3} + \frac{3a^3 \epsilon}{c_s^2} \zeta \dot{\zeta}^2 \right. \\ \left. + \frac{1}{2a} \left( 3\zeta - \frac{\dot{\zeta}}{H} \right) (\partial_i \partial_j \psi \partial_i \partial_j \psi - \partial^2 \psi \partial^2 \psi) - 2a^{-1} \partial_i \psi \partial_i \zeta \partial^2 \psi \right], \end{aligned} \quad (4.18)$$

where  $\dot{\zeta}$  is the derivative with respect to  $t$ . One can decompose the perturbations into momentum modes using

$$u_k = \int d^3x \zeta(t, \mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (4.19)$$

#### 4.1. The quadratic part

To solve the quadratic part of the action (4.17) we define

$$v_k \equiv z u_k, \quad z \equiv \frac{a\sqrt{2\epsilon}}{c_s}. \quad (4.20)$$

This brings the equation of motion for the perturbation  $\zeta$  to a simple form

$$v_k'' + c_s^2 k^2 v_k - \frac{z''}{z} v_k = 0, \quad (4.21)$$

<sup>8</sup> Note that there is a typo in  $(1/a^4)(\dot{\mathcal{R}}/H)$  term in equation (44) of [19], it should be  $(1/a^4)(\dot{\mathcal{R}}/H)\partial^2 \psi_1 \partial^2 \psi_1$  instead.

where the prime denotes the derivative with respect to the conformal time defined by  $dt = ad\tau$ ,  $\tau = -(aH)^{-1}(1 + \mathcal{O}(\epsilon))$ . The leading order of  $z''/z = 2a^2H^2(1 + \mathcal{O}(\epsilon))$  is contributed by the scale factor  $a$  which has the strongest time dependence. If the sound speed varies slowly enough, the leading behaviour of equation (4.21) is given by a Bessel function. We write it in terms of the Fourier modes of  $\zeta$ ,  $u_k$ , using (4.20),

$$u_k = u(\tau, \mathbf{k}) = \frac{iH}{\sqrt{4\epsilon c_s k^3}}(1 + ikc_s\tau)e^{-ikc_s\tau}. \quad (4.22)$$

Here we have made the approximation that the sound speed changes slowly, so this solution has oscillatory behaviour but with a frequency that is slowly changing due to the time dependence of  $c_s$ . This requires

$$-k\tau\Delta c_s \ll c_s k\Delta\tau. \quad (4.23)$$

That is, the phase change in (4.22) caused by the change of  $c_s$  is much slower than that caused by the change of  $\tau$ . This condition can be brought to the form

$$\frac{\dot{c}_s}{c_s} \ll H, \quad (4.24)$$

which is just the condition for small slow-variation parameter  $s$  in (2.7).

From equation (4.22) we can see that, before horizon exit  $c_s k > aH$ ,  $u_k$  is oscillating and its amplitude is decreasing proportionally to  $\tau$ . For  $v_k$ , this is the leading behaviour in flat space and we have chosen the standard Bunch–Davies vacuum. (We will discuss non-Gaussianities for other choices of vacua in section 6.) After the horizon exit  $c_s k < aH$ ,  $u_k$  remains constant [3]. This is most easily seen from (4.21) where we can neglect the second term, and we see that (for the growing mode)  $v_k \propto z_k$  so that  $u_k = \text{constant}$ . This conclusion is not going to be changed by the higher order interactions [7], because, as we can see from the interaction terms (4.18), they involve either spatial derivatives, which can be neglected at super-horizon scales, or powers of time derivatives starting from second order. So after horizon exit, the leading value of  $u_k$  is determined by (4.22) at  $\tau \approx 0$  with the rest of the variables evaluated at  $c_s k = aH$ . We emphasize here that the validity of our analysis only requires the *variation* of sound speed to be slow; the sound speed can be arbitrary. For our later purposes, the first-order corrections to the leading behaviour (4.22) are also important. We work this out in appendix A.

Now, we follow the standard technique in quantum field theory and write the operator in terms of creation and annihilation modes

$$\zeta(\tau, \mathbf{k}) = u(\tau, \mathbf{k})a(\mathbf{k}) + u^*(\tau, -\mathbf{k})a^\dagger(-\mathbf{k}) \quad (4.25)$$

with the canonical commutation relation  $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ .

## 4.2. The cubic part

The cubic effective action in (4.18) looks like order  $\mathcal{O}(\epsilon^0)$  in the slow-variation parameters. In slow-roll inflation, as emphasized and demonstrated in [7], one can perform a lot of integrations by parts and cancel terms of order  $\mathcal{O}(\epsilon^0)$  and  $\mathcal{O}(\epsilon)$ . The resulting cubic action is actually of leading order  $\mathcal{O}(\epsilon^2)$  in slow roll parameters. A similar analysis can be performed for the general Lagrangian in [19], as well as in the case of interest here where

the sound speed is arbitrary. Except for terms that are proportional to  $1 - c_s^2$  or  $\lambda$ , the rest of the terms can be cancelled to the second-order  $\mathcal{O}(\epsilon^2)$ ,

$$S_3 = \int dt d^3x \left\{ -a^3 \left( \Sigma \left( 1 - \frac{1}{c_s^2} \right) + 2\lambda \right) \frac{\dot{\zeta}^3}{H^3} + \frac{a^3 \epsilon}{c_s^4} (\epsilon - 3 + 3c_s^2) \zeta \dot{\zeta}^2 \right. \\ \left. + \frac{a\epsilon}{c_s^2} (\epsilon - 2s + 1 - c_s^2) \zeta (\partial\zeta)^2 - 2a \frac{\epsilon}{c_s^2} \dot{\zeta} (\partial\zeta) (\partial\chi) \right. \\ \left. + \frac{a^3 \epsilon}{2c_s^2} \frac{d}{dt} \left( \frac{\eta}{c_s^2} \right) \zeta^2 \dot{\zeta} + \frac{\epsilon}{2a} (\partial\zeta) (\partial\chi) \partial^2 \chi + \frac{\epsilon}{4a} (\partial^2 \zeta) (\partial\chi)^2 + 2f(\zeta) \frac{\delta L}{\delta \zeta} \Big|_1 \right\}, \quad (4.26)$$

where the variable  $\chi$  is defined in equation (4.9), and in the last term

$$\frac{\delta L}{\delta \zeta} \Big|_1 = a \left( \frac{d\partial^2 \chi}{dt} + H \partial^2 \chi - \epsilon \partial^2 \zeta \right), \quad (4.27)$$

$$f(\zeta) = \frac{\eta}{4c_s^2} \zeta^2 + \frac{1}{c_s^2 H} \zeta \dot{\zeta} + \frac{1}{4a^2 H^2} [-(\partial\zeta)(\partial\zeta) + \partial^{-2}(\partial_i \partial_j (\partial_i \zeta \partial_j \zeta))] \\ + \frac{1}{2a^2 H} [(\partial\zeta)(\partial\chi) - \partial^{-2}(\partial_i \partial_j (\partial_i \zeta \partial_j \chi))]. \quad (4.28)$$

Here  $\partial^{-2}$  is the inverse Laplacian,  $\delta L/\delta \zeta|_1$  is the variation of the quadratic action with respect to the perturbation  $\zeta$ , therefore the last term which is proportional to  $\delta L/\delta \zeta|_1$  can be absorbed by a field redefinition of  $\zeta$ . It can be easily shown that the field redefinition that absorbs this term is

$$\zeta \rightarrow \zeta_n + f(\zeta_n). \quad (4.29)$$

For the correlation function, only the first term in (4.28) contributes since all other terms involve at least one derivative of  $\zeta$  that vanishes outside the horizon. The three-point function after field redefinition  $\zeta \rightarrow \zeta_n + (\eta/4c_s^2)\zeta_n^2$  becomes

$$\langle \zeta(\mathbf{x}_1) \zeta(\mathbf{x}_2) \zeta(\mathbf{x}_3) \rangle = \langle \zeta_n(\mathbf{x}_1) \zeta_n(\mathbf{x}_2) \zeta_n(\mathbf{x}_3) \rangle \\ + \frac{\eta}{2c_s^2} (\langle \zeta_n(\mathbf{x}_1) \zeta_n(\mathbf{x}_2) \rangle \langle \zeta_n(\mathbf{x}_1) \zeta_n(\mathbf{x}_3) \rangle + \text{sym}) + \mathcal{O}(\eta^2 (P_k^\zeta)^3). \quad (4.30)$$

We proceed to calculate with the above cubic terms (4.26). The terms in the last line in (4.26) are all of subleading order in slow-variation parameters (2.7). The interaction Hamiltonian from the leading terms to  $\mathcal{O}(\epsilon^2)$  is

$$H_{\text{int}}(t) = - \int d^3x \left\{ -a^3 \left( \Sigma \left( 1 - \frac{1}{c_s^2} \right) + 2\lambda \right) \frac{\dot{\zeta}^3}{H^3} + \frac{a^3 \epsilon}{c_s^4} (\epsilon - 3 + 3c_s^2) \zeta \dot{\zeta}^2 \right. \\ \left. + \frac{a\epsilon}{c_s^2} (\epsilon - 2s + 1 - c_s^2) \zeta (\partial\zeta)^2 - 2a \frac{\epsilon}{c_s^2} \dot{\zeta} (\partial\zeta) (\partial\chi) \right\}. \quad (4.31)$$

One then computes the vacuum expectation value of the three-point function in the interaction picture that characterizes the primordial non-Gaussianities

$$\langle \zeta(t, \mathbf{k}_1) \zeta(t, \mathbf{k}_2) \zeta(t, \mathbf{k}_3) \rangle = -i \int_{t_0}^t dt' \langle [\zeta(t, \mathbf{k}_1) \zeta(t, \mathbf{k}_2) \zeta(t, \mathbf{k}_3), H_{\text{int}}(t')] \rangle, \quad (4.32)$$

where  $t_0$  is some very early time when the vacuum fluctuation of the inflaton is well within the horizon, and  $t$  is a time about several e-foldings after the horizon exit. Translated to the conformal time  $\tau = -1/(aH)$ , we can in a good approximation take the integral over conformal time  $\tau$  from  $-\infty$  to 0. We follow the standard procedure and compute the contributions from various terms. In the following, we first evaluate the leading contributions of each term.

(1) Contribution from  $\dot{\zeta}^3$  term. We denote  $K = k_1 + k_2 + k_3$ , and find

$$\begin{aligned} & -i \left( c_s^2 - 1 + \frac{2\lambda c_s^2}{\Sigma} \right) \frac{H^2 \epsilon}{c_s^4} u(0, \mathbf{k}_1) u(0, \mathbf{k}_2) u(0, \mathbf{k}_3) \int_{-\infty}^0 \frac{a d\tau}{H^3} \\ & \quad \times \left( 6 \frac{du^*(\tau, \mathbf{k}_1)}{d\tau} \frac{du^*(\tau, \mathbf{k}_2)}{d\tau} \frac{du^*(\tau, \mathbf{k}_3)}{d\tau} \right) (2\pi)^3 \delta^3 \left( \sum_i \mathbf{k}_i \right) + \text{c.c.} \\ & = -\frac{3H^4}{8\epsilon^2 c_s^4} \left( c_s^2 - 1 + \frac{2\lambda c_s^2}{\Sigma} \right) (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \left( \prod_{i=1}^3 \frac{1}{k_i^3} \right) \left( \frac{k_1^2 k_2^2 k_3^2}{K^3} \right). \end{aligned} \quad (4.33)$$

(2) Contribution from  $\zeta \dot{\zeta}^2$  term. We find

$$\begin{aligned} & i \frac{\epsilon}{c_s^4} (\epsilon - 3 + 3c_s^2) u(0, \mathbf{k}_1) u(0, \mathbf{k}_2) u(0, \mathbf{k}_3) \int_{-\infty}^0 a^2 d\tau \\ & \quad \times 2 \left( u^*(\tau, \mathbf{k}_1) \frac{du^*(\tau, \mathbf{k}_2)}{d\tau} \frac{du^*(\tau, \mathbf{k}_3)}{d\tau} + \text{sym} \right) (2\pi)^3 \delta^3 \left( \sum_i \mathbf{k}_i \right) + \text{c.c.} \\ & = \frac{H^4}{16\epsilon^2 c_s^4} (\epsilon - 3 + 3c_s^2) (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \left( \prod_{i=1}^3 \frac{1}{k_i^3} \right) \\ & \quad \times \left( \frac{k_2^2 k_3^2}{K} + \frac{k_1 k_2^2 k_3^2}{K^2} + \text{sym} \right). \end{aligned} \quad (4.34)$$

(3) Contribution from  $\zeta(\partial\zeta)^2$  term.

$$\begin{aligned} & \frac{H^4}{16\epsilon^2 c_s^4} (\epsilon - 2s + 1 - c_s^2) (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \left( \prod_{i=1}^3 \frac{1}{k_i^3} \right) \\ & \quad \times \left( (\mathbf{k}_1 \cdot \mathbf{k}_2) \left( -K + \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{K} + \frac{k_1 k_2 k_3}{K^2} \right) + \text{sym} \right). \end{aligned} \quad (4.35)$$

(4) Contribution from  $\dot{\zeta}(\partial\zeta)(\partial\chi)$  term.

$$-\frac{H^4}{16\epsilon c_s^4} (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \left( \prod_{i=1}^3 \frac{1}{k_i^3} \right) \left( \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2) k_3^2}{K} \left( 2 + \frac{k_1 + k_2}{K} \right) + \text{sym} \right). \quad (4.36)$$

(5) Contribution from field redefinition  $\zeta \rightarrow \zeta_n + (\eta/4c_s^2)\zeta_n^2$ .

$$\frac{\eta}{2} \frac{H^4}{16\epsilon^2 c_s^4} (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \left( \frac{1}{k_1^3 k_2^3} + \text{sym} \right). \quad (4.37)$$

As a first step, we add all of these leading contributions together. After some simplification, we find

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \rangle = (2\pi)^7 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) (P_k^\zeta)^2 \frac{1}{\prod_i k_i^3} \mathcal{A}, \quad (4.38)$$

where the above contributions to  $\mathcal{A}$  are organized as follows

$$\begin{aligned} \mathcal{A} \supset & \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right) \frac{3k_1^2 k_2^2 k_3^2}{2K^3} \\ & + \left( \frac{1}{c_s^2} - 1 \right) \left( -\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right) \\ & + \frac{\epsilon}{c_s^2} \left( -\frac{1}{8} \sum_i k_i^3 + \frac{1}{8} \sum_{i \neq j} k_i k_j^2 + \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 \right) \\ & + \frac{\eta}{c_s^2} \left( \frac{1}{8} \sum_i k_i^3 \right) \\ & + \frac{s}{c_s^2} \left( -\frac{1}{4} \sum_i k_i^3 - \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 \right). \end{aligned} \quad (4.39)$$

### 4.3. Correction terms

We notice that, in (4.39) for general  $c_s$ , the first two lines are not of the same order of magnitude as the last three lines. The former are  $\mathcal{O}(1)$  while the latter are  $\mathcal{O}(\epsilon)$ . So for  $c_s^2 \ll 1$ , it is clear that the first two terms dominate. If for completeness one is interested in the full result to  $\mathcal{O}(\epsilon)$ , however, small corrections to the first two lines may compete with the last three lines. This means that one must calculate the subleading terms (of order  $\mathcal{O}(\epsilon)$ ) for the first three integrations in section 4.2.

In obtaining (4.39), we treat all the slow-varying parameters in the integrand as constant, and we use the leading order solution (4.22). The corrections come from several sources.

Firstly, there are corrections to the leading order  $u(\tau, k)$  in (4.22), which we work out in appendix A,

$$\begin{aligned} u_k(y) = & -\frac{\sqrt{\pi}}{2\sqrt{2}} \frac{H}{\sqrt{\epsilon c_s}} \frac{1}{k^{3/2}} \left( 1 + \frac{\epsilon}{2} + \frac{s}{2} \right) \exp \left( i \frac{\pi}{2} \left( \epsilon + \frac{\eta}{2} \right) \right) y^{3/2} \\ & \times H_{(3/2)+\epsilon+(\eta/2)+(s/2)}^{(1)} ((1 + \epsilon + s)y), \end{aligned} \quad (4.40)$$

where  $y = c_s k / aH$ .

Secondly, various parameter in this solution as well as others in the integrand, including  $H$ ,  $c_s$ ,  $\lambda$ ,  $\epsilon$ , are all time-dependent subject to the slow-variation conditions. We Taylor-expand such functions as

$$\begin{aligned} f(\tau) &= f(t_K) + \frac{\partial f}{\partial t} (t - t_K) + \mathcal{O}(\epsilon^2 f) \\ &= f(\tau_K) - \frac{\partial f}{\partial t} \frac{1}{H_K} \ln \frac{\tau}{\tau_K} + \mathcal{O}(\epsilon^2 f), \end{aligned} \quad (4.41)$$

where the reference point  $\tau_K$  is chosen to be the moment when the wavenumber  $K = k_1 + k_2 + k_3$  exits the horizon.

Thirdly, the scale factor  $a$  also receives  $\mathcal{O}(\epsilon)$  correction,

$$a = -\frac{1}{H_K \tau} - \frac{\epsilon}{H_K \tau} + \frac{\epsilon}{H_K \tau} \ln(\tau/\tau_K) + \mathcal{O}(\epsilon^2). \quad (4.42)$$

We then consider all types of corrections in the first three integrations (4.33), (4.34) and (4.35) in section 4.2. We leave the details of these calculations to appendix B, and summarize the final results in the following subsection.

#### 4.4. Summary of final results

To first order in the slow-variation parameters  $\mathcal{O}(\epsilon)$ , the three-point correlation function of the gauge invariant scalar perturbation  $\zeta$  for a general single-field inflation model is given by the following:

$$\begin{aligned} \langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \rangle &= (2\pi)^7 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) (\tilde{P}_K^\zeta)^2 \frac{1}{\prod_i k_i^3} \\ &\quad \times (\mathcal{A}_\lambda + \mathcal{A}_c + \mathcal{A}_o + \mathcal{A}_\epsilon + \mathcal{A}_\eta + \mathcal{A}_s), \end{aligned} \quad (4.43)$$

where we have decomposed the shape of the three-point function into six parts

$$\mathcal{A}_\lambda = \left( \frac{1}{c_s^2} - 1 - \frac{\lambda}{\Sigma} [2 - (3 - 2c_1)l] \right)_K \frac{3k_1^2 k_2^2 k_3^2}{2K^3}, \quad (4.44)$$

$$\mathcal{A}_c = \left( \frac{1}{c_s^2} - 1 \right)_K \left( -\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right), \quad (4.45)$$

$$\mathcal{A}_o = \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right)_K (\epsilon F_{\lambda\epsilon} + \eta F_{\lambda\eta} + s F_{\lambda s}) + \left( \frac{1}{c_s^2} - 1 \right)_K (\epsilon F_{c\epsilon} + \eta F_{c\eta} + s F_{cs}), \quad (4.46)$$

$$\mathcal{A}_\epsilon = \epsilon \left( -\frac{1}{8} \sum_i k_i^3 + \frac{1}{8} \sum_{i \neq j} k_i k_j^2 + \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 \right), \quad (4.47)$$

$$\mathcal{A}_\eta = \eta \left( \frac{1}{8} \sum_i k_i^3 \right), \quad (4.48)$$

$$\mathcal{A}_s = s F_s. \quad (4.49)$$

The definitions of the sound speed  $c_s$ ,  $\Sigma$  and  $\lambda$  are

$$\begin{aligned} c_s^2 &\equiv \frac{P_{,X}}{P_{,X} + 2X P_{,XX}}, \\ \Sigma &\equiv X P_{,X} + 2X^2 P_{,XX}, \\ \lambda &\equiv X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX}. \end{aligned} \quad (4.50)$$

The definitions of the four slow-variation parameters are

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv \frac{\dot{\epsilon}}{\epsilon H}, \quad s \equiv \frac{\dot{c}_s}{c_s H}, \quad l \equiv \frac{\dot{\lambda}}{\lambda H}. \quad (4.51)$$

$\tilde{P}_K^\zeta$  is defined as

$$\tilde{P}_K^\zeta \equiv \frac{1}{8\pi^2} \frac{H_K^2}{c_{sK}\epsilon}. \quad (4.52)$$

Note that  $H$ ,  $c_s$ ,  $\lambda$  and  $\Sigma$  in this final result are evaluated at the moment  $\tau_K \equiv -(1/Kc_{sK}) + \mathcal{O}(\epsilon)$  when the wavenumber  $K \equiv k_1 + k_2 + k_3$  exits the horizon  $Kc_{sK} = a_K H_K$ , as indicated by the subscript  $K$ . Various  $F$  are functions of  $k_i$ , whose detailed forms are given in appendix B.1.

These results clearly illustrate that very significant non-Gaussianities  $f_{\text{NL}} \gg 1$  will arise most easily in models with  $c_s \ll 1$  or  $\lambda/\Sigma \gg 1$  during inflation (while conventional slow-roll models enjoy  $c_s = 1$  and  $\lambda/\Sigma = 0$ ). We also see that for this wide class of models, the functional form of the leading non-Gaussianity is completely determined in terms of 5 numbers:  $c_s$ ,  $\lambda/\Sigma$ , and the three slow-variation parameters  $\epsilon$ ,  $\eta$  and  $s$ . Note that in (4.39) we are assuming that these parameters do not vary over the few e-foldings we see close to the horizon. If they do, the parameter counting becomes a little more complicated, and one can in principle extract more information from these results (by studying the running of the non-Gaussianities). Indeed, after Taylor-expanding some slow-varying functions in working out all the correction terms in section 4.3, one more parameter  $l$  comes up. However, as illustrated in (4.44), it happens that one can absorb  $l$  in  $2\lambda/\Sigma$  as  $(\lambda/\Sigma)[2 - (3 - 2\mathbf{c}_1)l]$ , where  $\mathbf{c}_1$  is the Euler constant. (Later we will often refer to it as  $2\lambda/\Sigma$  for simplicity.) The error introduced to (4.43) after this absorption is of order  $\mathcal{O}(\epsilon^2)$ . So to the first-order  $\mathcal{O}(\epsilon)$  that we are interested in, our final result is still parameterized by five numbers.

## 5. Size, shape and running of the non-Gaussianities

In the previous section, we have obtained the most general form of the primordial three-point scalar non-Gaussianities up to first order in slow-variation parameters (4.51) in single-field inflationary models, where the matter Lagrangian is an arbitrary function of the inflaton and its first derivative. This non-Gaussianity is controlled by five parameters—three small parameters  $\epsilon$ ,  $\eta$  and  $s$ , the sound speed  $c_s$  and another parameter  $\lambda/\Sigma$ . There is a large amount of literature studying non-Gaussian features in models belonging to this class. One interesting feature of our result is that we can take different limits and smoothly connect various previous results. We will also explore regions which have not been studied before. Before giving several major examples, we first discuss some general features of the non-Gaussianity obtained in section 4.4.

The correlation function in section 4.4 is a function of three momenta forming a triangle. Therefore, generally there are three interesting properties—the magnitude of the function, its dependence on the shape of the triangle and its dependence on the size of the triangle. Namely, these quantities determine the size, shape and running of the non-Gaussianity.

To discuss whether a non-Gaussianity is large enough to be observed we first need to quote the experimental sensitivities. The non-Gaussianity of the CMB in the WMAP observations is analysed by assuming the following ansatz for the scalar perturbation

$$\zeta = \zeta_L - \frac{3}{5} f_{\text{NL}} \zeta_L^2, \quad (5.1)$$

where  $\zeta_L$  is the linear Gaussian part the perturbations, and  $f_{\text{NL}}$  is an estimator parameterizing the size of the non-Gaussianity. This assumption leads to the following three-point correlation function

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \rangle = (2\pi)^7 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \left( -\frac{3}{10} f_{\text{NL}} (P_k^\zeta)^2 \right) \frac{\sum_i k_i^3}{\prod_i k_i^3}. \quad (5.2)$$

Notice that this shape is different from any of the shapes in section 4.4 except for  $\mathcal{A}_\eta$ . But we can set up a similar estimator  $f_{\text{NL}}$  for each of those different shapes of non-Gaussianities to parameterize its magnitude. This matching is conventionally done for the equilateral triangle case  $k_1 = k_2 = k_3$ . We then have

$$\begin{aligned} f_{\text{NL}}^\lambda &= -\frac{5}{81} \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right) + (3 - 2c_1) \frac{l\lambda}{\Sigma}, \\ f_{\text{NL}}^c &= \frac{35}{108} \left( \frac{1}{c_s^2} - 1 \right), \\ f_{\text{NL}}^o &= \mathcal{O} \left( \frac{\epsilon}{c_s^2}, \frac{\epsilon\lambda}{\Sigma} \right), \\ f_{\text{NL}}^{\epsilon, \eta, s} &= \mathcal{O}(\epsilon). \end{aligned} \quad (5.3)$$

The bound on the parameter  $f_{\text{NL}}$  from data analysis depends on the shape of the non-Gaussianities which we will discuss shortly. The current bound is roughly  $|f_{\text{NL}}| < 300$  for the first two shapes [65], and  $|f_{\text{NL}}| < 100$  for the rest [4]. A non-Gaussianity is potentially observable in future experiments if  $|f_{\text{NL}}| > 5$  [66]–[68].

The magnitudes of  $\mathcal{A}_\epsilon$ ,  $\mathcal{A}_\eta$  and  $\mathcal{A}_s$  are unobservably small, of order  $\mathcal{O}(\epsilon)$ . The sizes of  $\mathcal{A}_o$  are determined by  $\epsilon/c_s^2$ ,  $\eta/c_s^2$  and  $s/c_s^2$ . So in order to make the magnitude of these functions larger, we need the denominator  $c_s^2$  to be less than at least one of the slow-variation parameters. (A similar conclusion for the  $\lambda/\Sigma$  term.) It is very interesting to construct such models, and we will show an example in this section. The most significant non-Gaussianities come from  $\mathcal{A}_\lambda$  and  $\mathcal{A}_c$  when  $c_s \ll 1$  and/or  $\lambda/\Sigma \gg 1$ .

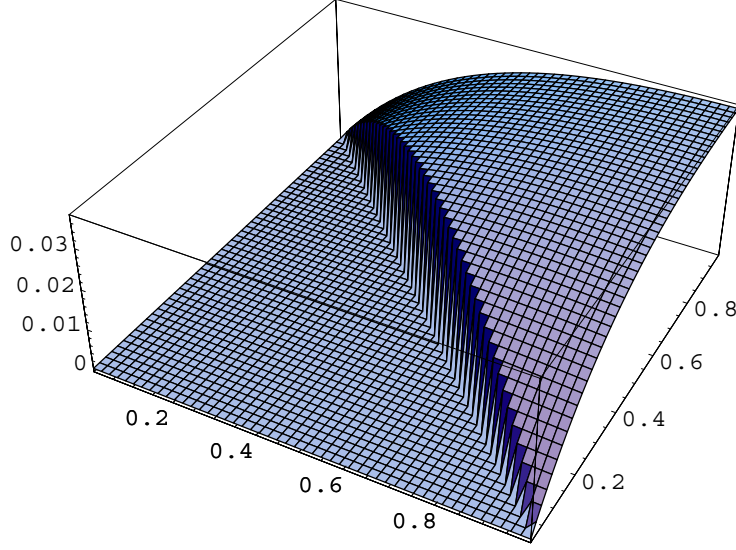
As in [20], to show the shape of  $\mathcal{A}(k_1, k_2, k_3)$ , we present the 3-d plot  $x_2^{-1} x_3^{-1} \mathcal{A}(1, x_2, x_3)$  as a function of  $x_2 = k_2/k_1$  and  $x_3 = k_3/k_1$ . The shapes of  $-\mathcal{A}_\lambda/k_1 k_2 k_3$  and  $\mathcal{A}_c/k_1 k_2 k_3$  are shown in figures 1 and 2. We can see that  $\mathcal{A}_\lambda$  and  $\mathcal{A}_c$  have overall similar shapes, but with opposite sign. The shape of  $-\mathcal{A}_\epsilon/k_1 k_2 k_3$  is shown in figure 3. The shapes of  $\mathcal{A}_\eta$ ,  $\mathcal{A}_s$  and  $\mathcal{A}_o$  are similar to that of  $\mathcal{A}_\epsilon$  up to a sign, in the sense that they all roughly approach to a pole in the squeezed limit, e.g. when  $k_1 = k_2$  and  $k_3 \rightarrow 0$ . (See appendix B.2.)

Interestingly, for a small sound speed, both the leading non-Gaussianity  $\mathcal{A}_\lambda$ ,  $\mathcal{A}_c$  and the subleading non-Gaussianity  $\mathcal{A}_o$  are potentially observable. Moreover, they have distinctive shapes. This is most dramatic in the squeezed limit where one of the momenta is relatively small. For example, in the limit  $k_1 = k_2$  and  $k_3 \approx 0$ ,

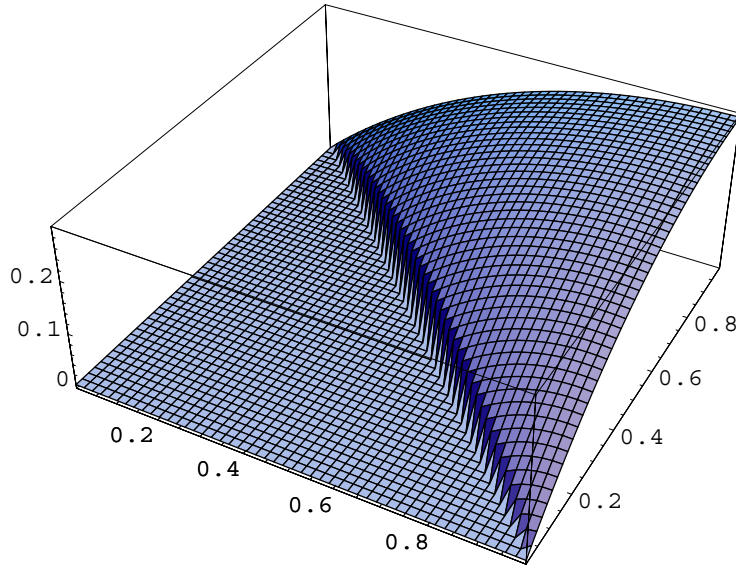
$$\frac{\mathcal{A}_\lambda}{k_1 k_2 k_3} \propto \frac{k_3}{k_1} = x_3, \quad (5.4)$$

while

$$\frac{\mathcal{A}_o}{k_1 k_2 k_3} \propto \frac{k_1}{k_3} = \frac{1}{x_3}. \quad (5.5)$$



**Figure 1.** The shape of  $-\mathcal{A}_\lambda / k_1 k_2 k_3$ .

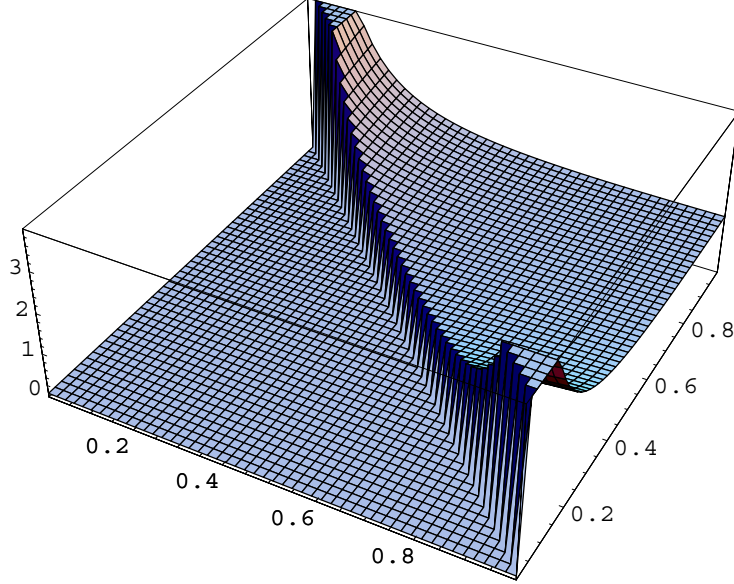


**Figure 2.** The shape of  $\mathcal{A}_c / k_1 k_2 k_3$ .

The size of the non-Gaussianity also depends on the scale that we measure. Analogous to the spectral index, we define [17]

$$n_{\text{NG}} - 1 \equiv \frac{\text{d} \ln |f_{\text{NL}}|}{\text{d} \ln k}. \quad (5.6)$$

For example, if the main contribution to  $f_{\text{NL}}$  comes from a small sound speed, then  $n_{\text{NG}} - 1 \approx -2s$ . So, in this case a measurement of the running of the non-Gaussianity directly tells us one of the slow-variation parameter  $s$ .



**Figure 3.** The shape of  $-\mathcal{A}_\epsilon/k_1 k_2 k_3$ .

### 5.1. Slow-roll inflation

We now reduce equation (4.43) to the slow-roll case. In this case, the deviation of the sound speed from one is very small. We denote  $u = 1 - (1/c_s^2) \ll 1$ . Assuming  $u = \mathcal{O}(\epsilon)$ , we can neglect both  $\mathcal{A}_s$  (4.49) since  $s \approx (\dot{u}/2H) \ll \mathcal{O}(\epsilon)$  and  $\mathcal{A}_o$  since  $\mathcal{A}_o = \mathcal{O}(u\epsilon)$ . As in [19], if we further assume  $P_{X\phi} = 0$ , the relations

$$\lambda = \frac{\epsilon}{6} \left( \frac{2\epsilon}{3\epsilon_X} (1-u)s - u \right), \quad \epsilon_X \equiv -\frac{\dot{X}}{H^2} \frac{\partial H}{\partial X} \quad (5.7)$$

follow. In this limit equations (4.44)–(4.49) become

$$\begin{aligned} \mathcal{A} = & - \left( \frac{\epsilon}{3\epsilon_X} s + u \right) \frac{k_1^2 k_2^2 k_3^2}{K^3} \\ & - u \left( -\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right) \\ & + \epsilon \left( -\frac{1}{8} \sum_i k_i^3 + \frac{1}{8} \sum_{i \neq j} k_i k_j^2 + \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 \right) \\ & + \eta \left( \frac{1}{8} \sum_i k_i^3 \right). \end{aligned} \quad (5.8)$$

Taking into account the convention difference  $\mathcal{A}_{\text{SL}} = 4\mathcal{A}_{\text{ours}}$ , we recover the result equation (83) of Seery and Lidsey.

Further setting the sound speed to be 1 (therefore  $u = 0$  and  $s = 0$ ), using the relation (3.5) and taking into account of the convention difference  $\mathcal{A}_{\text{Maldacena}} = 8\mathcal{A}_{\text{ours}}$ , we recover the result equation (4.6) of Maldacena.

## 5.2. DBI inflation

In [13, 17, 69], the three-point correlation functions in various models with potentially larger non-Gaussianities have been calculated by considering the perturbations in the matter Lagrangian only. In general, such a procedure is not valid because it is only the gauge invariant combination  $\zeta$  that will remain constant after the horizon exit. However (as realized by these authors) for the large non-Gaussianity case, due to large non-linear self coupling of the inflaton, the corrections coming from the gravitational part are relatively small, and this simpler method can be successfully used to obtain the leading behaviour of the non-Gaussianities. Indeed, in this limit our leading behaviour in  $\mathcal{A}_\lambda$  and  $\mathcal{A}_c$ , which is of order  $\mathcal{O}(c_s^{-2}, \lambda/\Sigma)$ , recovers the leading behaviour of the result in equation (6) of Gruzinov [69]. However, this procedure does not guarantee that the subleading term in [69], which is of  $\mathcal{O}(1)$ , will be correct. In fact, as we discussed at the beginning of this section, for  $0 < c_s^2 < \epsilon, \eta, s$ , the subleading terms actually come from  $\mathcal{A}_o$  and a subleading term in  $\mathcal{A}_\lambda$ , which are of order  $c_s^{-2}\mathcal{O}(\epsilon, \eta, s)$  and should be observable.

Now we consider the example of DBI inflation discussed in section 3.2. Interestingly for this type of inflation model, because of the Lagrangian (3.13), the parameter  $\lambda$  defined in (4.2) is

$$\lambda = X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX} = \frac{H^2 \epsilon}{2c_s^4} (1 - c_s^2). \quad (5.9)$$

So, the leading order contribution in  $\mathcal{A}_\lambda$  vanishes. The leading behaviour of  $\mathcal{A}_c$  reproduces the result of Alishahiha *et al* [13, 17]. Using the estimator defined in (5.3), we have

$$f_{\text{NL}}^c \approx 0.32 \frac{1 - c_s^2}{c_s^2} \approx 0.32 c_s^{-2}. \quad (5.10)$$

To estimate the subleading order corrections, let us look at both the UV and IR model. In the UV model, the sound speed is given by

$$c_s^{-1} \approx \sqrt{\frac{2\lambda}{3}} \frac{m M_{\text{pl}}}{\phi^2}. \quad (5.11)$$

It is easy to see that whether the subleading order is of order  $\mathcal{O}(1)$  or  $\mathcal{O}(\epsilon/c_s^2)$  depends on the value of  $\phi$ . In the IR model, the sound speed is related to the number of e-foldings  $N_e$  before the end of inflation by

$$c_s^{-1} \approx \beta N_e / 3, \quad (5.12)$$

where  $\beta$  parameterizes the steepness of the potential  $\beta \equiv m^2/H^2$ , which is generically of order one as we know from the usual eta-problem in slow-roll inflation. Since  $\epsilon, \eta, s = \mathcal{O}(N_e^{-1})$ , we see that this model falls into the region where  $c_s^2 \ll \mathcal{O}(\epsilon, \eta, s)$ , where both the leading non-Gaussianity  $\mathcal{A}_c$  and the subleading  $\mathcal{A}_o$  are observable. As we see from the discussions in section 5 and appendix B.2, their shapes are very different.

We can also compute the running of non-Gaussianities as considered in [17]

$$n_{\text{NG}} - 1 \approx -2s = -2 \sqrt{\frac{3M_{\text{pl}}^2}{f(\phi)V(\phi)}} \left( \frac{1}{2} \frac{V'}{V} - \frac{V''}{V'} + \frac{2}{\phi} \right). \quad (5.13)$$

Just like the spectral index, the running non-Gaussianities also provide a good probe of the effective potential  $V(\phi)$  and warp factor  $f(\phi)$ , and could (in an optimistic scenario) distinguish between different possible background geometries where the brane motion is occurring.

### 5.3. Kinetic inflation

We saw above that in one large class of models with measurable non-Gaussianities, the leading order of  $\mathcal{A}_\lambda$  vanishes. However, generally it does not vanish for some other models such as K-inflation, and could in principle be comparable to the second term  $\mathcal{A}_c$ . An experimental detection of the shape of large non-Gaussianities could therefore in principle distinguish between DBI inflation and more general models where the first term has significant contributions.

Here, we first outline a quick estimate of the non-Gaussianities in K-inflation, and then describe the exact result following section 4.

*5.3.1. Crude estimate of non-Gaussianities.* Given the fluctuation Lagrangians (3.38) and (3.39), we can do a simple estimate of the non-Gaussianity (following the general strategy also used in [13, 18]). While this is not strictly necessary in view of the detailed formulae in section 4, it is perhaps illuminating to understand in simple terms why these models have large  $f_{\text{NL}}$ .

The basic point is the following. We saw in (3.40) that

$$P_k^\zeta \sim \frac{1}{\gamma^{3/2}} \left( \frac{H}{M_{\text{pl}}} \right)^2. \quad (5.14)$$

Using the fact that

$$\zeta \sim \frac{H}{\dot{\Phi}} \pi \quad (5.15)$$

and evaluating this on the inflationary solution, we find

$$\zeta \sim \frac{\pi}{\gamma M_{\text{pl}}}. \quad (5.16)$$

It follows that

$$\langle \zeta \zeta \rangle \sim \frac{1}{\gamma^2} \langle \pi \pi \rangle. \quad (5.17)$$

Furthermore, given the overall factor of  $f(\gamma) \sim 1/\gamma^2$  in  $\mathcal{L}$ , which in particular multiplies the  $\pi$  kinetic terms, one has the relation

$$\langle \pi \pi \rangle \sim \gamma^2 (\delta\pi)^2 \quad (5.18)$$

where  $\delta\pi$  is a typical fluctuation of the  $\pi$  field during inflation. Therefore, estimating  $P^\zeta$  via

$$P^\zeta \sim \frac{1}{\gamma^2} \langle \pi \pi \rangle \sim (\delta\pi)^2 \quad (5.19)$$

and using  $P^\zeta \sim H^2/\gamma^{3/2}$ , we see that

$$\delta\pi \sim \frac{H}{\gamma^{3/4}}. \quad (5.20)$$

Using the form of the modes, one can also see that

$$\delta\dot{\pi} \sim H\delta\pi. \quad (5.21)$$

Now, we are interested in estimating the non-Gaussianity, say through a crude estimate of  $f_{\text{NL}}$ . Since naively

$$\frac{\tilde{\mathcal{L}}_3}{\tilde{\mathcal{L}}_2} \sim f_{\text{NL}}\sqrt{P^\zeta}, \quad (5.22)$$

we can plug (5.20) and (5.21) into the fluctuation Lagrangians to estimate the non-Gaussianity. The term  $2t\dot{\pi}Z$  in  $\tilde{\mathcal{L}}_3$  contributes fluctuations of size  $(1/M_{\text{pl}})(1/H\gamma)(H^2/\gamma^{3/4})^3 \sim (1/\gamma^{13/4})(H^5/M_{\text{pl}})$ . The largest terms in  $\tilde{\mathcal{L}}_2$  scale like  $H^4/\gamma^{3/2}$ . We then find

$$\frac{\tilde{\mathcal{L}}_3}{\tilde{\mathcal{L}}_2} \sim \frac{1}{\gamma^{7/4}} \frac{H}{M_{\text{pl}}}. \quad (5.23)$$

Because

$$\sqrt{P^\zeta} \sim \frac{1}{\gamma^{3/4}} \frac{H}{M_{\text{pl}}} \quad (5.24)$$

this translates into the rough estimate

$$f_{\text{NL}} \sim \frac{1}{\gamma} \sim \frac{1}{c_s^2}. \quad (5.25)$$

This in fact reproduces the more detailed results of section 4 when applied to K-inflation, though it does not give the (potentially very important) information about the detailed shape of the momentum dependence.

*5.3.2. Shape of non-Gaussianities in K-inflation.* An order of magnitude estimate for the general size of the non-Gaussian signatures arising in K-inflation appears in (5.25). Here, we refine this estimate using the formulae of section 4.

The Lagrangian of power-law K-inflation, at leading order in the expansion in  $\gamma$ , is given by

$$P(X, \phi) = \frac{16}{9\gamma^2} \frac{1}{\phi^2} (-X + X^2). \quad (5.26)$$

We then see that

$$\Sigma = \frac{16}{9\gamma^2} \frac{1}{\phi^2} (6X^2 - X) \sim \frac{16}{9\gamma^2} \frac{1}{\phi^2} \quad (5.27)$$

and

$$\lambda = \frac{32}{9\gamma^2} \frac{X^2}{\phi^2} \sim \frac{8}{9\gamma^2} \frac{1}{\phi^2}. \quad (5.28)$$

In each case, the estimate after  $\sim$  follows from the fact that  $X = \frac{1}{2} + \mathcal{O}(\gamma)$  on the inflationary solution.

We find that, to leading order in  $\gamma$ ,

$$\mathcal{A}_\lambda = \frac{12}{\gamma} \left( \frac{k_1^2 k_2^2 k_3^2}{K^3} \right) \quad (5.29)$$

and

$$\mathcal{A}_c = \frac{8}{\gamma} \left( -\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right). \quad (5.30)$$

So unlike the other higher derivative model we have carefully examined, DBI inflation, these models receive a leading-order contribution from the shape  $\mathcal{A}_\lambda$ . Also, in distinction to a general DBI inflation model in section 5.2, the constant speed of sound implies that the non-Gaussianity does not run in power-law K-inflation.

Evaluating on equilateral triangles and comparing to the ‘local’ form of non-Gaussianities, this translates to an estimated  $f_{\text{NL}}$  of

$$f_{\text{NL}} \approx \frac{170}{81} \frac{1}{\gamma}. \quad (5.31)$$

This gives  $f_{\text{NL}} \approx 125$  for realistic models, which is allowed by current experimental bounds but would be *easily* detectable in future experiments. Comparing to the rough estimate (5.25), which was  $f_{\text{NL}} \sim 8/\gamma$ , we see that the two results agree up to an  $\mathcal{O}(1)$  coefficient (whose moderate smallness prevents the model from being excluded by current data).

## 6. Non-Gaussianities as a probe of the inflationary vacuum

There has been some interest in the question of whether we can observe trans-Planckian physics in the cosmic microwave background radiation [11], [23]–[32]. In this context, the assumption that the Bunch–Davies vacuum is the unique initial state of inflation has recently been questioned. While several plausible alternatives for the initial state have been suggested, its precise form is highly dependent on how we model the behaviour of quantum fields at Planckian energies. What is universal however is that any deviation from the Bunch–Davies vacuum during inflation will result in modulations of the power spectrum [24, 25, 29], thus offering the exciting possibility of probing the initial state of the universe from cosmological measurements.

We hasten to stress that the microphysics that determines the choice of inflationary vacuum is by no means understood. Here we put aside the conceptual issues associated with the choice of vacuum and its consistency, and simply approach the problem of vacuum ambiguity from a phenomenological perspective. To be specific, we explore the possibility of using primordial non-Gaussianities to test any deviation from the standard Bunch–Davies vacuum. As it turns out, the effect of deviation from the Bunch–Davies vacuum on the shape of the primordial non-Gaussianities is quite simple to compute within our formalism. A general vacuum state for the fluctuation of the inflaton field during inflation can be written as follows

$$u_k = u(\tau, \mathbf{k}) = \frac{iH}{\sqrt{4\epsilon c_s k^3}} (C_+(1 + ik c_s \tau) e^{-ik c_s \tau} + C_-(1 - ik c_s \tau) e^{ik c_s \tau}). \quad (6.1)$$

In the standard Bunch–Davies vacuum we have  $C_+ = 1$  and  $C_- = 0$ . Now we allow a small deviation from the Bunch–Davies vacuum by turning on a small finite number  $C_-$ , and calculate the corrections to the shape of non-Gaussianities we found in the Bunch–Davies vacuum. For simplicity, we only consider the corrections to the leading order non-Gaussianities  $A_\lambda$ ,  $A_c$  in the small sound speed  $c_s \ll 1$  limit (the corrections to the other shapes of non-Gaussianities due to a non-standard choice of vacuum can be worked out by a similar procedure). The computations of the three-point functions of the primordial perturbations essentially go through as before. The first subleading correction to the Bunch–Davies vacuum result is simply to replace one of the three  $u(\tau, \mathbf{k})$  with its  $C_-$  component. (A correction of  $\mathcal{O}(C_-)$  in  $u(0, \mathbf{k})$  gives a term which has the same shape as in the Bunch–Davies vacuum case. We do not include them here.) This does not change the common factor  $1/k_1^3 k_2^3 k_3^3$  but will simply replace one of the  $k_i$  in the shapes  $A_\lambda$ ,  $A_c$  with  $-k_i$ . We denote the corresponding corrections  $\tilde{A}_\lambda$ ,  $\tilde{A}_c$ . We immediately find the corrections as

$$\begin{aligned} \tilde{A}_\lambda = \text{Re}(C_-) & \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right) \frac{3k_1^2 k_2^2 k_3^2}{2} \\ & \times \left( \frac{1}{(k_1 + k_2 - k_3)^3} + \frac{1}{(k_1 - k_2 + k_3)^3} + \frac{1}{(-k_1 + k_2 + k_3)^3} \right), \end{aligned} \quad (6.2)$$

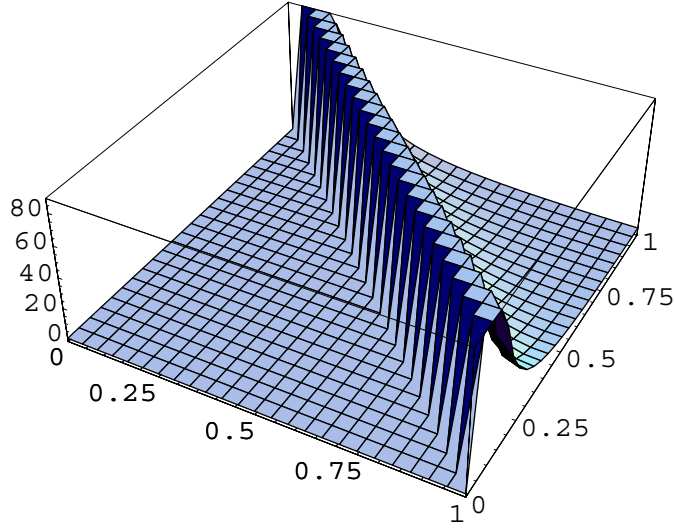
$$\tilde{A}_c = \text{Re}(C_-) \left( \frac{1}{c_s^2} - 1 \right) \sum_{p=1}^3 \left( -\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right) \Big|_{k_p \rightarrow -k_p}. \quad (6.3)$$

We can estimate the size of the non-Gaussianities  $\tilde{A}_\lambda$  and  $\tilde{A}_c$  according to the WMAP ansatz. This estimate is usually done in the equilateral triangle limit; we find

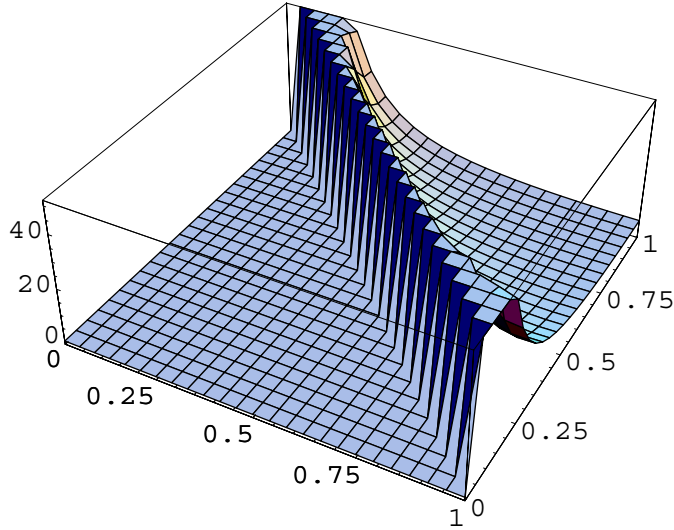
$$\begin{aligned} \tilde{f}_{\text{NL}}^\lambda &= -5 \text{Re}(C_-) \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right), \\ \tilde{f}_{\text{NL}}^c &= \frac{25}{4} \text{Re}(C_-) \left( \frac{1}{c_s^2} - 1 \right). \end{aligned} \quad (6.4)$$

If the sound speed  $c_s$  is sufficiently small, the effects of a slight deviation from the Bunch–Davies vacuum is potentially observable by future experiments. We plot the shapes of the non-Gaussianities  $\tilde{A}_\lambda$  and  $\tilde{A}_c$  in figures 4 and 5. We see the shapes of these corrections are very distinctive and in fact dramatically different from that of the DBI inflation or slow-roll inflation. In particular, these shapes are highly peaked at the ‘folded triangle’ limit where  $k_3 \approx k_1 + k_2$  for arbitrary values of  $k_1$  and  $k_2$ . This feature is not shared by other known sources of non-Gaussianities, and so measurements of the shape of non-Gaussianities could in principle be an excellent probe of the choice of inflationary vacuum.

Note that, while the rising behaviour of the non-Gaussianity in the folded triangle limit is the signal of the non-Bunch–Davies vacuum, the divergence at the limit, e.g.  $k_1 + k_2 - k_3 = 0$ , is artificial. This divergence is present because we have assumed that such a non-standard vacuum existed in the infinite past. Realistically there should be a cutoff at a large momentum  $M$  for  $k/a$ , where  $k$  is a typical value of  $k_{1,2,3}$ . This amounts to a cutoff for  $\tau$  at  $\tau_c = -M/Hk$ . Since the integrand is regulated at  $\tau = -1/Kc_s$  due to its rapid oscillation, if  $\tau_c < -1/Kc_s$ , the cutoff  $M$  has no effects on our calculation.



**Figure 4.** The shape of  $|\tilde{\mathcal{A}}_\lambda|/k_1 k_2 k_3$ .



**Figure 5.** The shape of  $|\tilde{\mathcal{A}}_c|/k_1 k_2 k_3$ .

That is, for  $K \gg kH/M_{c_s}$ , we will see the behaviours shown in figures 4 and 5 near the folded triangle limit. But within  $K < kH/M_{c_s}$ , the cutoff takes effect first, the divergence behaviour will be replaced. The details depend on the nature of the cutoff, e.g. a naive sharp cutoff will introduce oscillatory behaviour.

## 7. Conclusion

The forthcoming suite of cosmological experiments will nail down with ever greater precision the parameters of the inflationary model that yielded our homogeneous, isotropic universe. Some measurements, such as the value of the spectral index and the nature of its running, are guaranteed to occur. Others, such as a detection of primordial gravitational

waves, are not necessarily expected to occur on theoretical grounds (since models with very small  $r$  seem more natural as quantum field theories), but would be tremendously exciting and instructive if they do. The discovery of significant non-Gaussian scalar fluctuations falls into this latter category. While the simplest models of inflation do not produce this phenomenon, its discovery would tell us something qualitatively important about the inflationary epoch, and experiments sensitive enough to measure  $|f_{\text{NL}}| \geq 5$  will be launched in the next two years. For this reason, we feel it is worthwhile to parametrize the reasonable possibilities, and understand the qualitative physics of the models that produce them.

In this paper, we have taken some steps in this direction for generic single-field models. There are several clear directions for further work:

- It would be nice to derive the same formulae governing non-Gaussianities as arising directly from symmetry principles. Perhaps these would be encapsulated most neatly in a hypothetical dual, non-gravitational theory. For the models with  $c_s \ll 1$ , this theory may have novel properties.
- Higher derivative terms play a significant role in the dynamics of those single-field models which produce striking non-Gaussian signatures. One class of models where such terms are important, the DBI inflation [13]–[16], has a reasonable microscopic justification in string theory. It would be interesting to find other examples where one can microphysically justify the study of dynamics that is very sensitive to higher derivative terms.
- We have focused here on single-field models. It is a logical possibility that our 60 e-foldings arose from a multi-field inflationary model. This could be motivated if, for instance,  $r$  is measured to be non-negligible. In slow-roll models, measurable  $r$  implies an inflaton that traversed a super-Planckian distance in field space [71], as in chaotic inflation [72]. At least in string theory, this is difficult to accommodate in single-field models [73], but could conceivably happen in a multi-field setting [74]. For this and other reasons, it would be worthwhile to develop a general framework for analysing non-Gaussianities in multi-field models. Examples of multi-field models with significant non-Gaussianity appear in [75]–[77]. Formalisms to compute the non-Gaussianities in large classes of such models are developed in [78]–[81].

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## Appendix A. Corrections to $u_k$

In this appendix, we calculate the  $\mathcal{O}(\epsilon)$  correction to the solution of equation (4.21), generalizing the method of [82, 83].

We define

$$y \equiv \frac{c_s k}{aH} \quad (\text{A.1})$$

and write the equation of motion of the quadratic action

$$v_k'' + c_s^2 k^2 v_k - \frac{z''}{z} v_k = 0 \quad (\text{A.2})$$

in terms of  $y$ . Note that generally  $c_s$  is a (slowly varying) function of time. Using

$$\frac{z''}{z} = 2a^2 H^2 \left( 1 - \frac{1}{2}\epsilon + \frac{3}{4}\eta - \frac{3}{2}s \right) + \mathcal{O}(\epsilon^2) \quad (\text{A.3})$$

we get

$$(1 - 2\epsilon - 2s) y^2 \frac{dv_k}{dy^2} - sy \frac{dv_k}{dy} + y^2 v_k - \left( 2 - \epsilon + \frac{3}{2}\eta - 3s \right) v_k = 0. \quad (\text{A.4})$$

The solution of this differential equation is given by

$$v_k = y^{(1/2)(1+s)} \left[ C_1 H_\nu^{(1)}((1 + \epsilon + s)y) + C_2 H_\nu^{(2)}((1 + \epsilon + s)y) \right], \quad (\text{A.5})$$

where

$$\nu = \frac{3}{2} + \epsilon + \frac{\eta}{2} + \frac{s}{2}. \quad (\text{A.6})$$

The Bunch–Davies (BD) vacuum corresponds to  $C_2 = 0$ . To determine the coefficient  $C_1$ , we need to look at the large  $k$  behaviour of the equation (A.2),

$$v_k'' + c_s^2 k^2 v_k = 0. \quad (\text{A.7})$$

The behaviour is more general than the usual case where the sound speed is constant, because here we allow  $c_s$  to vary slowly. Using a similar approach and defining  $\tilde{y} \equiv -c_s k \tau = (1 + \epsilon + \mathcal{O}(\epsilon^2))y$ , we get the solution for (A.7) with positive energy (BD vacuum),

$$v_k \rightarrow \frac{1}{\sqrt{2c_s k}} \exp(i[(1 + s)\tilde{y} - (\pi/4)s]), \quad (\text{A.8})$$

up to a constant phase. Here the coefficient is determined by the quantization condition (Wronskian condition),  $v_k^*(dv_k/d\tau) - v_k(dv_k^*/d\tau) = -i$ , to first-order  $\mathcal{O}(\epsilon)$ . Notice that in (A.8), the sound speed  $c_s$  runs as a function of  $y$ . So we can expand it as

$$v_k \rightarrow \frac{1}{\sqrt{2c_{s0}k}} \left( \frac{y}{y_0} \right)^{s/2} \exp\left(i \left[ (1 + s)\tilde{y} - \frac{\pi}{4}s \right]\right), \quad (\text{A.9})$$

where the subscript 0 on  $c_s$  denotes the evaluation at  $y_0$ . Expanding (A.5) in the same limit,

$$v_k \rightarrow C_1 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1 + \epsilon + s}} y^{s/2} \exp\left(i \left[ y(1 + \epsilon + s) - \frac{\pi}{2}\nu - \frac{\pi}{4} \right]\right), \quad y \gg 1, \quad (\text{A.10})$$

we find

$$C_1 = -\frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{c_{sk}k}} \left(1 + \frac{\epsilon}{2} + \frac{s}{2}\right) \exp\left(i\frac{\pi}{2} \left(\epsilon + \frac{\eta}{2}\right)\right). \quad (\text{A.11})$$

Note a convention for the variables used here: the variables such as  $c_{sk}$  with the subscript  $k$  are evaluated at

$$y_0 = \frac{kc_{sk}}{a_k H_k} = 1; \quad (\text{A.12})$$

in some later formulae, the variables without the subscript such as  $c_s$  mean that they are functions of  $y$ .

From the definition

$$v_k \equiv zu_k, \quad z \equiv \frac{a\sqrt{2\epsilon}}{c_s}, \quad (\text{A.13})$$

we get the expression for  $u_k$  to order  $\mathcal{O}(\epsilon)$ ,

$$u_k(y) = -\frac{\sqrt{\pi}}{2\sqrt{2}} \frac{H}{\sqrt{\epsilon c_s}} \frac{1}{k^{3/2}} \left(1 + \frac{\epsilon}{2} + \frac{s}{2}\right) \exp\left(i\frac{\pi}{2} \left(\epsilon + \frac{\eta}{2}\right)\right) y^{3/2} H_\nu^{(1)}((1 + \epsilon + s)y). \quad (\text{A.14})$$

As an application, we use the  $y \rightarrow 0$  limit of equation (A.14) to derive an expression for the density perturbation to order  $\mathcal{O}(\epsilon)$ . To do this, we use the expansion of the Hankel function in the  $y \ll 1$  limit,

$$H_\nu^{(1)}(y) \rightarrow -i \frac{1}{\sin \nu \pi} \frac{1}{\Gamma(-\nu + 1)} \left(\frac{y}{2}\right)^{-\nu}, \quad (\text{A.15})$$

and get

$$u_k(0) = \frac{iH_k}{2\sqrt{c_{sk}\epsilon_k}} \frac{1}{k^{3/2}} \left(1 - (\mathbf{c}_2 + 1)\epsilon - \frac{\mathbf{c}_2}{2}\eta - \left(\frac{\mathbf{c}_2}{2} + 1\right)s\right) \exp\left(i\frac{\pi}{2} \left(\epsilon + \frac{\eta}{2}\right)\right), \quad (\text{A.16})$$

where

$$\mathbf{c}_2 \equiv \mathbf{c}_1 - 2 + \ln 2 \approx -0.73$$

and  $\mathbf{c}_1 = 0.577 \dots$  is the Euler constant. Hence the density perturbation is

$$\begin{aligned} \sqrt{P_k^\zeta} &= \sqrt{\frac{k^3}{2\pi^2}} |u_k(y=0)| \\ &= \frac{1}{\sqrt{8\pi^2}} \frac{H_k}{\sqrt{c_{sk}\epsilon_k}} \left(1 - (\mathbf{c}_2 + 1)\epsilon - \frac{\mathbf{c}_2}{2}\eta - \left(\frac{\mathbf{c}_2}{2} + 1\right)s\right) + \mathcal{O}(\epsilon^2). \end{aligned} \quad (\text{A.17})$$

This generalizes the result (31) of [82] to the case with running sound speed.

## Appendix B. Details on the correction terms

In this appendix, we provide details on the correction terms in section 4.3. Let us look at the first integration (4.33) in section 4.2,

$$-6i \int d\tau a f_1(\tau) \prod_i u(0, \mathbf{k}_i) \frac{d}{d\tau} u^*(\tau, \mathbf{k}_i) \cdot (2\pi)^3 \delta^3 \left( \sum_i \mathbf{k}_i \right) + \text{c.c.}, \quad (\text{B.1})$$

where

$$f_1(\tau) = \frac{\epsilon}{H c_s^4} (c_s^2 - 1) + \frac{2\lambda}{H^3}. \quad (\text{B.2})$$

We have evaluated the leading contribution of this integral in section 4.2. The corrections come from several different places.

The first is from the time variation in  $f_1(\tau)$ ,

$$\begin{aligned} f_1(\tau) &= f_1(t_K) + \frac{\partial f_1}{\partial t} (t - t_K) + \mathcal{O}(\epsilon^2 f_1) \\ &= f_1(\tau_K) - \frac{\partial f_1}{\partial t} \frac{1}{H_K} \ln \frac{\tau}{\tau_K} + \mathcal{O}(\epsilon^2 f_1). \end{aligned} \quad (\text{B.3})$$

We choose to eventually evaluate all the variables at the time  $\tau_K$ , which is defined as the moment when the wavenumber  $K = k_1 + k_2 + k_3$  exits the horizon  $K c_{sK} = a_K H_K$ , at which

$$\tau_K \equiv -\frac{1}{K c_{sK}} + \mathcal{O}(\epsilon). \quad (\text{B.4})$$

All the subscripts  $K$  denote the evaluation at the horizon exit point defined in (B.4). The  $\partial f_1 / \partial t$  can be expressed in terms of the slow-variation parameters

$$\frac{\partial f_1}{\partial t} = (\eta\epsilon + \epsilon^2)(c_s^{-2} - c_s^{-4}) + \epsilon s(-2c_s^{-2} + 4c_s^{-4}) + 2l\lambda H^{-2} + 6\epsilon\lambda H^{-2}, \quad (\text{B.5})$$

where we have defined

$$l \equiv \frac{\dot{\lambda}}{\lambda H}. \quad (\text{B.6})$$

We assume that the time variation of  $\lambda$  is slow,  $l = \mathcal{O}(\epsilon)$ . Plugging the correction terms of (B.3) into (B.1) and evaluating the rest in leading orders, we get<sup>9</sup>

$$\Delta\mathcal{A} \supset \frac{9}{4} \left( 1 - \frac{2}{3} \mathbf{c}_1 \right) \left( (-\epsilon - \eta) \left( \frac{1}{c_s^2} - 1 \right) + s \left( \frac{4}{c_s^2} - 2 \right) + (2l + 6\epsilon) \frac{\lambda}{\Sigma} \right)_K \frac{k_1^2 k_2^2 k_3^2}{K^3}. \quad (\text{B.7})$$

The second comes from the correction to the scale factor  $a \approx -1/H\tau$ . This can be obtained from the relation

$$d\tau = \frac{dt}{a} = -\frac{1}{H} d \left( \frac{1}{a} \right) \quad (\text{B.8})$$

<sup>9</sup> The integration  $\int_{-\infty}^0 dx \ln(-x) e^{ix} = i\mathbf{c}_1 - (\pi/2)$  has been used. Similar types of integrations will be used later.

and the expansion

$$\frac{1}{H} = \frac{1}{H_K} + \epsilon(t - t_K) + \mathcal{O}(\epsilon^2). \quad (\text{B.9})$$

Integrating (B.8) we can get the following expansion

$$a = -\frac{1}{H_K \tau} - \frac{\epsilon}{H_K \tau} + \frac{\epsilon}{H_K \tau} \ln(\tau/\tau_K) + \mathcal{O}(\epsilon^2). \quad (\text{B.10})$$

Plugging this correction term into (B.1), we obtain

$$\Delta\mathcal{A} \supset \left( \mathbf{c}_1 - \frac{1}{2} \right) \epsilon \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right)_K \frac{3k_1^2 k_2^2 k_3^2}{2K^3}. \quad (\text{B.11})$$

The third comes from the correction term to  $u(\tau, \mathbf{k}_i)$  which we obtained in appendix A. We first look at the corrections to the factor  $u(0, \mathbf{k}_i)$  in (B.1). The corrections to the final result come not only from the corrections in the bracket of (A.16), but also from the running from  $k_i$  to  $K$

$$\frac{H_{k_i}}{\sqrt{c_{si}\epsilon_i}} = \frac{H_K}{\sqrt{c_{sK}\epsilon_K}} \left( 1 - \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \ln \frac{k_i}{K} \right) + \mathcal{O}(\epsilon^2). \quad (\text{B.12})$$

So these add a correction term to the three-point correlation function<sup>10</sup>

$$\begin{aligned} \Delta\mathcal{A} \supset & \left( -3(\mathbf{c}_2 + 1)\epsilon - \frac{3\mathbf{c}_2}{2}\eta - 3 \left( \frac{\mathbf{c}_2}{2} + 1 \right) s - \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \ln \frac{k_1 k_2 k_3}{K^3} \right) \\ & \times \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right)_K \frac{3k_1^2 k_2^2 k_3^2}{2K^3}. \end{aligned} \quad (\text{B.13})$$

We next look at the corrections to the factor  $(d/d\tau)u^*(\tau, \mathbf{k}_i)$  in (B.1). To do this, we use (A.14) and expand the pre-factor around  $k = K$ ,

$$\begin{aligned} u_k(y) = & -\frac{\sqrt{\pi}}{2\sqrt{2}} \frac{H_K}{\sqrt{\epsilon_K c_{sK}}} \frac{1}{k^{3/2}} \left( 1 + \frac{\epsilon}{2} + \frac{s}{2} + \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \ln \frac{\tau}{\tau_K} \right) \exp \left( i\frac{\pi}{2}(\epsilon + \eta) \right) \\ & \times y^{3/2} H_\nu^{(1)}((1 + \epsilon + s)y). \end{aligned} \quad (\text{B.14})$$

<sup>10</sup> To make sure that the expansion such as (B.12) is perturbative, we need  $k_i \gg \mathcal{O}(K e^{-1/\epsilon})$ . Note that this condition still allows one of the momenta to be much smaller than the others, e.g.  $(k_3/k_1)^2 \ll \epsilon$ .

In order for the expansion such as (B.10) to be perturbative, we need  $\tau_K e^{-1/\epsilon} \gg \tau \gg \tau_K e^{1/\epsilon}$ . So it appears that the integration over  $\tau$  can only be taken from  $\tau_K e^{1/\epsilon}$  to  $\tau_K e^{-1/\epsilon}$ . We first look at the upper bound, since the mode  $k_i$  exits the horizon at  $\tau_i \approx -1/k_i c_{si}$ . At the upper bound of  $\tau$ , all modes have exited the horizon and their amplitudes are frozen. So the error introduced by including the integration from  $\tau_K e^{-1/\epsilon}$  to 0 is of order  $\mathcal{O}((k_i c_{si}/K c_{sK}) e^{-1/\epsilon}) \sim \mathcal{O}(e^{-1/\epsilon})$ . We next look at the lower bound. For the range of  $k_i$  that we are interested in, at  $\tau \sim \tau_K e^{1/\epsilon}$  all modes are well within the horizon. Their contributions are regulated away due to their rapid oscillation. Therefore, to order  $\mathcal{O}(e^{-1/\epsilon})$ , we can effectively take the integration range for  $\tau$  from  $-\infty$  to 0.

Denoting  $\Delta u(\tau, k_i)$  as the corrections to the leading order, we have

$$\begin{aligned} \Delta u^*(\tau, k_i) = & -\frac{1}{2} \frac{H_K}{\sqrt{c_{sK} \epsilon_K}} \frac{1}{k_i^{3/2}} \exp\left(-i\frac{\pi}{2}(\epsilon + \eta)e^{-ix}\right) \\ & \times \left[ -i(\epsilon + s) + (\epsilon + s)x + isx^2 \right. \\ & + \left( i\left(\epsilon + \frac{\eta}{2} + \frac{s}{2}\right) - \left(\epsilon + \frac{\eta}{2} + \frac{s}{2}\right)x - ix^2s \right) \ln \frac{\tau}{\tau_K} \\ & \left. + \frac{\sqrt{\pi}}{2} e^{ix} \left(\epsilon + \frac{\eta}{2} + \frac{s}{2}\right) x^{3/2} \frac{dH_\nu^*}{d\nu} \right], \end{aligned} \quad (\text{B.15})$$

where  $x \equiv -k_i c_{sK} \tau$ . The above corrections include those in the first line of (B.14), in  $y^{3/2}$  where  $y = -k_i c_{sK} \tau (1 - \epsilon - s \ln(\tau/\tau_K)) + \mathcal{O}(\epsilon^2)$  is used, and in  $H_\nu^{(1)*}((1 + \epsilon + s)y)$  which includes corrections in the index  $\nu$  and corrections in the variable  $y$ . Differentiating (B.15), we have

$$\begin{aligned} \frac{d}{d\tau} \Delta u^*(\tau, k_i) = & \frac{1}{2} \frac{H_K}{\sqrt{c_{sK} \epsilon_K}} \frac{1}{k_i^{3/2}} \exp\left(-i\frac{\pi}{2}(\epsilon + \eta)\right) k_i c_{sK} e^{-ix} \\ & \times \left[ -\left(\epsilon + \frac{\eta}{2} + \frac{s}{2}\right) + \left(\epsilon + \frac{\eta}{2} + \frac{s}{2}\right) \frac{i}{x} - i\epsilon x + sx^2 \right. \\ & + \left( i\epsilon + \frac{i}{2}\eta - \frac{3}{2}is - sx \right) x \ln \frac{\tau}{\tau_K} \\ & \left. + \frac{\sqrt{\pi}}{\sqrt{2}} e^{ix} \frac{d}{dx} \left( x^{3/2} \frac{dH_\nu^*}{d\nu} \right) \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \right]. \end{aligned} \quad (\text{B.16})$$

The first two lines in the square brackets in (B.16) contribute

$$\begin{aligned} \Delta \mathcal{A} \supset & \frac{3}{4} \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right)_K \left( \left( (3 - 6\mathbf{c}_1)\epsilon + \left( \frac{9}{2} - 3\mathbf{c}_1 \right) \eta + \left( 3\mathbf{c}_1 - \frac{17}{2} \right) s \right) \frac{k_1^2 k_2^2 k_3^2}{K^3} \right. \\ & \left. + \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \left( \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 - \frac{2}{K} \sum_{i > j} k_i^2 k_j^2 \right) \right). \end{aligned} \quad (\text{B.17})$$

The last term in (B.16) involves special functions and contributes

$$\Delta \mathcal{A} \supset \frac{3}{4} \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right)_K R_1(k_1, k_2, k_3) + \text{sym}, \quad (\text{B.18})$$

where

$$R_1(k_1, k_2, k_3) = \frac{k_2^2 k_3^2}{k_1} \text{Re} \left[ \int_0^\infty dx h^*(x) \left( 1 - i \frac{k_2 + k_3}{k_1} x \right) \exp \left( -i \frac{k_2 + k_3}{k_1} x \right) \right], \quad (\text{B.19})$$

$$h(x) = -2ie^{ix} + ie^{-ix}(1 + ix)[\text{Ci}(2x) + i\text{Si}(2x)] - i\pi \sin x + i\pi x \cos x. \quad (\text{B.20})$$

We have used the following relations,

$$xH_{\nu-2}^{(1)}(x) + xH_{\nu}^{(1)} = 2(\nu - 1)H_{\nu-1}^{(1)}(x), \quad (\text{B.21})$$

$$\left[ \frac{\partial J_{\nu}(x)}{\partial \nu} \right]_{\nu=1/2} = \left( \frac{1}{2}\pi x \right)^{-1/2} [\sin x \text{Ci}(2x) - \cos x \text{Si}(2x)], \quad (\text{B.22})$$

$$\left[ \frac{\partial N_{\nu}(x)}{\partial \nu} \right]_{\nu=1/2} = \left( \frac{1}{2}\pi x \right)^{-1/2} \{ \cos x \text{Ci}(2x) + \sin x [\text{Si}(2x) - \pi] \}, \quad (\text{B.23})$$

$$\left[ \frac{\partial J_{\nu}(x)}{\partial \nu} \right]_{\nu=-1/2} = \left( \frac{1}{2}\pi x \right)^{-1/2} [\cos x \text{Ci}(2x) + \sin x \text{Si}(2x)], \quad (\text{B.24})$$

$$\left[ \frac{\partial N_{\nu}(x)}{\partial \nu} \right]_{\nu=-1/2} = - \left( \frac{1}{2}\pi x \right)^{-1/2} \{ \sin x \text{Ci}(2x) - \cos x [\text{Si}(2x) - \pi] \}. \quad (\text{B.25})$$

The same procedure can be repeated for the second integration (4.34) in section 4.2,

$$\begin{aligned} -2i \int_{-\infty}^0 d\tau \, a^2 \, f_2(\tau) \left( u(0, \mathbf{k}_1) u(0, \mathbf{k}_2) u(0, \mathbf{k}_3) \right. \\ \left. \times u^*(\tau, \mathbf{k}_1) \frac{du^*(\tau, \mathbf{k}_2)}{d\tau} \frac{du^*(\tau, \mathbf{k}_3)}{d\tau} + \text{sym} \right) (2\pi)^3 \delta^3 \left( \sum_i \mathbf{k}_i \right) + \text{c.c.}, \end{aligned} \quad (\text{B.26})$$

where

$$f_2 = \frac{\epsilon}{c_s^4} (3 - 3c_s^2 - \epsilon). \quad (\text{B.27})$$

From the variation of  $f_2$ , we get the correction term

$$\Delta \mathcal{A} \supset \left( \left( \frac{3\eta}{4} - 3s \right) \left( \frac{1}{c_s^2} - 1 \right) - \frac{3s}{2} \right) \left( (1 - 2\mathbf{c}_1) \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 - (1 - \mathbf{c}_1) \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right). \quad (\text{B.28})$$

From the correction term to the scale factor  $a$ , we get

$$\Delta \mathcal{A} \supset \epsilon \left( \frac{1}{c_s^2} - 1 \right)_K \left( - \left( \frac{3}{2} + 3\mathbf{c}_1 \right) \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{3\mathbf{c}_1}{2} \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right). \quad (\text{B.29})$$

From the correction term to  $u(0, k_i)$ , we get

$$\begin{aligned} \Delta \mathcal{A} \supset \frac{3}{4} \left( 3(\mathbf{c}_2 + 1)\epsilon + \frac{3\mathbf{c}_2}{2}\eta + 3 \left( \frac{\mathbf{c}_2}{2} + 1 \right) s + \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \ln \frac{k_1 k_2 k_3}{K^3} \right) \\ \times \left( \frac{1}{c_s^2} - 1 \right)_K \left( \frac{2}{K} \sum_{i>j} k_i^2 k_j^2 - \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right). \end{aligned} \quad (\text{B.30})$$

The correction to  $u^*(\tau, k_i)$  in (B.15) contributes

$$\begin{aligned} \Delta\mathcal{A} \supset & -\frac{3}{4} \left( \frac{1}{c_s^2} - 1 \right)_K \left( (3 - 6\mathbf{c}_1)s \frac{k_1^2 k_2^2 k_3^2}{K^3} \right. \\ & + \left( -(1 + 2\mathbf{c}_1)\epsilon + \left( \frac{1}{2} - \mathbf{c}_1 \right) \eta - \left( \frac{3}{2} + \mathbf{c}_1 \right) s \right) \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 \\ & \left. + \left( \mathbf{c}_1\epsilon - \frac{1}{2}(1 - \mathbf{c}_1)\eta + \frac{1}{2}(1 + \mathbf{c}_1)s \right) \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right) \end{aligned} \quad (\text{B.31})$$

and

$$\Delta A \supset -\frac{3}{4} \left( \frac{1}{c_s^2} - 1 \right)_K \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \left( \frac{k_2^2 k_3^2}{k_1} G_1 + \text{sym} \right), \quad (\text{B.32})$$

where

$$G_1 \equiv \text{Re} \left[ \int_0^\infty dx h^*(x) \exp \left( -i \frac{k_2 + k_3}{k_1} x \right) \right]. \quad (\text{B.33})$$

The correction to  $(d/d\tau)u^*(\tau, k_i)$  in (B.16) contributes

$$\begin{aligned} \Delta\mathcal{A} \supset & \frac{3}{4} \left( \frac{1}{c_s^2} - 1 \right)_K \left( (2\epsilon - \eta + 3s) \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 \right. \\ & + (-2\mathbf{c}_1\epsilon + (1 - \mathbf{c}_1)\eta + (-6 + 6\mathbf{c}_1)s) \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \\ & + (2\epsilon + \eta + s) k_1 k_2 k_3 \\ & + (1 - 2\mathbf{c}_1)s \left( \frac{1}{K^3} \sum_{i \neq j} k_i^2 k_j^4 + \frac{2}{K^3} \sum_{i>j} k_i^3 k_j^3 \right) \\ & \left. - (2\epsilon + \eta + s) \left( \sum_i k_i^3 + \sum_{i \neq j} k_i k_j^2 + \sum_i k_i^3 \text{Re} \int_0^\infty dx \frac{e^{-ix}}{x} \right) \right) \end{aligned} \quad (\text{B.34})$$

and

$$\Delta\mathcal{A} \supset \frac{3}{4} \left( \frac{1}{c_s^2} - 1 \right)_K \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) (\tilde{M}_1 + \text{sym}), \quad (\text{B.35})$$

where

$$\tilde{M}_1 \equiv -k_1 \text{Re} \int_0^\infty dx \frac{1}{x} \left( k_2^2 + k_3^2 + i k_2 k_3 \frac{k_2 + k_3}{k_1} x \right) \exp \left( -i \frac{k_2 + k_3}{k_1} x \right) \frac{dh^*}{dx}. \quad (\text{B.36})$$

Notice that in (B.34), the last term is divergent. This divergence is cancelled by the divergence that appears in (B.36) using the limit  $h(x) \rightarrow (-2 + \mathbf{c}_1)i + i \ln 2x + \mathcal{O}(x^2)$  as  $x \rightarrow 0$ . So we can re-define

$$M_1 + \text{sym} \equiv \tilde{M}_1 - \text{Re}(k_2^3 + k_3^3) \int_0^\infty dx \frac{e^{-ix}}{x} + \text{sym} \quad (\text{B.37})$$

to absorb this divergence.

The following are the corrections to the third integration (4.35)

$$\begin{aligned}
& -2i \int_{-\infty}^0 d\tau a^2 f_3(\tau) (u(0, \mathbf{k}_1) u(0, \mathbf{k}_2) u(0, \mathbf{k}_3) \\
& \quad \times u^*(\tau, \mathbf{k}_1) u^*(\tau, \mathbf{k}_2) u^*(\tau, \mathbf{k}_3) (-\mathbf{k}_2 \cdot \mathbf{k}_3) + \text{sym}) \cdot (2\pi)^3 \delta^3 \left( \sum_i \mathbf{k}_i \right) + \text{c.c.}
\end{aligned} \tag{B.38}$$

with

$$f_3 = -\frac{\epsilon}{c_s^2} (1 - c_s^2 - 2s + \epsilon). \tag{B.39}$$

From the variation of  $f_3$ , we get

$$\begin{aligned}
\Delta \mathcal{A} \supset & \left( \left( \frac{\eta}{4} - \frac{s}{2} \right) \left( 1 - \frac{1}{c_s^2} \right) + \frac{s}{2} \right)_K \left( \frac{1 - \mathbf{c}_1}{2} \sum_i k_i^3 - \frac{1}{2} k_1 k_2 k_3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 \right. \\
& \left. + \frac{1 - 2\mathbf{c}_1}{K} \sum_{i > j} k_i^2 k_j^2 - \frac{1 - \mathbf{c}_1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right).
\end{aligned} \tag{B.40}$$

From the corrections to  $a$ , we get

$$\begin{aligned}
\Delta \mathcal{A} \supset & \frac{\epsilon}{2} \left( 1 - \frac{1}{c_s^2} \right)_K \left( \frac{-\mathbf{c}_1}{2} \sum_i k_i^3 - \frac{1}{2} k_1 k_2 k_3 + \frac{1}{2} \sum_{i \neq j} k_i k_j^2 \right. \\
& \left. - \frac{1 + 2\mathbf{c}_1}{K} \sum_{i > j} k_i^2 k_j^2 + \frac{\mathbf{c}_1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right).
\end{aligned} \tag{B.41}$$

From the corrections to  $u(0, k_i)$ , we get

$$\begin{aligned}
\Delta \mathcal{A} \supset & -\frac{1}{4} \left( 3(\mathbf{c}_2 + 1)\epsilon + \frac{3\mathbf{c}_2}{2}\eta + 3 \left( \frac{\mathbf{c}_2}{2} + 1 \right) s + \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) \ln \frac{k_1 k_2 k_3}{K^3} \right) \\
& \times \left( \frac{1}{c_s^2} - 1 \right)_K \left( \frac{1}{2} \sum_i k_i^3 + \frac{2}{K} \sum_{i > j} k_i^2 k_j^2 - \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \right).
\end{aligned} \tag{B.42}$$

Corrections from  $u^*(\tau, k_i)$  give

$$\begin{aligned}
\Delta \mathcal{A} \supset & -\frac{1}{8} \left( \frac{1}{c_s^2} - 1 \right)_K \\
& \times \left( (3\mathbf{c}_1\epsilon + \frac{3}{2}(-1 + \mathbf{c}_1)\eta + \frac{3}{2}(1 + \mathbf{c}_1)s) \sum_i k_i^3 \right. \\
& + (-3\epsilon + \frac{3}{2}(-1 + 2\mathbf{c}_1)\eta - \frac{3}{2}s) \sum_{i \neq j} k_i k_j^2 \\
& + (3(1 + 2\mathbf{c}_1)\epsilon - \frac{3}{2}\eta + (\frac{11}{2} + \mathbf{c}_1)s) k_1 k_2 k_3 \\
& \left. + (6\epsilon + 3(1 - 2\mathbf{c}_1)\eta + 2(1 + 2\mathbf{c}_1)s) \frac{1}{K} \sum_{i > j} k_i^2 k_j^2 \right)
\end{aligned}$$

$$\begin{aligned}
 & + (6\mathbf{c}_1\epsilon + 3(-1 + \mathbf{c}_1)\eta + (5 - 2\mathbf{c}_1)s) \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \\
 & - 2s \frac{1}{K} \sum_i k_i^4 + s \frac{1}{K^2} \sum_{i \neq j} k_i k_j^4 + (1 - \mathbf{c}_1) s \frac{1}{K^2} \sum_i k_i^5 \Big)
 \end{aligned} \tag{B.43}$$

and

$$\Delta\mathcal{A} \supset -\frac{1}{4} \left( \frac{1}{c_s^2} - 1 \right)_K \left( \epsilon + \frac{\eta}{2} + \frac{s}{2} \right) (N_1 + \text{sym}), \tag{B.44}$$

where

$$N_1 \equiv -3k_1(\mathbf{k}_2 \cdot \mathbf{k}_3) \text{Re} \int_0^\infty dx \frac{1}{x^2} \exp \left( -ix \frac{k_2 + k_3}{k_1} \right) \left( -1 + i \frac{k_2 + k_3}{k_1} x + \frac{k_2 k_3}{k_1} x^2 \right) h^*. \tag{B.45}$$

### B.1. Final results

Collecting all the results in section 4.2 and this appendix, we get the final result:

$$\begin{aligned}
 \langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \rangle &= (2\pi)^7 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) (\tilde{P}_K^\zeta)^2 \frac{1}{\prod_i k_i^3} \\
 &\times (\mathcal{A}_\lambda + \mathcal{A}_c + \mathcal{A}_o + \mathcal{A}_\epsilon + \mathcal{A}_\eta + \mathcal{A}_s)
 \end{aligned} \tag{B.46}$$

where we have decomposed the shape of the three-point function into six parts

$$\mathcal{A}_\lambda = \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} + (3 - 2\mathbf{c}_1) l \frac{\lambda}{\Sigma} \right)_K \frac{3k_1^2 k_2^2 k_3^2}{2K^3}, \tag{B.47}$$

$$\mathcal{A}_c = \left( \frac{1}{c_s^2} - 1 \right)_K \left( -\frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 + \frac{1}{8} \sum_i k_i^3 \right), \tag{B.48}$$

$$\begin{aligned}
 \mathcal{A}_o &= \left( \frac{1}{c_s^2} - 1 - \frac{2\lambda}{\Sigma} \right)_K (\epsilon F_{\lambda\epsilon} + \eta F_{\lambda\eta} + s F_{\lambda s}) \\
 &+ \left( \frac{1}{c_s^2} - 1 \right)_K (\epsilon F_{c\epsilon} + \eta F_{c\eta} + s F_{cs})
 \end{aligned} \tag{B.49}$$

$$\mathcal{A}_\epsilon = \epsilon \left( -\frac{1}{8} \sum_i k_i^3 + \frac{1}{8} \sum_{i \neq j} k_i k_j^2 + \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 \right), \tag{B.50}$$

$$\mathcal{A}_\eta = \eta \left( \frac{1}{8} \sum_i k_i^3 \right), \tag{B.51}$$

$$\mathcal{A}_s = s F_s. \tag{B.52}$$

The definitions of the sound speed  $c_s$ ,  $\Sigma$  and  $\lambda$  are

$$\begin{aligned}
 c_s^2 &\equiv \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}, \\
 \Sigma &\equiv XP_{,X} + 2X^2P_{,XX}, \\
 \lambda &\equiv X^2P_{,XX} + \frac{2}{3}X^3P_{,XXX}.
 \end{aligned} \tag{B.53}$$

The definitions of the four slow-variation parameters are

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv \frac{\dot{\epsilon}}{\epsilon H}, \quad s \equiv \frac{\dot{c}_s}{c_s H}, \quad l \equiv \frac{\dot{\lambda}}{\lambda H}. \quad (\text{B.54})$$

$\tilde{P}_K^\zeta$  is defined as

$$\tilde{P}_K^\zeta \equiv \frac{1}{8\pi^2} \frac{H_K^2}{c_{sK} \epsilon}. \quad (\text{B.55})$$

Note that  $H$ ,  $c_s$ ,  $\lambda$  and  $\Sigma$  in this final result are evaluated at the moment  $\tau_K \equiv -(1/Kc_{sK}) + \mathcal{O}(\epsilon)$  when the wavenumber  $K \equiv k_1 + k_2 + k_3$  exits the horizon  $Kc_{sK} = a_K H_K$ . So,  $\tilde{P}_K^\zeta$  in (B.55) is defined differently from (A.17). The various functions  $F$  are given by the following:

$$F_{\lambda\epsilon} \equiv \left( \frac{3}{2}\mathbf{c}_1 - \frac{9}{2}\mathbf{c}_2 - \frac{39}{4} - \frac{3}{2} \ln \frac{k_1 k_2 k_3}{K^3} \right) \frac{k_1^2 k_2^2 k_3^2}{K^3} + \frac{3}{4K^2} \sum_{i \neq j} k_i^2 k_j^3 - \frac{3}{2K} \sum_{i > j} k_i^2 k_j^2 + \frac{3}{4} R(k_1, k_2, k_3), \quad (\text{B.56})$$

$$F_{\lambda\eta} \equiv \frac{1}{2} F_{\lambda\epsilon} - \left( 3\mathbf{c}_1 - \frac{33}{4} \right) \frac{k_1^2 k_2^2 k_3^2}{K^3}, \quad (\text{B.57})$$

$$F_{\lambda s} \equiv \frac{1}{2} F_{\lambda\epsilon} + \left( \frac{3}{2}\mathbf{c}_1 - 6 \right) \frac{k_1^2 k_2^2 k_3^2}{K^3}, \quad (\text{B.58})$$

$$F_{c\epsilon} \equiv -\frac{1}{8}(\mathbf{c}_1 + 3\mathbf{c}_2 + 15) \sum_i k_i^3 - \frac{11}{8} \sum_{i \neq j} k_i k_j^2 + \frac{1}{8}(11 - 6\mathbf{c}_1) k_1 k_2 k_3 + \left( -\frac{\mathbf{c}_1}{2} + 3\mathbf{c}_2 + \frac{7}{2} \right) \frac{1}{K} \sum_{i > j} k_i^2 k_j^2 - \left( 2\mathbf{c}_1 + \frac{3}{2}\mathbf{c}_2 + \frac{3}{2} \right) \frac{1}{K^2} \times \sum_{i \neq j} k_i^2 k_j^3 + \left( \frac{9}{2} - 3\mathbf{c}_1 \right) \frac{k_1^2 k_2^2 k_3^2}{K^3} + \ln \frac{k_1 k_2 k_3}{K^3} \left( -\frac{1}{8} \sum_i k_i^3 + \frac{1}{K} \sum_{i > j} k_i^2 k_j^2 - \frac{1}{2K^2} \sum_{i \neq j} k_i^2 k_j^3 \right) + \frac{3}{4} Q(k_1, k_2, k_3), \quad (\text{B.59})$$

$$F_{c\eta} \equiv -\left( \frac{\mathbf{c}_1}{16} + \frac{3}{8}\mathbf{c}_2 + \frac{17}{16} \right) \sum_i k_i^3 - \left( \frac{3\mathbf{c}_1}{8} + \frac{11}{16} \right) \sum_{i \neq j} k_i k_j^2 + \frac{17}{16} k_1 k_2 k_3 + \left( \frac{\mathbf{c}_1}{2} + \frac{3\mathbf{c}_2}{4} - \frac{5}{2} \right) \frac{1}{K} \sum_{i > j} k_i^2 k_j^2 + \left( -\mathbf{c}_1 - \frac{3}{8}\mathbf{c}_2 + \frac{7}{4} \right) \frac{1}{K^2} \times \sum_{i \neq j} k_i^2 k_j^3 + \left( -\frac{9}{4} + \frac{3}{2}\mathbf{c}_1 \right) \frac{k_1^2 k_2^2 k_3^2}{K^3} + \ln \frac{k_1 k_2 k_3}{K^3} \left( -\frac{1}{16} \sum_i k_i^3 + \frac{1}{2K} \sum_{i > j} k_i^2 k_j^2 - \frac{1}{4K^2} \sum_{i \neq j} k_i^2 k_j^3 \right) + \frac{3}{8} Q(k_1, k_2, k_3), \quad (\text{B.60})$$

$$\begin{aligned}
 F_{cs} \equiv & -\left(\frac{7\mathbf{c}_1}{16} + \frac{11}{16}\right) \sum_i k_i^3 - \frac{5}{16} \sum_{i \neq j} k_i k_j^2 - \left(\frac{\mathbf{c}_1}{8} + \frac{3}{16}\right) k_1 k_2 k_3 \\
 & + \left(\frac{21}{4}\mathbf{c}_1 + \frac{9}{4}\mathbf{c}_2 + \frac{41}{8}\right) \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 + \left(\frac{15}{8}\mathbf{c}_1 - \frac{9}{8}\mathbf{c}_2 - \frac{21}{4}\right) \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 \\
 & + \frac{1}{4K} \sum_i k_i^4 - \frac{1-\mathbf{c}_1}{8} \frac{1}{K^2} \sum_i k_i^5 - \frac{1}{8K^2} \sum_{i \neq j} k_i k_j^4 \\
 & + \left(\frac{27}{4} - \frac{3}{2}\mathbf{c}_1\right) \frac{k_1^2 k_2^2 k_3^2}{K^3} + \left(\frac{3}{4} - \frac{3}{2}\mathbf{c}_1\right) \frac{1}{K^3} \sum_{i \neq j} k_i^2 k_j^4 + \left(\frac{3}{2} - 3\mathbf{c}_1\right) \frac{1}{K^3} \sum_{i>j} k_i^3 k_j^3 \\
 & + \ln \frac{k_1 k_2 k_3}{K^3} \left( -\frac{1}{16} \sum_i k_i^3 + \frac{1}{2K} \sum_{i>j} k_i^2 k_j^2 - \frac{1}{4K^2} \sum_{i \neq j} k_i^2 k_j^3 \right) \\
 & + \frac{3}{8} Q(k_1, k_2, k_3),
 \end{aligned} \tag{B.61}$$

$$\begin{aligned}
 F_s \equiv & \frac{1}{4}(1 - \mathbf{c}_1) \sum_i k_i^3 + \frac{1}{4} \sum_{i \neq j} k_i k_j^2 - \frac{1}{4} k_1 k_2 k_3 + (-1 + 2\mathbf{c}_1) \frac{1}{K} \sum_{i>j} k_i^2 k_j^2 \\
 & + (1 - \mathbf{c}_1) \frac{1}{K^2} \sum_{i \neq j} k_i^2 k_j^3 + \left(\frac{9}{2} - 3\mathbf{c}_1\right) \frac{k_1^2 k_2^2 k_3^2}{K^3},
 \end{aligned} \tag{B.62}$$

where  $\mathbf{c}_1 = 0.577 \dots$  is the Euler constant and  $\mathbf{c}_2 \equiv \mathbf{c}_1 - 2 + \ln 2 = -0.73 \dots$ . The functions  $R(k_1, k_2, k_3)$  and  $Q(k_1, k_2, k_3)$  involve special functions,

$$R(k_1, k_2, k_3) \equiv \frac{k_2^2 k_3^2}{k_1} \text{Re} \left[ \int_0^\infty dx \left( 1 - i \frac{k_2 + k_3}{k_1} x \right) \exp \left( -i \frac{k_2 + k_3}{k_1} x \right) h^*(x) \right] + \text{sym}, \tag{B.63}$$

$$\begin{aligned}
 Q(k_1, k_2, k_3) \equiv & -k_1 \text{Re} \left[ \int_0^\infty dx \frac{1}{x} \left( k_2^2 + k_3^2 + i k_2 k_3 \frac{k_2 + k_3}{k_1} x \right) \exp \left( -i \frac{k_2 + k_3}{k_1} x \right) \frac{dh^*}{dx} \right] \\
 & - (k_2^3 + k_3^3) \text{Re} \left[ \int_0^\infty dx \frac{e^{-ix}}{x} \right] \\
 & - \frac{k_2^2 k_3^2}{k_1} \text{Re} \left[ \int_0^\infty dx \exp \left( -i \frac{k_2 + k_3}{k_1} x \right) h^*(x) \right] \\
 & + \frac{k_1}{2} (k_1^2 - k_2^2 - k_3^2) \\
 & \times \text{Re} \left[ \int_0^\infty \frac{dx}{x^2} \exp \left( -i \frac{k_2 + k_3}{k_1} x \right) \left( -1 + i \frac{k_2 + k_3}{k_1} x + \frac{k_2 k_3}{k_1^2} x^2 \right) h^*(x) \right] \\
 & + \text{sym},
 \end{aligned} \tag{B.64}$$

$$h(x) \equiv -2ie^{ix} + ie^{-ix}(1 + ix)[\text{Ci}(2x) + i\text{Si}(2x)] - i\pi \sin x + i\pi x \cos x. \tag{B.65}$$

In all formulae, the ‘sym’ stands for two other terms with cyclic permutation of the indices 1, 2 and 3.

## B.2. The squeezed limit

It is interesting to look at the behaviours of various functions in the squeezed limit (for example,  $k_1 = k_2$  and  $k_3 \rightarrow 0$ ), because from them one can roughly know whether the shape of a non-Gaussianity is closer to the DBI type (figure 2), or the slow-roll type (figure 3).

In slow-roll inflation, Maldacena has argued that the three-point function in the squeezed limit goes to [7]

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \rangle \rightarrow -(2\pi)^7 \frac{1}{4k_1^3 k_3^3} (n_s - 1) P_{k_1}^\zeta P_{k_3}^\zeta \quad (\text{B.66})$$

to all orders in the slow-roll parameters. This condition was generalized to general single-field inflation in [84].

Here we check this condition in the general single-field inflation model with large non-Gaussianities. We first consider the case where  $c_s = 1$ . For a large  $\lambda/\Sigma \gg 1$ , the right-hand side of the consistency condition starts from order  $\mathcal{O}(\epsilon)$  (does not include the terms  $P_{k_1}^\zeta P_{k_3}^\zeta$ ), while the left-hand side starts from  $\mathcal{O}(\lambda/\Sigma)$  in  $\mathcal{A}_\lambda$  and  $\mathcal{O}(\epsilon\lambda/\Sigma)$  in  $\mathcal{A}_o$ . So, in order for the condition to hold, both the leading and subleading order terms in the three-point function have to vanish in the squeezed limit. It is not difficult to see that  $\mathcal{A}_\lambda$  vanishes in this limit. Interestingly, the subleading terms (B.56), (B.57) and (B.58) also vanish in this limit due to a cancellation from the special function  $R$  (see appendix B.3).

But the case with a small  $c_s \ll 1$  is more subtle. The condition (B.66) similarly requires the leading and subleading orders of the left-hand side to vanish. It is easy to see the leading order contribution  $\mathcal{A}_c$  satisfies this condition. For the subleading order, without going into the full details of the special function  $Q$ , it is easy to see that (B.59), (B.60) and (B.61) cannot vanish simultaneously by taking differences between them (to get rid of  $Q$ ). So overall  $\mathcal{A}_o$  goes as

$$\frac{A_o}{k_1 k_2 k_3} \propto \frac{k_1}{k_3}. \quad (\text{B.67})$$

Hence, the shape  $\mathcal{A}_o/k_1 k_2 k_3$  has similar poles to the slow-roll shapes  $\mathcal{A}_\epsilon$  and  $\mathcal{A}_\eta$  in the squeezed limit. In addition, since

$$\frac{F_s}{k_1 k_2 k_3} \rightarrow \frac{k_1}{k_3}, \quad (\text{B.68})$$

the order  $\mathcal{O}(\epsilon)$  terms on both sides of the condition cannot match either. It will be interesting to have a more intuitive understanding as to why the subleading terms deviate from the consistency condition when the sound speed deviates from one.

## B.3. Some details

In this section, we demonstrate some details to show that  $F_{\lambda\epsilon}$  vanishes in the squeezed limit. To calculate the squeezed limit of the R-term, we analytically continue the integrand in the convergent direction by  $x \rightarrow -ix$  and note the asymptotic behaviour

$$\text{Ci}(-2ix) - i\text{Si}(-2ix) \sim -\frac{i}{2}\pi - \frac{1}{2x}e^{-2x} \left( 1 + \mathcal{O}\left(\frac{1}{x}\right) \right), \quad x \rightarrow +\infty. \quad (\text{B.69})$$

So the  $e^x$  terms cancel in the asymptotic behaviour of  $h^*(x)$ , and we find

$$h^*(x) \sim x e^{-x}, \quad x \rightarrow +\infty. \quad (\text{B.70})$$

One can also compute the asymptotic of  $h^*(x)$  near  $x \sim 0$ , and we find

$$h^*(x) = (c - i \log(x)) + \mathcal{O}(x^2), \quad (\text{B.71})$$

where  $c$  is a constant that will not be important for us.

We can now take a squeezed limit  $k_3 \rightarrow 0$  in the R-term. There are 3 terms in the symmetric rotation of the indices. Two terms are proportional to  $k_3^2$  and the integral is convergent because of the good asymptotic behaviour (B.70), (B.71), so they vanish in the squeezed limit. The remaining term is

$$R(k_1, k_2, k_3) = \frac{k_1^2 k_2^2}{k_3} \text{Re} \left[ -i \int_0^\infty dx \left( 1 - \frac{K}{k_3} x \right) e^{-(K/k_3)x} h^*(x) \right]. \quad (\text{B.72})$$

Here  $K = 2k_1 = 2k_2$ . We only keep the leading term in the squeezed limit and drop higher powers of  $k_3$ . We can then change integration variable  $x \rightarrow (k_3/K)x$  and expand around  $k_3 = 0$  using the formula (B.71). We find

$$\begin{aligned} R(k_1, k_2, k_3) &= \frac{k_1^2 k_2^2}{K} \text{Re} \left[ -i \int_0^\infty dx (1 - x) e^{-x} \left( c - i \log \left( \frac{k_3}{K} x \right) \right) \right] + \mathcal{O}(k_3^2) \\ &= \frac{k_1^2 k_2^2}{K} + \mathcal{O}(k_3^2) = \frac{k_1^3}{2} + \mathcal{O}(k_3). \end{aligned} \quad (\text{B.73})$$

So the squeezed limit of the term  $F_{\lambda\epsilon}$  vanishes

$$F_{\lambda\epsilon} = \left( \frac{3}{8} - \frac{3}{4} + \frac{3}{8} \right) k_1^3 + \mathcal{O}(k_3) = \mathcal{O}(k_3). \quad (\text{B.74})$$

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